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ABSTRACT

This document contains the proceedings of the 25th annual conference of the Mathematics Education Research Group of Australia (MERGA) held at the University of Auckland, New Zealand. The focus of this meeting is mathematics education in the South Pacific. Presentations are centered around the topic of numeracy in primary or elementary school. Papers include: (1) "A Samoan Perspective on Pacific Mathematics Education" (Karoline Afamasaga Fuata'i); (2) "The Process of Introducing New Tasks Using Dynamic Geometry into the Teaching of Mathematics" (Colette Laborde); (3) "Effectiveness and Efficiency of Calculator Use for Computation in Years 8-10" (Jack Bana and Rachel Shipley); (4) "A Problem-Based Schema Analysis in Algebra" (Mohan Chinnappan); (5) "An Investigation of Disjuncture between Graphing in School Mathematics and School Physics" (Patricia A. Forster); (6) "Single or Combination Grades in Mathematics in the Early Years" (Marj Horne); (7) "A Model for Teaching Numeracy Strategies" (Peter Hughes); (8) "Teacher Development for Inquiry Based Instructional Practice in Mathematics: A Poststructuralist Postscript" (Mary Klein); (9) "Patterns of Misperceptions in Linear Transformations: Four Illustrations" (Martin Lamb, Sam Leong, and John Malone); (10) "Common Errors in Mental Computations of Students in Grades 3-10" (Alistair McIntosh); (11) "Rethinking Curriculum: An Ethical Perspective" (Jim Neyland); (12) "Monitoring Effective Use of Computer Algebra Systems" (Robyn Pierce and Kaye Stacey); (13) "Professional Development of Teachers of Mathematics through Extended Collegial Dialogue: The ACT Mathematics Quality Teacher Program" (Steve Thornton and Kerrie Blain); and (14) "Validation of the New Zealand Number Framework" (Jenny Young-Loveridge and Vince Wright). The second volume also includes 22 short communications. (KHR)

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MERGA 25

Mathematics Education in the South Pacific

Proceedings of the 25th Annual Conference of the Mathematics Education Research Group of Australasia Incorporated, held at The University of Auckland, July 7-10, 2002

Volume I

Bill Barton, Kathryn C. Irwin,
Maxine Pfannkuch & Michael O. J. Thomas
(Editors)

THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

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Preface

These are the proceedings of the Twenty-fifth Annual Conference of the Mathematics Education Group of Australasia. During the twenty-five years in which this group has met, research in mathematics education has grown markedly in this part of the world and is now a field of research in nearly every university in New Zealand and Australia.

The focus of this conference is Mathematics Education in the South Pacific, an appropriate focus for a conference for which authors come from Australia, New Zealand, Samoa, and several countries that border the northern, western and eastern Pacific. It is a particularly appropriate focus for this conference in Auckland, which is reported to be the city with the largest population of people who come from Pacific nations. We are pleased to welcome keynote speakers from France, the United States, Samoa and Dunedin to this city for this conference.

Papers in proceedings for different years reflect changing areas of concern in mathematics education. The first conference in 1977 included an early paper on problem solving which then became a major focus of mathematics education in the 1980s. In the 1980s and 1990s there were many papers on this topic in MERGA Proceedings. However, in 2002 there is only one paper on problem solving, and there are many papers related to numeracy in primary or elementary school. This change of focus reflects the emphasis that mathematics educators and several governments are placing on numeracy at this time.

Each of the 81 research papers accepted for these proceedings has been reviewed, without the author being identified, by two experienced researchers in mathematics education who followed strict guidelines. Where reviewers disagreed about the acceptability of the paper, a third reviewer read the paper without seeing the previous reviews. Only those papers that were accepted by two reviewers have been included.

Some papers were submitted as entries for the Practical Implications Award, and reviewed by a committee of four other researchers. In 2002, this has been awarded to the paper written by Len Sparrow and Sandra Frid. Other papers were submitted for consideration for the Early Achievement Awards. The winner of this award will be decided after the presentations at the conference.

A considerable amount of time and effort has gone into reviewing these papers, making arrangements for the conference, and preparing these proceedings. As chairperson of the Proceedings Committee, I would particularly like to thank Mike Thomas, Maxine Pfannkuch, and Bill Barton for their many hours of labour.

Kathryn C. Irwin
for the Proceedings Committee
Auckland, New Zealand
June 2002

The Conference Committee would like to thank the following who reviewed papers for the MERGA 25 conference:

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Keynote Papers

A Samoan Perspective on Pacific Mathematics Education

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The status of mathematics education in schools in Samoa is a major concern to post-secondary mathematics educators. Incoming students from secondary schools are not adequately prepared for advanced studies in mathematics for lack of a solid understanding of basic mathematics principles which is essential for further studies in calculus and linear algebra. This paper provides a general overview of national examinations results and selected comments from examiners' reports to illustrate the state of mathematics education in Samoan schools. The quality of mathematics learning in schools directly impacts on first year programs in post-secondary institutions as time is often taken out to bridge students up to the appropriate mathematics entry level. The paper briefly describes some of the research and professional development activities organized by the University in an effort to impact on the teaching of mathematics within its teacher training programs.

This paper focuses on the state of mathematics education in Samoa. However, because of similar historical developments in their political, economical and educational systems, and common traditional and cultural backgrounds, many of the issues plaguing Samoa's educational sector are also common in the rest of the island countries. This paper is organized into four parts. The first part discusses the general state of school mathematics as indicated by national examinations results in the last five years (1996-2000) to provide the general mathematics environment which nurture students for further studies at tertiary level including the National University of Samoa (NUS). Included also are selected comments from examiners' reports that are indicative of main concerns for each national examination. The second part defines the problem that is prevalent in mathematics classrooms in Samoa as indicated by national results and manifested through the performances of students that enroll in NUS mathematics courses. The third part provides a brief discussion of various philosophical perspectives on mathematics and mathematics education to illustrate how these views can influence the way mathematics may be taught and assessed in the classroom. The fourth part presents a synopsis of on-going research and professional development activities conducted and organized by NUS as part of its on-going efforts to re-dress low pass rates in its mathematics courses and to meet the training needs of Samoan mathematics teachers.

National Examinations - Primary Level

National literacy and numeracy testing started recently as a direct result of the UNESCO PILL Report (UNESCO, 1992) which raised the alarm on Samoa's national literacy rates. Subsequently, Samoa's Primary Educational Literacy Levels (SPELL) Tests I (Year 4) and II (Year 6) were established and designed to monitor national standards. The tests are also important for the identification of students who are not achieving minimum competencies in both literacy and numeracy in schools. (Department of Education (DOE), 1999a, 2000a). The results (recorded as "at-risk percentages") track the proportion of total students at each level that are not achieving prescribed minimum competencies. The trend to date (1996 – 2000, Figure 1) shows that the Year 4 'at risk' percentage usually doubles

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B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 1–13). Sydney: MERGA. ©2002 MERGA Inc.

by the time the same cohort is tested again in Year 6 (DOE, 1999a, 2000a). *This is a concern to the DOE including those at NUS.* Of particular importance to NUS is the implication in terms of both pre-service and in-service mathematics teacher education. Within the pre-service teacher education program, mathematics content courses are the responsibility of the Mathematics & Statistics Department (MSD). Thus, the state of mathematics education in schools directly impacts on the mathematics "readiness" or lack thereof, of incoming students to cope with pre-degree and degree content courses in mathematics.

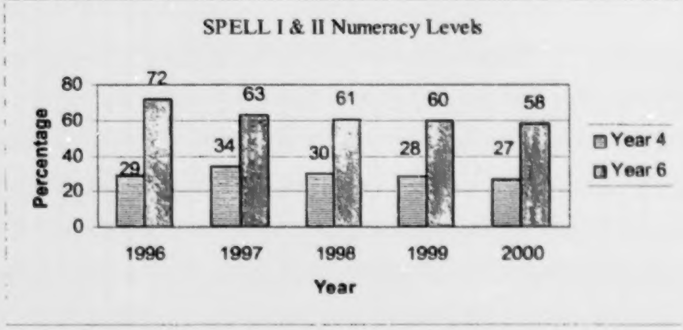

Figure 1. Years 4 & 6 Numeracy at-risk percentages.

At primary level, in spite of the decreasing Year 6 'at-risk' percentages over the period, see Figure 1, the amount of more or less 60% is still too high. This means that more than half the students in Year 6 each year in the last five years continue to underachieve. In fact, the examiner's report suggested that regular professional training for Year 4 and 6 teachers should be conducted to create awareness and develop effective strategies to help students improve their basic literacy and numeracy skills. Furthermore, the high 'at-risk' percentage in Year 6 may also be indicative of limited access to, or lack of supplementary and remedial materials for students (DOE, 2000a). A host of factors may be contributing to this poor performance such as poor mathematics knowledge and pedagogical skills of primary teachers, unavailability of relevant resources to supplement classroom teaching, and low level of institutional support for teachers and students; collectively they contribute to teaching and learning styles that promote rote-memorization. (DOE, 1986, 1995).

At the end of primary level is the Year 8 national examination established mainly for the selection of students to enter the top five government secondary schools. Of concern is the continuous low pass rates (those with raw scores ≥ 50%) in mathematics. In particular, it decreased from 21% (1998) to 16% (1999) with a slight improvement to 18% in 2000. These low pass rates are symptomatic of problems in the teaching and learning of mathematics either through lack of resources such as textbooks for both teachers and students, and/or poor content knowledge and pedagogical skills of teachers which translates to incomplete coverage of prescribed syllabus. Under these circumstances, students often resort to rote-memorization of procedures and entire answers that are reproduced at random during exams. In fact Year 8 Examiners' Reports continuously point out that the majority of students "(are) guessing what the answers might be" oftentimes quite unrelated to given questions, and "solving word problems (is) the most common weakness." Moreover, students appear to lack "basic understanding of mathematics operations and their sequence … to get the correct answers" (DOE, 1999b, 2000b).

National results up to this level consistently show that girls perform better than boys. Raw means are consistently low at < 35% as shown in Figure 2 below.

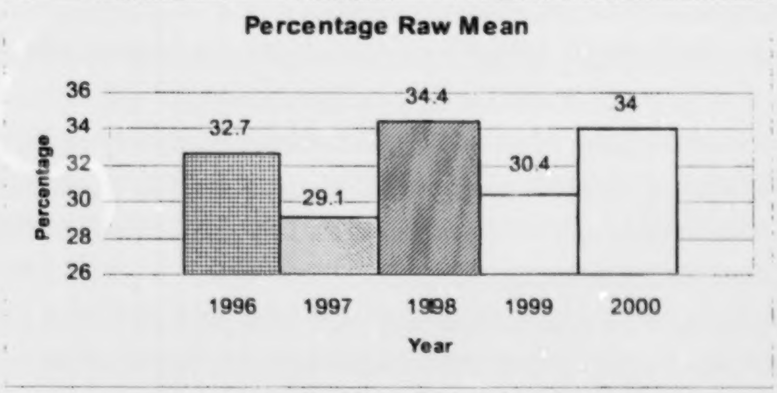

Figure 2. Year 8 National Examination Raw Means 1996 – 2000.

Given the poor numeracy results at Year 6, the consistently low pass rates and raw means at Year 8 is further confirmation of the level of mathematics learning in most primary classrooms. The 2000 Examiner's Report (DOE, 2000b) further re-iterates this poor performance particularly in both mathematics and science as well as the continual lowest ranking of mathematics' raw means out of all examinable school subjects each year.

National Examinations - Secondary Level: Years 11 - 12

The Years 11, and 12 national mathematics examination results in Figure 3 show that the raw means are all below 36% (DOE, 1999c, 1999d, 2000c, 2000d). Of the 2473 Year 11 students in 1999, only 14% passed whilst only 37% of the schools achieved raw means above the national mean. The Examiners' Reports pointed out that in sections where students had to show their working, most scripts were either left blank or had answers that were totally different from questions. According to the examiners, these actions "... reflect the lack of ... skills like knowledge, understanding, interpreting and application." It was recommended "that the (above) learning skills (should be) developed to improve (students') language ability in problem solving" (DOE, 1999c, 2000c).

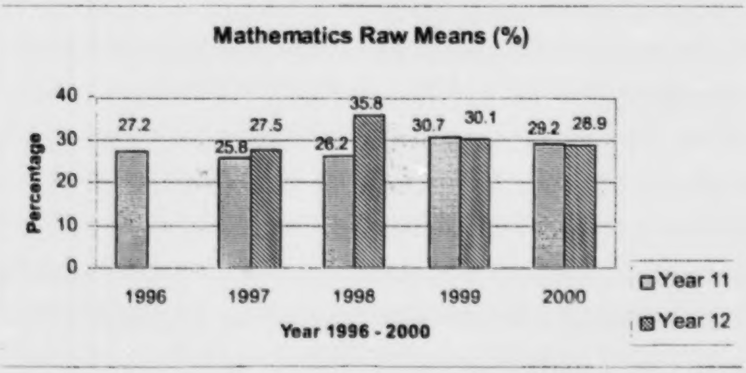

Figure 3. Years 11 & 12 Mathematics National Examinations.

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On the other hand, the Year 12 national results showed low varying raw means (27% to <36%) in the last five years. It was reported also that only 14% of 1913 students passed in 1999 (DOE, 1999d), decreasing to 10% (of 1845) in 2000 with a slight increase to 13% (of 1956) in 2001 (DOE, 2000e, 2000f, 2001). The poor results depicted in Figure 3 is alarming and a major concern to those teaching mathematics at Year 13 and beyond. Evidence from examination scripts showed that students had difficulties transferring their skills with numerical expressions to algebraic ones, effectively applying mathematics concepts such as ratio and proportions to real life situations, correctly expressing problems in mathematical statements, and misunderstanding/misinterpreting basic mathematical terms. For the majority of students, their performances appear to deteriorate as they advance from primary through to secondary level. Common difficulties and misconceptions appeared to be carried over to subsequent levels as evidenced by similar recurring comments across examinations. For those entering pre-degree programs at NUS, their weak understanding and ineffective application of fundamental principles to solve problems permeate their study of mathematics and impede their abilities to cope with more advanced mathematics.

Regional PSSC Examinations Year 13 – Secondary Level

Students who satisfy an institution's selection criteria (based on Year 12 results) continue onto Year 13, the final year of secondary level. However, unlike other Pacific Island countries which have restricted intakes for the Regional Pacific Senior Secondary Certificate (PSSC) examinations at Year 13, Samoa gives as many students as possible the opportunity to study at this level. Subsequently, its national scaled means are often 2% to 5% below regional scaled means and consistently below the 50% pass mark over the years, see Figure 4 below. Gender performance in Samoa in terms of grade averages show that boys consistently do better than girls over the years (South Pacific Board of Educational Assessment (SPBEA), 1999, 2000). (Note that the reverse trend characterises national primary examination results.) The regional raw means over the years are also shown in Figure 4 as they may "indicate the changing teaching and learning standards of the region or that the papers are changing in their level of difficulty" (SPBEA, 2000). The consistently low scaled means (under-50%) suggest that in spite of increasing enrolments in the PSSC examinations, many students continue to under-achieve in mathematics.

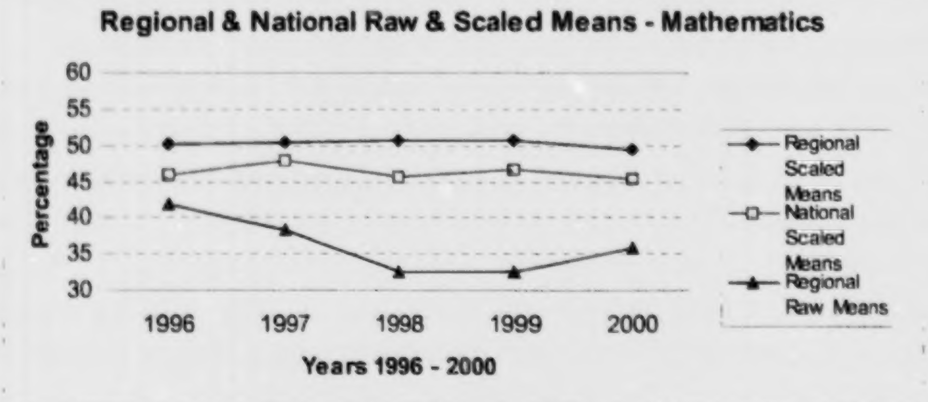

Figure 4. PSSC Mathematics Raw & Scaled Means.

University Preparatory Year (UPY) - NUS

After PSSC, admission into the UPY level at NUS, is determined by attaining a minimum aggregate score (usually 15) on the PSSC examinations. In the last two years, about 20% of Year 13 students qualified to enter UPY. About two-thirds of this group usually form the science and commerce streams in which mathematics and/or statistics are requirements. In particular, the required sequence for science students is MAU01: Algebra & Trigonometry (Semester 1) and MAU02: Calculus (Semester 2) whilst STU01:Statistics is available as an option. In comparison, the required sequence for commerce students is MAU03:Algebra and STUOI:Statistics. Although both sequences are available to social science students as electives, very few choose mathematics/statistics at this level. Figure 5 shows pass rates for all mathematics and statistics courses in the period 1996 to 2000. It should be noted that pass rates for MAU02 were much higher when eligibility was conditional upon a pass in MAU01 (up to 1997).

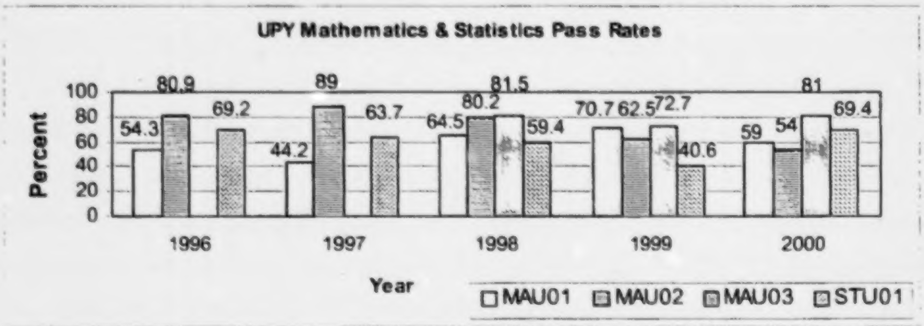

Figure 5. UPY Science, Mathematics & Statistics Pass Rates.

Although UPY students enjoy a high national ranking, these same students often experience difficulties in effectively applying basic mathematics concepts, axioms, and principles to solving given problems thus hindering performance at recommended levels in algebra, trigonometry, statistics, and calculus. As a consequence, the prescribed two-semester curriculum to prepare them for university studies is severely comprised as time is taken out for remedial work. Most of the misconceptions at this level are well ingrained and often difficult to overcome in the first semester without some sacrifices for both student and lecturer for extra work outside of scheduled class time. It is desirable therefore for students to develop a more solid understanding of fundamental mathematics concepts and principles while at primary and secondary levels in addition to being proficient in applying operations and executing procedures.

In summary, and on the basis of examiners' reports, it appears that students right across the levels have difficulties understanding, and interpreting the language of given problems. It is desirable to be proficient in executing a sequential procedure. However, when it is applied indiscriminately contrary to the intention of the problem then the action signifies a misinterpretation of the problem context and may be suggestive of rote-memorization learning styles. These weaknesses are symptomatic of a lack of a deep understanding of the conceptual structure of mathematics and poor meta-cognitive skills. Meta-cognitive skills include the abilities to control and monitor the activity of problem solving, to identify appropriate strategies and methods required by problems, and to know how and when to use these methods to successfully solve problems. Given that metacognitive activities improve achievement but that students may not automatically use them, it is suggested that students be taught these activities and encouraged to use them (Belmont, 1989). There is, therefore, a need to undertake research in Samoan classrooms to examine ways in which students understanding of mathematics could be improved, and subsequently conduct professional development programs for Samoan mathematics teachers that are informed and guided by research findings.

The Problem

The general problem is, therefore, the pervasive poor state of mathematics teaching and learning that seems to exist at all levels in Samoa's educational system. A number of interacting factors may have contributed to the existing situation ranging from ministerial, community and institutional support for classroom teaching and learning to teacher's mathematical beliefs and background, mathematics curriculum, classroom socio-cultural practices, national and school assessment procedures, and the students themselves. However, this paper will focus only on students, their learning environment within the classroom, and the problem of how their understanding of mathematics can be improved. It is acknowledged that students' mathematical beliefs and learning styles are very much influenced by what is regarded as important and legitimate by the ministry, institution and community at large as enacted within their school environment. In particular, national, regional and post-secondary mathematics results suggest that Samoan students have difficulties interpreting and effectively solving word problems, justifying and rationalizing solutions consistent with problem contexts; in effect a predominantly procedural, and narrow view of mathematics. Overall, they are mathematically illiterate in the view of Richards (1991) as defined below. Subsequently, some concerns and issues arising from students' performances in national examinations and at NUS may be collectively categorized into four types namely problem interpretation, transfer and application of knowledge and skills to new situations, and rote-memorization of procedures, and a fourth category that is more related to teachers' role concerning the incomplete coverage of prescribed syllabus during the school year. This categorization facilitates discussion but all four interweave and are mutually dependent on each other. For example, problem misinterpretation connotes language problems, which may be due to unfamiliar terms students may have never seen before thus they cannot comprehend the question. This may mean that the topic was not properly taught if at all (type 4) thus the tendency may be to just write down anything for an answer. However, even if the terms are familiar, students may not possess the relevant knowledge and meta-cognitive skills to effectively evaluate the applicability or not of identified methods and strategies and past experiences to successfully solve problems (types 2 and 3).

According to Richards (1991), "... the traditional structure of classroom instruction (and practice) are patterned after a faulty perception of mathematical activity. Students are not able to conduct and maintain a mathematical discussion; communicate *(discuss, evaluate, explain, describe)* the mathematics as they may not understand the appropriate language, syntax or rhythm of a mathematical discussion; and no sense of making conjectures, or evaluating mathematical assumptions" (Richards, 1991). In the case of Samoa and for the purpose of this paper, the most important questions are: (1) What can be done to improve the quality of mathematics teaching and learning in Samoan schools? (2) How can the communication between researchers at NUS and practitioners in schools be maintained? Answers to these questions would offer guidelines to improve the teaching and learning of mathematics courses offered for teacher education programs. Doing so would at least impact on the quality of teacher preparation and continuing professional development programs in mathematics.

The next section briefly discusses various beliefs and philosophies of mathematics as a discipline, which could assist mathematics teachers in re-assessing the way mathematics ought to be taught and learnt in Samoan classrooms.

Philosophies of Mathematics and Mathematics Education

Mathematics is often viewed as a formal logical deductive system from given primitive axioms in which truth is established by a formal set of syllogisms. The meaning or significance of a mathematical statement is extracted by interpreting its relationships to the formal system (Schoenfeld, 1991). Overall, a discipline is generally viewed as having its own "superordinate, general concepts and its categorical and causal relationships" (White, 1982); "… its within-field and outside-the-field values; ... its agents and audiences (the so-called "community of scholars"); and … its phenomena of interest (that it) deals with, the occasions which give rise to the *quest* for knowledge" (Gowin, 1981). Hence, mathematics knowledge may be seen as a cultural product, in the sense that its overall context, its direction of movement, respond to the pressures of society (Hersh, 1994). Through publications, conferences and professional discussions and communications, mathematicians engage in warranting mathematics knowledge. This is a social process (Manin, 1977). According to Ernest (1999, 1994a), a revolutionary new tradition in the philosophy of mathematics is primarily naturalistic concerned with describing the nature of mathematics, its history and applications including the practices of mathematicians, and the place of mathematics in human culture. Like science, mathematics can advance by making mistakes, correcting them and recorrecting them.

In contrast, from the socio-linguistic perspective, if language is viewed as a conveyor of information and a means of situating people in a social system, then mathematics learning may be viewed as a task practiced by communities in terms of linguistic discourse enacted by its members. Richards (1991) defines four such types of communities as *research maths, inquiry maths, journal maths, and school maths*. *Research maths* is spoken mathematics of the professional mathematician and scientist while *inquiry maths* is the mathematics used by mathematics literate adults. In comparison, *journal maths* is the language of mathematical publications and papers whilst *school maths* is the discourse of the mathematics classroom in which mathematics is taught consisting of a body of facts and procedures to be learnt. In this view, if students are to have a view of *mathematics as a socially constructed body of knowledge* then they should be enculturated into the practice of mathematicians by learning to be mathematics literate (*inquiry maths*). That is, students need to *think, discuss, and act mathematically*. Teaching and learning mathematics therefore involves being initiated into mathematical ways of knowing, ideas and practices of the mathematical community and making these ideas and practices meaningful at an individual level (Ernest, 1999, 1998, 1994b). Thus, if mathematics learning is viewed both as a process of *active individual construction*, and a *process of enculturation* into the mathematical practices of a wider society, then the aim of mathematical classroom practices ought to include *equipping students with the appropriate language and skills to enable them to construct* for themselves the mathematics that is taught, and critically analyze and *justify* their constructions in terms of the structure of mathematics. Hence, students must engage in a mathematical discussion with others thereby practicing or being enculturated into the community of mathematical literate adults. In general, individuals (such as teachers) use their personal knowledge of mathematics and mathematics education to direct and control mathematics learning as conversation both (a) to present mathematical knowledge to learners directly or indirectly (i.e. teaching); and (b) to participate in the dialectical processes of criticism and warranting of other's mathematical claims (i.e. assessment) (Ernest, 1999). Thus, teachers' beliefs of what mathematics is can influence the way he or she teaches and assesses mathematics in the classroom. For example, depending on the teacher's beliefs mathematics may be taught as a linguistic discourse in which two-way conversations and discussions are a major part of the lesson or it may be a one-way delivery of information with very little participation on the part of students.

From the discipline's perspective, if mathematics is viewed as a subject with a conceptual structure that guides its methods of solving problems then an effective problem solver is one that has a deep understanding of the structure of interconnections between mathematics concepts, principles, and formulas. Furthermore, such a problem solver can subsequently construct and demonstrate how this structure is linked to relevant methods of solutions, and vice-versa. This implies that students can actively construct their mathematical ways of knowing as they strive to make sense of mathematics. Thus far, teaching and learning a subject with a unique knowledge structure as a social process necessitates that both teachers and students can communicate competently and effectively as well as perform efficiently. Hence all participants need to understand the language that is used and possess the appropriate skills to assess tasks and perform competently. If students are not able to solve problems competently, this implies that they may have misinterpreted given problems, do not understand the mathematics involved, and/or are unable to systematically apply their knowledge to problems (White & Gunstone, 1992, 1989). Therefore, the ability to solve a problem successfully is dependent on how well a student understands the relevant knowledge structure and whether or not he or she has good meta-cognitive skills. The later make students aware of the purpose of problems and enable them to monitor and control their own progress during the problem solving process.

In accordance with the socio-cultural perspective of mathematics learning as a process of enculturation into a community of practice and the discipline's constructivist view of a learner, the primary responsibility of mathematics teachers then becomes that of teaching "students to think, to question, and to probe, to get to the heart of the matter, to be able to employ ideas rather than to simply regurgitate them" (Schoenfeld, 1991). This is the challenge for all mathematics teachers and educators. Students at various levels should be mathematics literate. They should be able to understand the language of mathematics and communicate mathematics competently. Hence mathematics teaching and learning should include discussions of not only *how* (procedures) problems are solved but *why* (concepts, principles) methods are appropriate. The regular justifications of methods in terms of the structure of mathematics in primary and secondary classrooms can facilitate the development of a practical understanding of interconnections between mathematics concepts/principles and procedures. These elaborations would necessarily involve questions and answers as teacher and students strive to understand each other thereby encouraging students to become more mathematically literate; that is, become more able to think, discuss and act mathematically (Richards, 1991). Therefore, it is important to undertake research that investigate ways in which students can better understand the structure of mathematics and how this understanding could be effectively applied in solving problems. For example, we need to determine the mechanisms through which students effectively retain, control and apply their understanding of mathematics concepts and principles to the identification of multiple methods of solving problems.

Mathematics Education Research at NUS

Part of the University's efforts to redress the problem of high failure rates and poor mathematics performance was to conduct research to investigate ways in which students' mathematics literacy could be improved. The first research area examined the potential usefulness of concept maps and vee maps as meta-cognitive tools in learning mathematics. Concept maps are schematic diagrams of hierarchical inter-connecting concepts with linking words on lines describing the nature of the interrelationship. Vee maps are diagrams that illustrate both the conceptual framework of a problem and its methods of solutions. In completing vee diagrams, students are required to identify and provide relevant mathematics principles to guide the transformation of given information to determine methods of solutions. This act necessarily requires that students be well versed with the conceptual structure of relevant mathematics topics. Different levels of understanding may be manifested through students' completed concept maps and vee diagrams. The main aim is for students to systematically analyze the structure of mathematics knowledge with a view of using the conceptual analyses to identify (multiple) methods to solve difficult and novel problems. Findings from this research are useful in developing strategies to overcome the over-reliance of students on rote learning of procedures, which are often incorrectly applied when solving new problems. Instead, students are encouraged to search for viable connections between relevant concepts, principles and methods across related topics (Afamasaga-Fuata'i, 1998a, 1998b, 1999a, 1999b).

The second research area examines students' strategies for determining modeling functions directly from data values through an identification of characteristic patterns of rates of changes for polynomial and transcendental functions. The use of data values as the starting point of investigation is mainly to provide students with another means of identifying functions other than the over-emphasized algebraic and graphical approaches prevalent in mathematics classrooms. Furthermore, this numerical approach has wider applications for modeling experimental data from scientific and statistical investigations. Results from this research would be useful in designing more challenging application problems to improve and enrich students' understanding of functional relationships that is crucial in school and advanced mathematics (Afamasaga-Fuata'i, 1998e, 1995).

The results of these on-going research are fed back to mathematics teachers and lecturers as part of their professional development activities in the form of publications, seminars, workshops, and upgrading programs within the Science Faculty (Afamasaga-Fuata'i, 1999a, 1999b). Public seminars are offered as part of the Science Faculty's development programs. Workshops are offered upon requests from the respective Directors of Education (such as the Congregational Christian Church's Director) and occasionally during school holidays sponsored by NUS to which local mathematics teachers and specialists are invited to attend.

Mathematics Professional Development Programmes at NUS

The University has been offering professional development programs in mathematics (both at pre-degree and degree levels) since 1992 which practicing mathematics teachers have utilized to upgrade their content knowledge (Afamasaga-Fuata'i, 1998c, 1998d). The pre-degree program provides an alternative pathway for practicing teachers (and since 1997, pre-service teacher trainees) to meet pre-requisites for entry into degree courses available for the Certificate, Diploma, Bachelor of Arts (BA) or Bachelor of Science (BSc) majoring in Mathematics. Undertaking further studies in mathematics by practicing teachers had been and still is for a number of teachers, on an individual basis and self-funded. However, it was not until 1998, that the DOE launched a Samoa-Peace Corp Scheme which enabled Samoan school teachers to undertake professional development programs at NUS for two years while Peace Corp Volunteers take over their teaching responsibilities in the schools. In addition to formal courses in mathematics, teachers were also invited to attend subject workshops that focussed on particular problems related to mathematics teaching on a fortnightly basis during the semester. The first and second group of science and mathematics teachers have already graduated and are now back in their former schools.

On the other hand, the 1997 merger of the Secondary Teachers College with NUS as the Faculty of Education meant that the MSD was directly responsible for teaching the mathematics content courses for the pre-service teacher trainees. This involvement provided the vital link for the MSD to be involved in the pre-service program to complement their departmental efforts with practicing schoolteachers. For example, mathematics lecturers not only teach the teacher trainees mathematics but they also coordinated and offered workshops for some mathematics school teachers and curriculum specialists from the DOE (Afamasaga-Fuata'i, 1999b).

The impact of these development efforts on student learning in the classroom remains to be seen. There is still a lot that needs to improve in the classroom in terms of student educational resources and support before any major improvements in national results can be seen in the near future.

Conclusions and Recommendations

National results in mathematics can be improved if the quality of teaching in the classroom improves. This necessitates providing enough institutional support and educational resources to teachers and students so that they could get on with the job of teaching and learning respectively. The DOE is currently engaged in an institutional strengthening project mainly focussing on the personnel, and facilities in and around the main educational compound in Apia in an effort to improve the delivery of efficient services to schools around the country. Other sub-projects include those developing national strategies for the provision of special needs education, and writing of student materials for primary and secondary levels to supplement teachers' curricular manuals. Some student booklets have been distributed already to schools. These are all necessary developments. However, it is vital that needs of students in the classroom are adequately met and supported by ensuring that sufficient quality educational resources (textbooks and related reading materials) are accessible to supplement blackboard work by teachers. There is a scarcity of reference materials and basic mathematics textbooks in most classrooms.

Secondly, teachers need to participate in professional development activities on a regular basis to refresh, upgrade and review their outlook on teaching mathematics. The opportunity to share similar concerns and experiences with other mathematics teachers in Samoa can contribute positively to teachers' own self-esteem and confidence as educators. Hence, there is an urgency for the DOE and NUS to work collaboratively in developing guidelines and implementing strategies whereby the two institutions can fully utilize the expertise available locally in Samoa to develop professional development activities tailored to impact on mathematics learning in local classrooms. As a nation, we should refrain from imposing readymade programs exported from overseas without first undertaking extensive research concerning their applicability and suitability in the Samoan context. It is recommended therefore that there be adequate institutional and financial support available for conducting research in Samoan mathematics classrooms to guide the development of suitable professional development programs.

The current arrangement for workshops mostly funded by the MSD is mainly the result of aggressive efforts by enthusiastic members of the department to share their experiences with the practitioners in the field. This is largely an effort to develop closer links between the school system and the University to minimize the widening gap between Year 13 and UPY mathematics. As the standard of NUS degree courses in mathematics are comparable to those offered elsewhere in overseas universities, lowering the entry requirement for UPY mathematics will be detrimental to those wishing to undertake further studies. Therefore, there is a need to review the entire national curricula in mathematics at all levels to avoid the overloading of topics at the senior secondary end. For example, some of the topics that are currently in the national Year 12 and Year 13 mathematics curricula should be re-distributed further down to Years 9 to 11. Similarly, the primary mathematics curricula ought to be reviewed to facilitate transition into secondary level by ensuring that pre-requisites are adequately covered by the end of Year 8. This review may be effectively done if one national mathematics committee is set up to monitor and oversee this exercise. At the moment, there seems to be two separate groups of individuals working independently, one working with primary and the other with secondary mathematics curricula. Furthermore, even within the secondary level curriculum writing projects, there is a lack of real communication between those working on the Year 12 curriculum development and those for the Year 13. This results in confusion and misinformation about what needs to be done to narrow the gap between Year 13 mathematics and UPY minimum entry requirements. A similar gap also exists between Year 12 and Year 13 mathematics curricula. This concern led the MSD in 1997 to initiate a series of meetings with secondary school teachers to discuss a possible national curriculum in mathematics that meets both the needs of the PSSC examination and UPY entry requirements (Lei Sam, 1998). Various members of the MSD department are involved in the curriculum writing projects for both primary and secondary levels as individuals.

The MSD has initiated a textbook writing project geared at the Year 13 level, which incorporates some of the ideas and findings from the concept mapping and vee mapping research. This initiative is slow as those involved have other full-time commitments they could not be released from. Some lecturers are also examiners and markers for national and the PSSC examinations on an individual basis. The challenge for all concerned is to put in place an institutional agreement between DOE and NUS whereby the professional expertise available at NUS is fully utilized by the DOE. Such available manpower may be utilized to develop national mathematics curricula for all levels, organize and coordinate regular mathematics workshops in conjunction with the MSD's involvement in the pre-service program. Such an involvement (pre-service plus in-service) represents a more concerted and integrated approach to improving mathematics education in Samoa. It also provides an ideal opportunity for mathematics lecturers to provide regular feedback and offer expert assistance directly to those who are teaching at primary and secondary levels based on their research and experiences as post-secondary mathematics teachers, examiners, and markers of national and regional examinations. After all, the products of primary and secondary schooling eventually end up in post-secondary classrooms.

In conclusion, the existing research and professional development programs already established at NUS by the MSD department is only a start. It needs to expand nationally to include all those who teach mathematics nationwide. Having said that, there is then a need for adequate institutional financial support for offering such workshops. These professional gatherings would allow teachers to exchange experiences and ideas on how mathematics learning could be improved with those teaching and researching at post-secondary level. Finally, it is recommended that all mathematics teachers undertake professional development programs at NUS for two to three years supplemented with regular participation in workshops and seminars to maintain the free flow of information that will be useful for their classroom teaching. The formal institutional collaboration sought between DOE and NUS may not solve all problems in the short-term but it is certainly a positive step in the right direction in trying to improve the quality of mathematics teaching and learning in Samoan schools.

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The Process of Introducing New Tasks Using Dynamic Geometry Into the Teaching of Mathematics

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Since 1997, there has been a very strong incentive from the French Ministry of Education for promoting the integration of information technologies in education at all levels. Among various actions, the Ministry provided a financial support for organizing pre and in service teacher education (IUFM) at national and local levels. The renewed curricula offer a real integration of technology into the teaching of mathematics and they claim the necessity of this integration: the use of dynamic geometry software programs and spreadsheets is compulsory.

It is quite difficult to estimate the proportion of teachers making a real use of technology. But it should be around 20%. One can be surprised by the gap between the institutional situation promoting technology and the real situation in classroom. One can seek the reasons for this reluctance by the changes of several kinds introduced by technology:

- changes in the concepts as mediated by the environment
- changes in the tasks given to students
- changes in the management of classroom, due to a growing autonomy of the student due to technology

Technology allows new kinds of tasks revealing the meaning of theoretical objects. But creating these new tasks is not easy and requires time. We will present below the learning tasks of a new type allowed by dynamic geometry software and try to explain why it takes time for teachers to design and use these tasks. Examples will be given in Cabri-geometry, existing both as a software program or application of the calculators TI 92, TI 89 and TI 83.

I - The Spatial and the Theoretical

Dynamic geometry software based on direct manipulation offer a microworld in which theoretical objects and relations (sometimes very complex from a conceptual point of view) can be visualized and physically manipulated. Such kind of environments offers the possibility for students of constructing knowledge in action and not only by having recourse to language. At first, some theoretical distinction will be made.

1.1 - The Distinction Between Spatial and Theoretical Properties

Physical space and geometry as a theory are two separate domains. Space is considered here as part of reality and geometry as a set of theories partly modelling space but also developing its own questions and solutions.

Diagrams in 2D geometry play an ambiguous role: one the one hand they refer to theoretical objects whereas on the other hand they offer *graphical - spatial* properties which can give rise to a perceptual activity from the individual. For example, the diagram in Figure 1 represents a parallelogram.

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 15–34). Sydney: MERGA. ©2002 MERGA Inc.

Figure 1. Diagram representing a parallelogram.

It presents several graphical-spatial properties like two sides are horizontal, the two other are oblique with a given direction (bottom left to top right), the opposite sides are parallel, the horizontal sides have a given length. Note that these properties are selected among a larger set of properties like the colour or the width of the sides. Some of these graphical-spatial properties can be interpreted in a geometrical way, some of them would not be considered as interesting from a geometrical point of view. For example, the position of the diagram in the sheet of paper is generally considered as not relevant in geometry. The slope of the side may also be not relevant, it depends on the problem in which the parallelogram occurs. Some graphical-spatial properties of the diagram are *incidental* with respect to the geometrical problem to be solved, some other are *necessary* like the parallelism properties. Furthermore some graphical-spatial properties necessarily follow from other ones. So there is a link of *necessity* between the parallelism of opposite sides and the fact that the intersecting point of the diagonals also is their midpoint. The teaching of geometry deals with these links of necessity between graphical-spatial properties. But one can understand the nature of these links if and only if one also can understand that some other links are incidental. Necessity makes sense as opposed to contingence. Geometry may appear as useful if it allows you to predict, to produce or to explain graphical-spatial properties of diagrams thanks to these necessity links. But it requires the awareness of the distinction between graphical-spatial properties and theoretical geometrical properties.

This ambiguous role of diagrams is completely implicit in the traditional teaching of geometry in which theoretical properties are assimilated into graphical ones. It is as if it is possible to abstract from the diagram the properties of the theoretical object that is represented by this diagram. One of the consequences is that pupils often draw the conclusion that it is possible to construct a geometrical diagram using only visual cues; or to deduce a property empirically by checking on the diagram. When pupils are asked by the teacher to construct a diagram, the teacher expects them to work at the level of geometry using theoretical knowledge whereas pupils very often stay at the graphical level and try to satisfy only visual constraints.

The task of drawing a tangent line to a circle passing through a given point is frequently viewed by pupils as the physical task of rotating a straight edge passing through the given point and adjusting it in order to "touch" the circle (Figure 2).

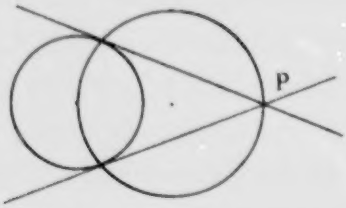

Figure 2. Tangents to circles task.

The teacher is expecting a drawing process based on geometrical relations – the tangent line is perpendicular to the radius and the locus of points from which it is possible to see a segment under a right angle is a circle.

The problem is that the final result may not be better from a visual point of view in this latter case than in the former one… Some traditional construction problems may fail to call for geometrical knowledge.

Therefore diagrams instead of helping students may become an obstacle to geometrical thinking in the sense that they avoid reasoning in theoretical terms (Fishbein, 1993; Mariotti, 1995; Salin, & Berthelot, 1994; Duval, 1998). Several investigations show that it is not easy for beginners in geometry to identify in a diagram the properties relevant from a geometrical point of view. Argaud (1998, pp. 290-297) could observe how it is difficult for children at the end of primary school to recognize a quadrilateral, parallel or perpendicular segments:

a- Is D a quadrilateral ?

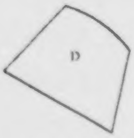

Figure 3.

14 children over 26 answer yes.

b – Is J a quadrilateral ?

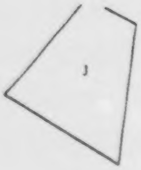

Figure 4.

10 children over 13 answer yes. The given arguments are: it has 4 sides even it is not finished (8 children), it has 4 sides (1 child), it has 4 edges (1 child).

c – Are the segments parallel ?

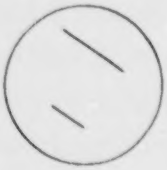

Figure 5.

16 children over 24 say that they are not parallel. The given arguments are: the segments are not opposed, or one is shifted, or they are not of same length.

I.2 - Diagrams in Computer Based Environments

Spatial-graphical and geometrical aspects are very much interrelated in the new kind of diagrams provided by geometry microworlds: these microworlds offer diagrams whose behaviour is controlled by theory. In dynamic geometry microworlds like Cabri-géomètre (Laborde & Straesser 1990), diagrams result from a sequence of primitives expressed in geometrical terms chosen by the user. When an element of such a diagram is dragged by means of the mouse , the diagram is modified preserving all geometric relations used in its construction. These artificial realities could be compared to entities of the real world: it is as if they react to the manipulations of the user by following the laws of geometry just like material objects react by following the laws of physics. A crucial feature of these realities is their quasi-independence of the user as soon as they have been created. When the user drags one element of the diagram, this latter is modified according to the geometrical way it has been constructed and not to the wishes of the user. This is not the case in paper and pencil diagrams which can be slightly distorted by the pupils in order to meet their expectations. Computers diagrams are external objects whose behaviour and feedback is no longer controlled by the user as soon as they have been created. Their behaviour requires the construction of an interpretation by the pupils. Geometry is a means, among others, of interpreting the behaviour of these computer diagrams.

II - Tasks in Dynamic Geometry Software Fostering the Learning of Geometry

The link between the spatial graphical and the geometrical properties of a diagram are reinforced in a dynamic geometry environment, as said above. This feature may be used to design tasks in which students will learn to link visual aspects to geometrical properties.

We propose to categorize the possible tasks to be offered in dynamic geometry environment in four categories of tasks with respect to visual phenomena produced by the environment

- tasks for interpreting visual phenomena
- tasks for producing or reproducing visual phenomena
- tasks for predicting visual phenomena
- tasks for explaining visual phenomena

In all these tasks, geometry provides a tool of solution.

II.1 - Tasks for Interpreting Visual Phenomena

This kind of tasks is adequate for beginners, especially at primary school. A dynamic Cabri- diagram is given at the screen of the computer and the children must describe it in terms of geometrical properties. An example is given in Figure 6, in which a quadrilateral is displayed on the screen. It looks like a square; if one of its vertices is dragged, it is visible that its sides are no longer equal but that it still has four right angles. If another vertex is dragged, it becomes visible that some of its angles are no longer right. The children must observe more precisely the behaviour of the diagram to identify the invariant properties in the drag mode: one of the pairs of opposite sides remains parallel in the drag mode.

In this kind of tasks, children learn to recognize a geometrical property from its various spatial-graphical representations.

At another school level, we also used this type of tasks for introducing new transformations. A point P and its image P' through the unknown transformation were given to the students (Jahn 1998). They could move P and observe the subsequent effect on P'. Students were asked to find the properties of the unknown transformation by means of this black box. In such a task students must themselves ask questions about the transformation :

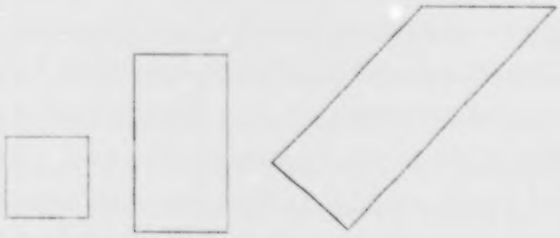

Figure 6.

Does it preserve collinearity ? Does it preserve distance ? Does it have invariant points ?

Cabri can be used to design experiments and get empirical answers. For example, one may redefine P as belonging to any given straight line and obtain the image of this line as the Locus of P' depending on the variable point P. Two specific tools of Cabri are used "Redefinition" and "Locus". It presupposes that the students not only master their use but also decide to use them. This decision actually involves mathematical knowledge: the fact that the image of a figure is the set of images of points of a figure; this is often completely implicit in our curricula but it presents a conceptual cut (even obstacle probably from both cognitive and didactical origin) with the view of a figure as an entity and not as a set of points.

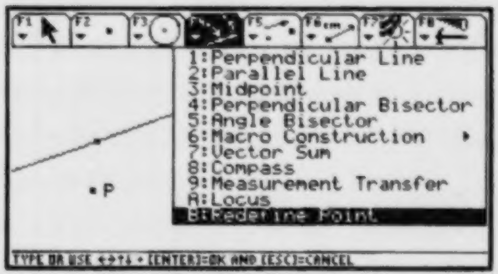
F1
F2
F3
F5
F6
F7
F8
1:Perpendicular Line
2:Parallel Line
3:Midpoint
4:Perpendicular Bisector
5:Angle Bisector
6:Macro Construction ▸
7:Vector Sum
8:Compass
9:Measurement Transfer
A:Locus
B:Redefine Point
• P
TYPE OR USE ←→↑↓ + [ENTER]=OK AND [ESC]=CANCEL

Figure 7. – Redefining the given point P…

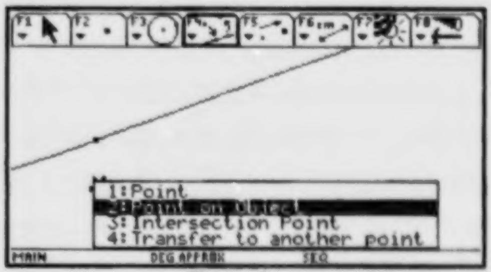
F1
F2
F3
F5
F6
F7
F8
1:Point
2:Point on Object
3:Intersection Point
4:Transfer to another point
MAIN
DEG APPROX
SEQ

Figure 8. – as a point on …

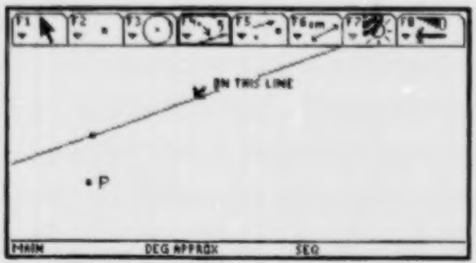
F1
F2
F3
F5
F6
F7
F8
ON THIS LINE
• P
MAIN
DEG APPROX
SEQ

Figure 9. – a line

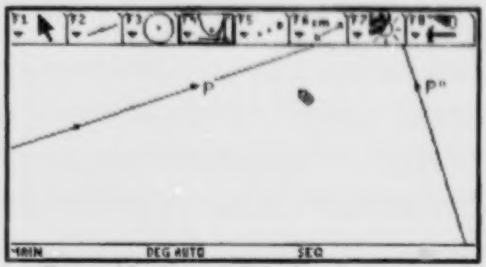
F1
F2
F3
F5
F6
F7
F8
P
P"
MAIN
DEG AUTO
SEQ

Figure 10. –Locus of P' when P moves on the line

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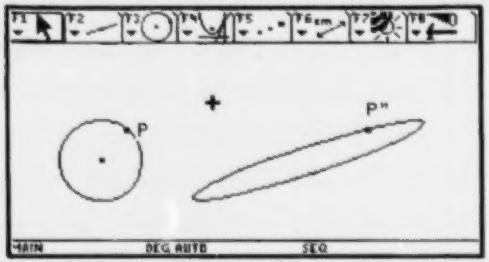

Figure 11. Locus of P' when P is moving on a circle.

Such a task offers a very different point of view on the notion of geometrical transformation. Instead of studying the effects of a known transformation, students are asked to characterize the transformation by means of its properties. Of course this may be an attractive task only if some exotic transformations and not only the usual ones are given to students. Theorems of invariance receive a new meaning in this kind of task: they are tools for identifying the category which the unknown transformation belongs to. An effect of this kind of task is that students may understand why to study all these theorems about invariant elements of transformations. The invariance properties become remarkable phenomena instead of being the routine.

II.2 - Tasks for Producing or Reproducing Visual Phenomena

The construction tasks in Cabri-geometry differ from construction tasks in paper and pencil environment in that

- the constructed diagram must keep all the expected geometrical properties in the drag mode
- to obtain drag mode proof properties, the construction must be done using the primitives of Cabri, given in form of geometrical terms : "parallel line", "perpendicular bisector", "reflection"...

Producing a diagram requires thus in Cabri-geometry explicit knowledge of geometry .

Cabri-geometry offers construction tools that are not existing in paper and pencil environment like vectors or transformations. Construction tasks may receive new efficient solving strategies.

Example with the tool vector:

Construct a triangle ABC from the given points A, B and M centroid of triangle ABC.

Point C is theoretically determined by the relation :

vector MA + vector MB + vector MC = vector 0. (1)

In paper and pencil environment, it is impossible to make direct use of this relation. The sum MS of vectors MA and MB must be constructed through the parallelogram construction and point C is then constructed so that M is midpoint of CS. This strategy is not simpler than a purely geometric strategy consisting in considering M as a point at two thirds of a median. Thus the vector relation is not efficient for the task in paper and pencil environment.

In Cabri, C can be constructed only by two operations as the symmetrical point with respect to M of the endpoint of the vector sum of the two vectors MA and MB (see Figure 12).

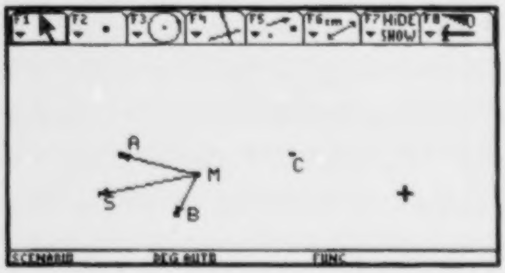

Figure12.

This solution requires to change relation (1) into
vector MC = -(vector MA + vector MB) (2)
and to identify a point symmetry in relation vector MC = - vector MS.

Cabri contributes thus to linking the algebraic aspects of vectors to the geometrical aspects. Relation (1) is restricted in paper and pencil environment to algebraic calculations, whilst in Cabri, it also receives a geometrical meaning since it is a tool of construction. It offers a new connection in the conceptual field of vectors (Vergnaud, 1991) or in the web of vectors (Noss & Hoyles, 1996).

In the same way, a geometrical transformation like point symmetry can be used as a tool for constructing a parallelogram. When students are given the task of constructing a parallelogram, they all do it by using parallel lines and the parallelogram collapses when moving a vertex until three vertices are collinear. The construction by point symmetry appears as more powerful because preserving the parallelogram even when it is flat. The environment Cabri-geometry reveals thus the power of point symmetry under two aspects: the operational aspect as a construction tool and the generality of this tool allowing a construction resisting to a limit case.

The property of symmetry of the parallelogram in a paper and pencil environment is mainly used for proving, dynamic geometry software allows the students to experience this property in action before using it at a more formal level (Laborde, 1995).

In paper and pencil environment, vectors and transformations are only used in reasoning and proofs on theoretical objects. It has been often observed that it is quite difficult for students to have recourse to them in their proofs. They prefer to use Euclidean arguments. We assume that Cabri allows tasks in which theoretical objects like vectors or transformations receive a kind of reification by being tools of construction.

Tasks for Reproducing Visual Phenomena

In this kind of tasks, the students are given a diagram on the screen of the computer and they have to reproduce it. These tasks are an extension of the first kind of tasks. They combine interpretation and construction.

Example

A triangle ABC and a point U depending on the triangle are given (Figure 13). Another triangle A'B'C' is given on the screen of the computer and the task of the students is to reconstruct point U' depending on A'B'C' as U depends on ABC. The students must identify the geometrical relationships between U and ABC in order to be able to construct U'. It can be done by dragging A, B or C and observing the behaviour of U (Fig.14). As students assume that it must be a remarkable point (effect of didactical contract), they can observe the limit cases and infer from them whether U is the orthocenter of ABC, the center of its circumscribed circle or its centroid. If U is the orthocenter, it must be coinciding with a vertex of ABC when the corresponding angle is right. If U is the center of the circumscribed circle, it must coincide with the midpoint of a side when ABC is a right angled triangle. Checking can be done by drawing lines AU, BU and CU and identifying their properties. Mathematical reasoning based on mathematical knowledge of the definition of the points is involved in this recognition task : "If it is this point, it should behave this way".

The construction of U' requires the use of the identified properties determining point U.

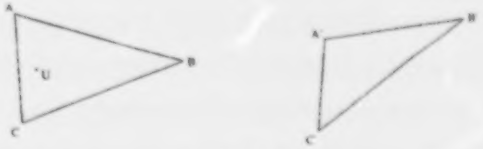

Figure13.

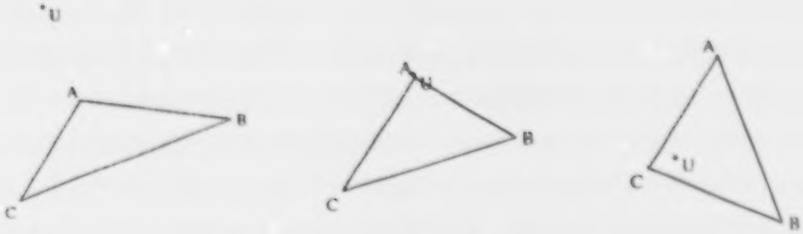

Figure 14.

The Possibility of Inverse Problems

All these tasks, interpretation, (re-)production, differ from usual tasks in which an object is defined and the definition is to be used for finding the properties of the object. Here the problem is inverse, an unknown mathematical object is given under the form of a dynamic "intelligent" diagram and one has to find its definition by experimenting mathematically on it.

Theoretical knowledge is a means of modelling the behaviour of spatial graphical objects, like in the LEGO experiment described in Isoda & al. (2001) in which students have to explain why a mechanism made in LEGO provided a rotation of 60°. This kind of activities requires the use of reasoning mathematically and thus is fostering learning since LEGO mechanisms or Cabri-geometry offers feedback invalidating wrong assumptions.

II.3 – Prediction tasks

Prediction activities are also interesting for the learning of geometry thanks to feedback provided by Cabri: the student may become aware of the inadequacy of his/her expectations when confronting his/her prediction with the observed result on the computer. Ex : what will happen to the image of a polygon through a translation if you move the vector of translation ? or how to modify the vector of translation so that a circle and its image become tangent ? We must stress that the facility of relaxing or modifying conditions (in Cabri "redefine an object") is a very good means of asking students to make predictions: "what will happen if this condition is no longer satisfied ?" or "what will happen if this condition changes into that one?"

II. 4 - Tasks for Explaining Visual Phenomena

Dynamic geometry software may be used for creating intriguing visual phenomena which are not expected by students. The only way of explaining those phenomena is the recourse to theory.

For example, students are asked to observe the behaviour of the sum of vectors MA and MB when M is dragged. They usually do not see anything remarkable. It they are asked to activate the trace of the sum when M is dragged, they immediately see that it is passing through a fixed point (Figure 15). This is not without surprising the students who do not expect this phenomenon even if they know that the sum of two vectors is the diagonal of the parallelogram constructed on the two vectors. It creates the opportunity for the teacher to ask why this intriguing phenomenon.

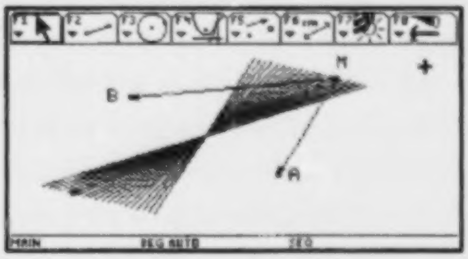

Figure 15.

In a computer environment, the need for proof cannot any longer be favoured by the uncertainty of the result. It may arise for intellectual motives because the student wants to know why a phenomenon takes place. As pointed out by the Piagetian perspective, a means of provoking this intellectual curiosity may be caused by conflict between what the learner believes or predicts and what actually happens. In terms of tasks, this may be done by asking the students to predict properties of the diagram before allowing them to check on the computer, as in the following example (Abd El All, 1996).

Students were given a rectangle ABCD and the quadrilateral IJKL of the midpoints of the sides of ABCD in a paper and pencil environment. They had to determine the nature of IJKL and to justify their answer. All students found that it is a rhombus. Then they had to predict whether IJKL would remain a rhombus in any movement of B which does not preserve ABCD as a rectangle. All students predicted that IJKL would not be any longer a rhombus. In the third step they were given in Cabri (Figure 16) a quadrilateral ABCD having two diagonals equal (B is moving on a circle with center D and radius AC, A, C and D being fixed).

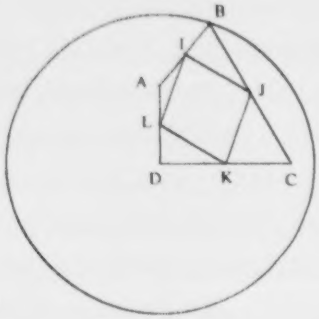

Figure16.

Almost all students recognized that IJKL is a rhombus and were very surprised by this fact; they expressed their desire to know why. This did not lead to a successful proof for all students because it turned out that some of them were unable to mobilize the property of the midpoints line in a triangle. As in a paper and pencil environment, the interventions of the teacher if they come at a right time may play a critical role not only in the solving process of the specific problem but also on the students themselves: after the students have clearly defined a question (here for example, how to prove that the sides of IJKL are equal) and have been confronted to several unsuccessful trials to solve it. In such conditions, even though the property of the midpoints line is brought by the teacher, this property may receive the meaning for the students to be a means of proving that a segment length is half of another one because it answers a question they could determine on their own and express.

Problems of inexistence of objects seem to foster the move from spatial-graphical level to the theoretical level. In absence of existence of any instance of an object, only the recourse to theory may justify the inexistence. However the observations of students in such a problem showed that the recourse to theory may not take place.

The question of existence of a triangle in which two perpendicular bisectors are perpendicular which was asked to 12-13 year old students in both environments (paper and pencil and Cabri-géomètre) seemed for example to raise more justifications in a Cabri environment than in a paper and pencil environment in which students only tried to solve by checking on diagrams. In a paper and pencil environment, they did not try to have recourse to a proof because they were uncertain of the inexistence whereas in Cabri they were visually convinced of the inexistence of such a triangle and try to find reasons to this surprising visual phenomenon (Abrougui, 1995). Finally the intriguing phenomenon led also in this situation to a need of proof. As stressed by De Villiers (1990, p. 18), "proof is not necessarily a prerequisite for conviction, conviction is far more frequently a prerequisite for proof".

For students, conviction certainly comes to a great part from the observation of spatial-graphical properties. This brings us back to the importance of the ability of recognizing these properties… We would like to conclude that learning geometry should be achieved by learning in interaction to deal with diagrams, to make experiments on them. The new generation of computer dynamic diagrams renews dramatically the situation.

III – The Process of Integration: Two Case Studies

III.1 – Design of Teaching Scenarios Based on Cabri by Teachers

Our experience of designing teaching learning scenarios with teachers based on Cabri-geometry and integrating them in the regular course of the teaching shows that it takes a long time before reaching the adequate tasks that take advantage of the technology environment and find the good time management in the classroom.

Over several years, our team of teachers designed successive versions of the same scenarios. The evolution of their features can be summarized as follows.

The first versions called often for immediate visual observations and generalization by inductive reasoning. One of the teacher who was a novice teacher used mainly technology as a provider of data and the solving of the tasks did not require the use of Cabri. These tasks gave a great role to measuring and did not use the animation facility of numbers, or even the continuous dragging possibility.

In the first versions by experienced teachers, Cabri was supposed to facilitate the mathematical task that was considered as unchanged: Cabri was used as a visual amplifier (Pea, 1985). For instance, in tasks of identifying properties, it was assumed that it was easier to observe that three lines always intersect in one point during the deformation of the diagram by the drag mode than in a static paper and pencil diagram. The need for proof was less important than in a paper and pencil environment.

A teacher designed a task about the composition of point symmetries and he wrote about it:

> It is just about grasping at an intuitive level a possible generalisation of results just observed in particular cases. It is necessary to remain modest at the level of Seconde (Grade 10) and to take into account the real possibilities of students. The computer environment turns out to be useful here as a means of exploration and help to conjecture: If the number of the polygon is even, it is the image of MNPQ through a translation. If it is odd, it is the image through a point symmetry.

He wrote about a task of conjecturing properties of dilations:

> The animation of the screen (which results from the use of the tool "Dilation" in conjunction with dragging) implies that the students acquire without too much investment a global visualisation of a dilation: the effect of Dilation on usual figures and their main properties which are conjectured after multiple trials that they can be achieved in a short period of time. [...] In paper and pencil environment, the properties are immediately required to construct the image of a figure.

The role of the software was to save time, to avoid complex constructions requiring the use of properties that are exactly the properties to be discovered, and to favour visualisation. The declared intention of the teacher was to keep the demands of the task at a modest level. The role of Cabri was mainly to facilitate conjecturing and not to cause a problem, as in construction tasks, where the solving strategies have to be constructed with the Cabri tools.

Only the later versions introduced two new kinds of tasks:

- tasks in which the environment allows efficient strategies which are not possible to perform in a paper and pencil environment, such as construction tasks in which tools offered by Cabri were used for constructions;
- tasks raised by the computer context, i.e. tasks which can be carried out only in the computer environment, such as reproducing visual phenomena or prediction tasks.

One of the teachers who introduced the task of constructing a triangle from its centroid and two vertices did it because he was aware of the change of strategy and of the efficiency of vectors and transformations as tools of construction. He commented on it in one of the research meetings, stressing that it was a completely new kind of task belonging to a new culture in which theoretically complex objects become construction tools. He supported the idea that this culture needs time to be developed and must be introduced early in the school year. It was clear from his comments, that this teacher was aware of the conceptual change involved in this kind of task. It is interesting to note the evolution of this teacher, who at the beginning was mainly using Cabri as provider of dynamic imagery for facilitating formulations on the theoretical objects evoked by the diagrams. In this case, the theoretical objects are tools *operating* on the diagrams.

The inverse problems of studying a transformation not from its definition but from the behaviour of a point and of its image on the screen was introduced very late by the teachers and mainly under influence of researchers.

III.2 – Epistemological Beliefs

The design of tasks is based on implicit assumptions about the way students learn mathematics and about the mathematical content itself. The reactions of the teachers when integrating technology could be used as a window on their own epistemology (Noss & Hoyles, 1996, ch. 8).

From these reactions some hypotheses could be made about their beliefs about mathematics and their conceptions of learning. Three types of reactions will be commented below in terms of possible beliefs:

- dichotomy between conjecture and explanation or proof
- repetition of the same tasks in Cabri and in paper and pencil environment
- complexification of tasks

Dichotomy Between Conjecture and Explanation or Proof

The most obvious contribution of Cabri is the possibility of dynamic visualisation of geometrical relations preserved by the drag mode. Teachers (even the novice in using technology) immediately exploited this possibility by asking students to conjecture properties from what they could see. However, when the students were asked to justify, the teachers did not mention the possibility of using Cabri to find a reason or to elaborate a proof. It is as if there was no interaction between visualisation and proving. Technology was used in these tasks, as facilitating the formulation of conjectures but its role did not go beyond that. Quoting the formulation of Hölzl (2001, p. 65), we would say that the drag mode was "used only in a verifying manner" and that "learners are just supposed to vary geometric configurations and confirm empirically more or less explicitly stated facts". A heuristic context was not really created in which the dynamic geometry environment supported a solving process of proving based on experimentation in the environment (trials of particular cases with the drag mode, change of conditions on the givens by using redefinition).

It was as if the process of elaborating a proof should deal with theoretical objects unrelated to their representations, not modified by actions on these representations. Bosch and Chevallard (1998) argue that mathematicians have always considered their work as dealing with non-ostensive objects and that the treatment of ostensive objects (expressions, diagrams, formulas, graphical representations) plays just an auxiliary role for them. This conception, according to which mathematical concepts exist independently of their representations, and which does not take into account interactions and mutual controls between non-ostensive and ostensive objects, seems to underpin this dichotomy.

It is not without relation to another conception: the intrinsic link of geometry with paper and pencil that is presented below.

Repetition of the Same Tasks in Cabri and in Paper and Pencil Environment

One of the teachers who was novice in using technology did not rely on technology based activities for learning geometry and, in addition to technology based activities, proposed similar paper and pencil tasks seemingly unaware that a paper and pencil task may be less demanding in terms of knowledge, by allowing perceptive strategies instead of strategies based on theoretical properties. It seems that she had an epistemological view of geometry as intrinsically linked to paper and pencil. This belief of the canonical form of mathematics linked to paper and pencil environment is widely shared. Povey and Ransom (2000) report about an inquiry carried out among undergraduate students in mathematics in UK. Each of them seemed to refer to a single specific mode of understanding mathematics, the paper and pencil mode (pp.52-3). "Technology can help if you have a paper and pencil understanding" told one of the students and this formulation could be taken as summarizing the philosophy of the scenarios written by this teacher. As expressed by Povey and Ransom, the underlying learning assumption is that "doing maths by hand indicates that one understands it". This is exactly the type of claim made by the teacher when she explained to us that, without the material action of drawing the image of a straight line through dilation with a straight edge, students would not appropriate this invariant of dilation.

This point of view is often linked with the conception of a paper and pencil environment as 'not a context'. Knowing how to carry out a construction in paper and pencil environment would be the warrant of de-contextualised knowledge. Noss and Hoyles (1996, p. 48) propose an alternative view of abstraction as not necessarily linked to de-contextualisation and "as a process of connection rather than ascension". They add that the "situated, the activity based, the experiential can contain within it the seeds for something more general" (*ibid*, p. 49). In the interaction with the computer, learners may construct what Noss and Hoyles call *situated abstractions*. Situated abstractions are invariants that are shaped by the specific situation in which they are forged by the learner. Although those invariants are situated, they simultaneously contain the seed of the general that could be valid in other contexts:

"Within a computational environment, some at least of these objects and relationships become real for the learner (we are using 'real' here to mean something other than simply ontologically existent-perhaps meaningful or broadly connected are better descriptions): learners web their own knowledge and understandings by action within the microworld, and simultaneously articulate fragments of that knowledge encapsulated in computational objects and relationships-abstracting *within*, not *away from*, the situation. In computational environments, there can be an explicit appreciation of the form of generalized relations within them (the relational invariants) while the functionality and semantics of these invariants-their meanings- is preserved and extended by the learner" (p. 125).

Such linkage between understanding and paper and pencil may also be explained by the institutional context. Even if all kinds of calculators are allowed in our French national examination, all examination tasks are given in a paper and pencil environment. The teacher thus prioritises this context to be sure that students are able to perform the tasks in the examination environment.

Complexification of Tasks

Experienced teachers involved in teacher education, such as the teachers we worked with, very often have a constructivist view of learning based on two assumptions:

- Students learn when they are faced with tasks for which mathematics notions are efficient tools of solutions;
- Feedback coming from the situation may favour an evolution of solving strategies more than a judgement coming from the teacher.

Feedback coming from dynamic geometry software may from this point of view be very rich in that it allows an interaction between the visual and the theoretical aspects of geometry. If a constructed diagram in the drag mode does not keep the shape that was expected, it means that the construction process must be wrong. The drag mode can also invalidate a conjectured property and thus lead the students to abandon it.

The teacher may rely too much on feedback from the calculator/computer and propose tasks of a greater complexity than corresponding paper and pencil tasks. The teacher underestimates the complexity of the task, and the time needed for the student to solve the task because he has little reference in his experience. He overestimates the possibility of interpretations by the student of feedback given by the software.

We observed this in the first version of a scenario on vectors in which students were asked to construct all diagrams for the tasks in Cabri. Instead of teaching vectors for two weeks, it took two months! It is also a common phenomenon that any kind of teaching innovation provokes time inflation. Schneider (1999) reported on teaching about logarithms and exponentials based on the use of the TI 92 which took 40 hours of teaching instead of the usual nine hours.

Returning to the project, after the first year, the teachers attempted to find an optimal balance

- between what is prepared and demonstrated by the teacher on the LCD display and what is done by students,
- between what is ready made and given on the calculators to students and what has to be done by the students with the software.

For example, after one year the teachers preferred to give the macro-construction of the multiplication of a vector by a number for students to explore and interpret rather than for the students to construct themselves. Even apparently minor aspects may slow down the construction of a diagram. In the scenario "Vectors", the first task was about polygons that students had to draw. To this end, they had to designate the successive vertices of the polygon and, at the end of the sequence, again the first vertex. Actually it turned out that students tried to do polygons with a large number of sides, and sometimes had difficulties in designating at the end exactly the first vertex and not a close point. If they had used a double click on the last vertex of the polygon, it would have avoided difficulties. But the teacher did not anticipate the long time spent on drawing the polygons and mentioned this shortcut only orally during the activity. This meant that only some students paid attention to his remark.

Evaluating the complexity of a task with technology requires taking into account not only the conceptual difficulties but also the use of the technology by the students. This is not easy and the wrong a priori evaluation by the teacher of the complexity of the task came from an absence of reference about students' behaviour in the tasks. The complexification of tasks may also come from the uncertainty of the teacher about what students would learn from the tasks. They tended to presuppose that technology would facilitate the solving process, so that in order to be sure that students learned something from the new kind of tasks, the teachers increased the level of complexity. In all cases, it is clear that a deep and precise knowledge of students' behaviour and strategies in the Cabri environment is essential for evaluating a priori the degree of difficulty of a task.

III.3 The Design of Tasks by Prospective Teachers[1]

In order to evaluate the effect of teacher education to the use of Cabri-geometry in the teaching, we asked prospective teachers to design tasks allowing students of grade 8 or 9 to overcome difficulties in proof[2] tasks.

The prospective teachers were in their last year of professional education. They were in charge of a class (4 to 6 hours teaching per week) and followed two days of professional development sessions per week at the university institute for teacher education. They receive education in pedagogy of mathematics and "didactique" as well as help for practical problems in teaching. They also have to carry out a small research project about a teaching problem they define themselves.

They followed general sessions about the use of Cabri during six hours introducing them to the features of Cabri and helping them to solve mathematics problems with Cabri. These sessions were purely devoted to the learning of how to use Cabri. After these sessions, four prospective teachers were given fictive proofs to problems supposed to be written by students of grade 8-9. They were asked

- to analyse the errors occurring in the solutions
- and to design tasks based on Cabri and meant for helping students overcome the difficulties they encounter when writing proofs.

Then these four students attended sessions (altogether 12 hours) on the way to integrate Cabri into the mathematics teaching in which they had to reflect on the changes brought by Cabri on the notion of figure as well as on new tasks made possible by Cabri.

Two months after the sessions, they were given the same type of tasks as in the first experiment with the only change that after they proposed some tasks, they were requested by the experimenter to design tasks

- using the ambiguity, or the replay of the Cabri construction
- and tasks of reproduction of a dynamic Cabri-diagram.

Examples of tasks and of their fictive solutions given to the prospective teachers

Task 1:

The midsegment theorem says: "In a triangle, if a line is passing by the midpoint of a side and is parallel to the third side, then it cuts the second side in its midpoint."

Its reciprocal says: "In a triangle, if a line is passing by the midpoints of two sides, then it is parallel to the third one."

Explain all differences that you see between the two statements.

Solution 1

For both statements, there is a triangle and there is a straight line which cuts the two sides in their midpoint. For the first statement, this line cuts the second side in its midpoint and for the second statement, this line is parallel to the third side.

Solution 2

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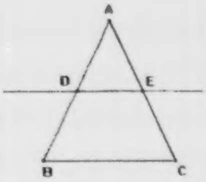
[1] This work is a master thesis (called in French DEA) of Seden Tapan supervised by H. Chaachoua and myself that will be defended in June 2002 (Tapan 2002)

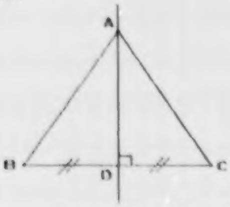
[2] Proof is introduced in the curriculum in France at the beginning of secondary school (grade 6) and becomes an usual task at grade 8.



These both statements say almost the same thing but they only turn the sentence the other way. Because in the two statements, there is a triangle ABC and there is a line (DE) that is passing through the midpoints of segments AB and AC and which is parallel to line (BC).
Hence I can say that if the one is true then the other one is also true.

Solution 2 does not distinguish statements in function of their status in a proof, hypothesis or conclusion of a step and considers only the content of each statement. Solution 1 distinguishes statements trough their position in the statement (in the first part or in the second part after "then") but no more than that. As solution 2, solution 1 does not express the status of each part of a deductive step.

Task 2:



In the above diagram, BD = DC and (AD) ⊥ (BC.)
Prove that ABC is an isocèles triangle.

Solution 1
In an isosceles triangle, the height is also the perpendicular bisector. As line (AD) is height and perpendicular bisector, ABC is isosceles.
Solution 2
As ABC is an isosceles triangle, line (AD) is height and perpendicular bisector. Thus ABC is isosceles since its height is also perpendicular bisector.

Solution 1 seems to express the status of a statement in a deductive step but mentions the reciprocal theorem instead of the adequate one. Solution 2 may be interpreted in that the student is unable to recognize the status of a statement and to distinguish between hypothesis and conclusion, he/she just formulates statements by using external signs of deduction.

To students who do not consider the status of statements but only their content, Cabri offers several ways of externalising this difference:

- Students who do not see the difference between hypotheses and conclusions may infer that properties are true just because they seem true on the diagram (confusion between spatial and theoretical relations, cf. §I). The drag mode allows to eliminate properties purely taken from the diagram that seem to be true on the particular case of the diagram
- The drag mode can also be used to show by relaxing a property P that as soon as P is satisfied, property Q is also satisfied. When dragging the point on the first side, as soon as it is the midpoint, the line parallel to the third side cuts the second side in its midpoint (task 1). The tool Redefinition can also be used in this way.

- In the construction process are used only the given of the problem whilst the conclusions appear as staying true in the drag mode although not used for the construction of the diagram.
- The tool ambiguity shows that two geometrical objects have the same spatial position. This is exactly what a theorem expresses. In task 1, in the statement "In a triangle, if a line is passing by the midpoint of a side and is parallel to the third side, then it cuts the second side in its midpoint", the intersecting point of the side and the parallel line (hypotheses) is also the midpoint of the second side (conclusion). It results that in some cases, ambiguity reveals the difference between the given used in a construction and the conclusion.
- The replay of a ready made construction allows the user to see the objects used in the construction, in particular the objects that have been hidden by the designer of the construction. It also may be a means of externalising the difference between hypotheses and conclusions.
- Finally when trying to reproduce Cabri diagrams, the student must be able to identify those which characterize the construction and then imply the other properties he could observe.

From the observations the two pairs of prospective teachers designing tasks, it appears that before the specific teaching on the pedagogical use of Cabri, they were only able to use the construction process and the drag mode to show the difference between properties just been taken as granted from the diagram and proved properties. Only one pair could also use construction to show the difference between hypotheses and conclusion. After this teaching, both pairs could use construction and drag mode to make visible the difference between hypothesis and conclusion. They could use ambiguity only when requested to do this and about a specific task which was formulated to favour this use (The task consisted of proving that two points were coinciding). The drag mode was not used by them to make students aware of the sufficiency of a property with respect to another one. It was difficult for these teachers to use successfully the replay of a construction and situations of reproduction of a Cabri-diagram. This shows clearly the gap between the use of a tool for solving tasks and the use for designing tasks. The possible uses of Cabri described above for overcoming difficulties in proving were not taught in an explicit manner and had to be constructed by the teachers.

IV. Conclusion

In our introduction we claimed that the process of integrating technology into mathematics teaching is a long and complex process. In analysing the types of tasks developed by teachers over the three years of the project and their evolution, we can formulate tentative explanations for the length of this process.

As the didactic system as a complex system, technology is not just an additional element in the system since it interacts with all the components of the system, which are subject to change. This point of view is based on two theoretical approaches, the notion of instrument as developed by Rabardel and Vérillon and the mediating function of a computational learning environment (Noss & Hoyles, 1996). Vérillon and Rabardel (1995) stressed how an artefact is not taken as such by the learner but reconstructed by him/her. The learner constructs both a representation of the artefact (the instrument) and the structures that allow him/her to perform activities with the artefact (schemes of utilisation of the artefact). Both types of constructs depend on previous knowledge of the learner and affect this knowledge. According to a Vygotskian perspective, Rabardel claims that the "instruments" constructed by the learner constitute forms which structure the relationships with situations and knowledge and thus may have a considerable influence on the construction of knowledge. Noss and Hoyles (1996) investigated many years how learners construct situated abstractions dependent on the means of action and expression offered by the environment. Students construct an instrument in function of the tasks they have to solve, i.e. mathematical tasks. We assume that the tasks of the teachers differ from those of students. The teachers must be able to use Cabri for creating tasks to be solved by students. They must be able not only to consider the Cabri tools as tools for solving problems but also to consider them as mediating mathematical knowledge: construction combined with drag mode may mediate the status of a statement with respect to deduction; the locus or trace may mediate the notion of a geometrical object as a set of points.

We interpret the behaviour of novice teachers in the design of scenarios as resulting from their perspective that technology is an additional component of the teaching system but external to the learning processes. Technology was facilitating material aspects of the actions of the students (teacher novice in teaching), technology was used in observation and construction tasks but activities in paper and pencil environment were given in addition by the teacher who was a novice in the use of technology. It is interesting to note that this latter teacher planned what might be interpreted as a more verifying or test way of using the drag mode than search way (in Hölzl's terms). In the observation tasks that she gave, all steps of the conjectures were given explicitly.

A second interesting feature of the design process of the scenarios by the experienced teachers can also be interpreted in terms of instrumentation and mediating function of the environment. These teachers offered more open exploration activities involving more a search use of the drag mode; they did it in two kinds of circumstances: at the beginning of sessions in observation tasks to introduce new properties and at the end of sessions in open problems to be solved. But the comments they added, expressed clearly that the drag mode was for them more facilitating visualisation than acting in the solving process, even for the open-ended problem. It took one or two years for them to accept that investigating the invariants of an unknown transformation under the form of a black box situation through the drag mode and the tool "Redefinition" could be part of a scenario. The difference between a reproduction of a Cabri diagram and an observation situation for conjecturing must be stressed here. A situation of reproducing a Cabri diagram is a problem situation and the invariants are the tools of solution of this problem. In an observation situation, in which students are asked to conjecture properties, the question is more to satisfy a contract of finding properties relevant from the perspective of the teacher.

We assume that really integrating technology into teaching takes time for teachers because it takes time for them to accept that learning might occur in computer-based situations without reference to paper and pencil environment and to be able to create appropriate learning situations. But it also takes time for them to accept that they might lose part of their control over what students do. Povey and Ransom (*op.cit.*) concluded from their inquiry among undergraduate students (already cited above) that the plea for learning by doing 'by hand' could be related to a "desire to feel in control"(p.56). As they mentioned, speaking about technology as "taking over" and depriving the human of control is usual in a wider social context. The situation is far more complex for a teacher who must not only understand what the computer does but also what the students do with the computer.

Cabri covers a broad domain of knowledge and action. It is a microworld allowing multiple ways of exploring, experimenting and solving a problem. If the basic use of Cabri can be learned rapidly because of its friendly interface, constructing a global and structured representation of all of its possibilities requires time. It requires even more time to analyse the possible uses of Cabri in terms of mediation of knowledge, and to construct correspondings tasks.

In the same way as teachers do not have to reconstruct all exercises and problems that they give to students, it is not expected that teachers should on their own find the adequate situations to use technology. Research and investigation should be carried out in order to have a better knowledge of students learning with technology. The results and data of these investigations could then be transferred to teacher education. This is why we consider that research on the integration of technology into maths teaching is important.

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Seeking Interventions to Improve Adult Numeracy Instruction in the United States: Hybrids Only Need Apply

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In adult numeracy instruction, the gap between current and potential instructional realities is of great concern. Narrowing that gap might best be accomplished with creative, sly, and surprising combinations of practice, research, theory, and/or policy. In this paper, I describe current policies, practices, and research connected with adult numeracy instruction in the United States, propose a different model, and depict three multi-dimensional interventions which create powerful opportunities to find the way from what currently exists to what might be.

An estimated 90 million adults in the United States demonstrate skills that are less than adequate to meet the demands of a technological society (Kirsch, Jungeblut, Jenkins, & Kolstad, 1993). My work and this paper focus on a subset of that population - the approximately 4 million adults who enrol in publicly funded adult basic education programs in the United States each year.

According to one analysis, the reasons for participation in adult basic education programs can be categorised as follows: accessing information, giving voice to one's ideas, acting more independently, and keeping up with a rapidly changing world (Stein, 1995). Whatever an adult's reasons for participating, their return to school typically includes reading, writing, and mathematics instruction (Beder, 1999). Participants in adult basic education programs are mostly poor and female. Many of them are immigrants seeking to improve their command of English. Some participants want to improve their employment prospects. About a third are young adults who have dropped out of formal education and may enrol in order to pass a high school equivalency exam.[1]

Adult basic education programs in the United States are operated locally by a variety of organisations and agencies. Adult students attend approximately two to twenty hours of instruction a week in classes run by school systems, community colleges, workplaces, libraries, correctional institutions, shelters, or community-based organisations. Program staff members are usually employed on a part-time or volunteer basis. One out of seven is a professional adult basic education teacher working on a full-time basis (United States Department of Education, 1999).

This segment of the population, in these non-traditional educational contexts, should pique the interest of educational researchers in the literacy, language, and mathematics fields - even those researchers whose primary focus is the education of children or university students. Alan Bishop has wisely observed:

> Adults are not like school students but they too are mathematics learners. So we can expect to find from research on adult learners, data and thoughts which will inform and extend our constructs and concepts of mathematics learning in general (Bishop, 1997, p.3).

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[1] 1997-1998 data as reported on the United States Department of Education website: http://www.ed.gov/offices/OVAE/datahome.html.

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The Current Status of Adult Numeracy in the United States

While mathematics education reform has been at the forefront of K-12 school reform in the United States (with leaders, resources, curriculum, and staff development aimed at implementing the principles and standards articulated by the National Council of Teachers of Mathematics and supported by research), the topic of numeracy education is almost non-existent in the national debate concerning adult basic education reform. In adult numeracy instruction, the gap between current and potential instructional realities is disturbing with an increasingly wider gap between K-12's teaching of mathematics and adult basic education's approach. Below I describe the current backdrop for adult numeracy instruction in the United States, propose a different model, and depict three multi-dimensional interventions which create powerful opportunities to narrow the gap between existing practices and the promises of reform mathematics.

Current Numeracy Policy

Literacy - the ability to read and write - is accepted by policymakers as an essential goal in adult education and training while numeracy - the ability to work with quantity, space, and relationships - is ignored. For more than a decade, advocates for adult basic education have lobbied hard to make sure that literacy becomes a national priority. The success of their efforts is evident in the titles of major organizational structures created in the 1990s: the National Center for the Study of Adult Learning and Literacy, the National Institute for Literacy, and the Division of Adult Education and Literacy (which prior to 1991 was named the Division of Adult Education).

While some policymakers insist that numeracy has been included under the literacy umbrella, numeracy is nearly absent from the mission statements and reports of the aforementioned institutions. Despite the minimal attention that numeracy receives in policy circles and public debate in the United States, some analysts estimate that more than 80% of adults enrolled in adult basic education programs receive some math-related instruction (Gal & Schuh, 1994). Where mathematics is expressly referenced, such as within the new accountability system, a limited view of numeracy as arithmetic procedures prevails. The nature and content of math-related instruction provided to adult basic education students should be of critical importance to math reformers and researchers as well as policymakers and funders who care about adults' ability to participate in society and to perform on math-related tasks in everyday life.

Current Curriculum Model

At present, the two forces that mould adult basic education mathematics curriculum are the high school equivalency exam (i.e. the GED) and commercially published workbooks. While the workbooks are *de facto* curriculum guides for teachers, the GED exam is the key motivator for teaching mathematics. "Is this going to be on the GED?" is the adult student's equivalent of her high school counterpart's question: "When are we ever going to use this?" In their genuine desire to respond to students and in their quest for outcomes, teachers are pressured to gear their classes to the GED.

The GED consists mainly of multiple-choice items presented in adult-like contexts. The mathematics questions of the GED are divided among four areas: (a) number operations and number sense; (b) measurement and geometry; (c) data analysis, statistics, and probability; and (d) algebra, functions, and patterns (GED Testing Service, 2001).

Commercial workbooks reflect these categories and generally present computational routines, with opportunities for repeated practice of these routines in direct preparation for the test.

In the future, adult numeracy curriculum may be primarily shaped by the National Reporting System (NRS), a national assessment system developed by the United States Department of Education. The potential danger of this system is that curriculum will be geared predominantly to commercial standardised tests. Thus, the goals of adult students who want to address the mathematical demands of their daily life, to prepare for further education, or to train for specific employment purposes may be overshadowed by the need to demonstrate high standardised test scores.

The Adult Numeracy Teachers and their Pedagogy

Most adult education teachers are trained in the fields of literacy or language. Fewer than 5% of adult education teachers in the United States are certified as mathematics specialists or instructors (Gal and Schuh, 1994). Surveys indicate, however, that literacy teachers often have to teach mathematics as well. For example, the majority of adult basic education teachers in Arkansas teach all subjects, including mathematics, but most of them have a bachelor's degree in elementary education (Ward, 2000). Similarly, a significant percent of adult basic education teachers in Massachusetts became math instructors either "by accident" or because it was a requisite of their employment (Mullinix, 1994).

Individualised instruction and repeated practice are the staple of mathematics instruction. Teachers commonly use individualised instruction for several reasons. First, certain factors of the adult basic education environment, specifically multi-levelled classes and irregular attendance, warrant such instruction. Second, teachers want to avoid embarrassing moments that could discourage students by awakening unpleasant memories of past educational experiences. Further, they want to respond to the students' stated goals. Thus, many teachers believe that each student should follow an individualised program. Whether it take the form of a folder with a checklist of pages to complete or computer-assisted instruction, individualised instruction results in a classroom culture defined by isolated silence.

A hallmark of individualised instruction is the reliance on repeated practice of routine procedures outlined in workbooks (Kloosterman, Hassan, & Wiest, 2000). However, when used as the primary tool for mathematics instruction, workbooks often "discourage intuitive approaches and promote a mathematics that comes from an outside authority rather than a personal mathematics that can be applied in many situations" (Tout & Schmitt, 2002, p. 162).

Research on Numeracy

Research on numeracy in adult basic education is minimal. Further, the research that does exist focuses mainly on affective issues, such as math anxiety, rather than cognitive issues (Safford-Ramus, 2000). Some limited attempts, however, have been made to promote research related to adult numeracy in the United States. For example, the National Center on Adult Literacy (NCAL) published technical reports that lay out the need for a research program on adult numeracy (Gal, 1993; Gal and Schmitt, 1994). Further, the National Center for the Study of Adult Learning and Literacy (NCSALL) coordinated with international research efforts by hosting a research forum on adults learning mathematics in

July 2000 (ALM7 Conference). To date, however, there is no coordinated research program being conducted.

Proposed Improvements to Adult Numeracy Instruction

This bleak profile of American adult numeracy education must change. While adult basic education is viewed as a second chance for so many people who have dropped out of the educational system as children, this second chance need not be second rate. Adults returning to education deserve quality experiences that reward their courageous and arduous efforts.

An Improved Numeracy Policy

In contrast to literacy campaigns in the United States that have omitted numeracy, similar campaigns for adult access to basic education in other English-speaking countries, such as the United Kingdom and Australia, speak of an adult *literacy and numeracy* educational system. The differences in policy go beyond the name as well, the United States might do well to emulate these models.

The strength of the Australian model is its numeracy standards that are based on a contextual definition of numeracy. The most widely adopted approach is the Certificates in General Education for Adults (Kindler, Kenrick, Marr, Tout, & Wignall, 1996). This nationally accredited, competency-based curriculum framework takes the view that numeracy is about making meaning of mathematics and has developed a set of learning outcomes that are organised around the purpose and use of mathematics in social contexts: numeracy for practical purposes, numeracy for interpreting society, numeracy for personal organization, and numeracy for knowledge.

The United Kingdom offers standards, which provide a richer mathematics for all levels. In the United States, only number computation is targeted for those at the beginning level, whereas the British standards include geometry and data topics at all. An improved national policy in adult education would, like in Australia and the United Kingdom, include numeracy and contextualise and enrich topics to be covered at all levels.

An Improved Numeracy Curriculum

Adult basic education mathematics instruction should be less concerned with school mathematics and more concerned with the mathematical demands of the everyday world, namely the demands that adults face in their roles as workers, family members, and community members. Thus, numeracy curriculum should not be synonymous with school mathematics but emerge as a new discipline linking mathematics with the real world. Johnston and Yasukawa (2001) have come to believe the focus is on the link itself:

> In our teaching of numeracy … it is *the relationship, the negotiation*, between mathematics and the world that has become the core concern (p. 292).

As opposed to remedial basic mathematics, the subject matter of this new curriculum might be defined as an "at homeness"[2] with quantity, space, and relationships. An ideal numeracy curriculum might follow the tenets proposed by the Adult Literacy and Life Skills Survey:

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[2] In 1982 , Cockcroft used the phrase "at homeness with number" to describe what it means to be numerate. (Cockcroft Report, 1982, paragraph 39).

> Numerate behavior is observed when people manage a situation or solve a problem in a real context; it involves responding to information about mathematical ideas that may be represented in a range of ways; it requires the activation of a range of enabling knowledge, behaviors, and processes (Gal, van Groenestijn, Manly, Schmitt, and Tout, 1999, p. 11).

Math content areas–number, data, algebra, and geometry–would be integrated at all levels of progress and with all relevant contexts, for people who are at the beginning levels of literacy as well as those readying for further education.

An Improved Pedagogy of Numeracy

The pedagogy of numeracy appropriates the best of approaches from K-12, but aligns itself with the principles of andragogy, namely respect for adults' experiences in the workforce/world. In practice, we think this would look something like this:

> Students in our classrooms work in small groups, interacting, puzzling over problems, strategizing about solutions, sharing these solutions with other groups in the class, and listening to others' solutions and strategies. As they collaborate, students 'talk math.' Their discourse includes mathematical vocabulary and explanations of how they have solved a problem and why the solution makes sense. The teacher acts as a facilitator, to guide the learning and to make the mathematics explicit … In the classrooms where we are trying to develop communities of mathematical investigation and discourse, we hear students' ideas and understandings that we had not heard before. (Steinback, Schmitt, Merson & Leonelli, in press).

Ginsburg and Gal's (2001) suggested instructional strategies concur with the spirit of what we consider a healthy instructional environment: opportunities for explorations, group work, and the sharing of multiple strategies. This vision is further bolstered by Susan Imel (1999) who has suggested that established adult education principles be used to redesign programs. Such principles include involving learners in planning and implementing learning activities; drawing on their experiences as a resource; and encouraging self-direction along with spirit of collaboration.

Improved Training for Adult Numeracy Teachers

It is not realistic to require a pre-service program for the adult basic education system, but it is feasible to provide ongoing opportunities for teacher development. Communities of teachers working together to improve both content and pedagogical knowledge should be the norm, rather than the exception. The reflective practitioner described by Shön (1983), one who learns and responds *in situ,* is preferable to one who comes armed with all the knowledge she needs. The tools of the classroom - the curriculum, the pedagogy, the instruction, the discourse, and the assessments - should be seen as supporting not only the every student's learning, but every teacher's learning as well. The following section on multiple interventions fills out this picture in a bit more detail.

An Improved Research Model for Adult Numeracy

A research program that would inform practice and policy and build theory should focus on adults' numerate thinking in and out of classroom settings. The research on the development of children's mathematical thinking and the body of work on everyday cognition can serve as starting points, but a separate line of research should focus on adults learning mathematics. In 1994, British adult numeracy practitioners and researchers convened Adults Learning Maths - A Research Forum. This group has extended internationally and serve as a site for sharing and building a research program. In the following section, I outline one way I hope the main research enterprise on adult mathematical thinking might occur.

Multi-dimensional Interventions

Clearly, those who want to see a first-rate education available for adults need to be catalysts for change with stamina to face tremendous obstacles. Given numeracy's stepchild status within an already marginalised field, the need to be creative and sly, maximising existing resources leads us to consider surprising combination of practice, research, theory, and/or policy. Below I depict three multi-dimensional interventions that create powerful opportunities to find the way from what currently exists to what might be.

Where Practitioners Become Researchers

I first got an inkling about how powerful a hybrid intervention could be when a teacher group became researchers. From 1992-1994, sixteen of us came together and formed what became known as the Massachusetts Adult Basic Education Math Team. We taught math in adult basic education centres across our small state, Massachusetts. Some taught GED preparation, others beginning literacy/numeracy; and some taught English to speakers of other languages (ESOL) or to workers in industrial workplace education programs. These distinctions soon blurred as we created and embarked upon a two-year collaborative teacher-research project to test and adapt the NCTM *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) to our adult basic education classrooms.

As researchers, we posed questions that challenged one aspect our current teaching approaches. These questions were often rhetorical, but they provided focus. For example,

- Can (my) fear of manipulatives be overcome?

> I sincerely hope that no longer will the tools for teaching math be considered a book, a pencil, a paper, and an assignment to do the next 20 problems and check your answers in the back. After doing this project I now realize why so many people do not like math. Who would like trying to rearrange numbers with absolutely no concept as to what they are doing? (Richards, 1994, p. 10).

- What happens to students' definitions of math and science as they explore the connections between them through an integrated unit approach?

> [Describing the class being taped for a video] Under bright lights with cameras in our faces we were scaling down the Solar System, the Milky Way, and the distances within. Calculators were buzzing, zeros were flying, and mistakes were being made comfortably.In the video, I had really faded into the distance (DeCoster, 1994, p 26).

We were on alert for changes in our students' attitudes, but also ended up uncovering what had not been obvious before about what adults in our classes were thinking. Consider this striking example - one teacher's account of how learners left their numerate selves at the door of the classroom:

> I asked a group of my GED math students to tell me how much it would cost if I bought four shirts for $7.98 each. They were told they could figure it out anyway they wanted, except they could not use paper and pencil. I watched as they used their fingers in the air or "wrote" on the desk. Most were able to multiply and get the right answer. When I asked HOW they got their answer, all agreed that they needed to multiply $7.98 by four.

> I then asked if they were in a store and had to figure out the same problem would they have done it the same way. All agreed they probably would NOT solve it the same way in "real life." Some said they would have multiplied four by seven plus four by one and then subtracted eight cents from that total. Others said they would have rounded $7.98 to $8.00, multiplied that by four and then subtracted $.08 from the product. I then asked why no one admitted to solving the problem like that in class. The response was that this was math class they need to do it out. (Moses, 1994, p. 60).

The teachers' research questions and ensuing reflections re-aligned instruction with the highest principles espoused by adult education theorists and NCTM: teacher as facilitator and co-learner rather than know-it-all, and the importance of meaningful mathematics rather than rote learning. The contradictions we had all grown accustomed to and lived with were no longer willingly tolerated compromises in our heads or classrooms.

Team members saw this as transformative on several levels. The changes played out in the ways they taught, what they taught, how they approached doing mathematics themselves, the ways their students approached doing mathematics, their approach to teaching other subjects, and the ways they looked at their workplaces and at themselves professionally and personally. In the mid-project evaluation, one member was quoted as saying, "It's been sort of a religious experience for me to be on this math team. I've changed so much … I know I've only just started" (Francis, 1993, p. 4).

In retrospect, at the root of the transformative experience that many math team participants describe is the powerful combination of agency/activism on so many fronts. Teachers were at once researchers, learners, leaders and advocates. They felt overwhelmed at times, but also powerful. Revisiting the stories of their transformations is useful. It gives us hope and it provides us with a successful model to replicate. However, not every project can be a math team. Nor should it. As discussed above, the issues are located at the policy level and within the curriculum as well as at the level of staffing: teacher approach, background, and math competence. While Massachusetts math team teachers made reform mathematics come alive in their classrooms and took a message about reform into frameworks and workshops, at the state level, I dreamed that we could make that much more change happen by building a national community of adult numeracy educators and by infiltrating curriculum. To the degree that any of these interventions has been successful in changing policy or shaping curriculum, I now ascribe the success to the same causes as the original math team success: the power of activism on multiple fronts.

Where Practitioners Build a National Community and Influence Policy

While the Massachusetts ABE Math Team was meeting, activity was also brewing in other places: at the National Center on Adult Literacy's Numeracy Project, at the Math Exchange Group in New York City, and at the GED Testing Service. Teachers were not only experimenting in their classes; they were talking about it with other practitioners. With funding from the Department of Education and the National Council of Teachers of Mathematics, over 100 practitioners and policymakers came together. At this meeting, the practitioners formed the Adult Numeracy Practitioners Network (now the Adult Numeracy Network or *ANN*). This group has collaborated on a framework for adult numeracy standards (Curry, Schmitt, & Waldron, 1996) and continues to serve as a forum for those involved in adult numeracy.

An open question for ANN is where and how best to make its presence known. Pockets of reform are wonderful, but do not affect the majority of adults in the adult basic education system. Where are the most effective points for leverage? Is it within K-12 organizations to gain visibility for adult education, within literacy circles to gain recognition for numeracy? Is it better to go it alone? These are decisions for a group with an eye on the national rather than regional implications. ANN members refuse to be isolated like the learners in individualised classes; they will talk to each other and be heard on a national level.

The EMPower Curriculum Project

Over the past ten years, adult basic education math teachers have built their grass roots networks. Numeracy is somewhat more visible at statewide conferences and in states' curriculum frameworks initiatives. A vibrant root system supports a plant's reach for the sun. To that end, we stretched teachers and asked them to keep firmly planted in their classrooms, but also to dip into policy and research. What would happen if we stretched curriculum beyond its usual job of regulating the time spent in classrooms? What if it is stretched to be a vehicle for staff development and a conduit for research?

At TERC, in Cambridge, Massachusetts, I am part of a team of educators currently developing a curriculum for adults and out-of-school youth. We call it *EMPower - Extending Mathematical Power*[3] because its purpose is to extend the K-12 reforms to adult basic skills programs. Our working hypothesis is that adults will benefit from some of the pedagogical tactics of the reform movement. The *EMPower* Project, too, is a hybrid intervention because it combines curriculum development with research and the curriculum with staff development.

Where Curriculum Is a Tool for Staff Development

The concept of curriculum serving as a vehicle for staff development fits well with the staff of adult education. As mentioned at the beginning, adult education teachers are likely to have little mathematics training. In the ideal world, adult basic education teachers would come to their jobs with some training in mathematics teaching, but unlike K-12 schools, we have no teacher preparation system and no mechanism to reach teachers who typically find employment in adult education accidentally. Even in a best case scenario, where the training was available and programs and staffs could afford to participate, the problem of teacher turnover plagues the field. Writing a curriculum that explicitly takes on the role of staff development is unique for adult education. We have learned about this model from the authors of TERC's K-5 curriculum, *Investigations in Number, Data, and Space* (Mokros, J. & Russell, S.J. 1995). These authors take the stance that teachers form a kind of partnership with the curriculum:

> The link between curriculum and teacher decision-making is a focus on mathematical reasoning. Neither curriculum nor teacher can fully anticipate the complex and idiosyncratic nature of the mathematical thinking that might go on among thirty students in a single classroom during any one mathematics class. However, both teacher and curriculum contribute to a repertoire of knowledge about student thinking that leads to better mathematics teaching and learning (Russell, 1997, p. 248).

When curriculum supports teacher development in this way, aside from the suggested lists of materials, objectives, and steps, it assumes that teachers could benefit from ideas for leading mathematics discussions, how to respond to learners' ideas, and what to look for when students are working in small groups on investigations.

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[3] Extending Mathematical Power (EMPower) is funded by the National Science Foundation under Grant Number ESI-9911410. The website is http://empower.terc.edu.

To foster students' independent thinking, problem solving, reasoning, and communication, there must be opportunities in class for learners to express their ideas and their logic. Most teachers are accustomed to a format where they ask questions that call for a short answer. Conducting a math discussion requires more than a good question. At least three ingredients are key: an open-ended question, wait time, and a culture of listening. The curriculum can encourage wait time and can describe a classroom culture where learners' strategies are valued, and it is well poised to deliver carefully crafted questions that spark exchange. Within a given lesson, learners are often asked "How did you know?" rather than "What did you get for the answer?" In this way the curriculum helps make the students' mathematical thinking audible and visible, so that the teacher can make decisions about appropriate next steps.

Authentic questions ask for learners' ideas or past experiences where the teachers do not know the answer already. Learners' responses to such questions will often catch teachers off guard. Relating to the question in their own idiosyncratic way, learners may open the door for conversation or math that is only tangentially related to what the teacher had prepared. Teachers often fear such questions and the loss of control of the math in the lesson. Having an inkling of what learners might say helps teachers keep the math on track. One teacher whose students interpreted an investigation about the average price of a used four-door car to mean "which car would be the best car for someone of my means to buy?" asked that the math lessons include a section called "Learners might say." By providing this section, teachers are better able to anticipate and incorporate learners' contributions.

The curriculum encourages teachers and students get behind the meaning of the algorithms, formulae, tricks and short cuts. In order to teach this way, teachers need to be secure in their own understanding. One teacher this year asked for more information about the order of operations. She wondered why multiplication and division precede addition and subtraction. In order to help students appreciate the "why", she herself needed more background information. Most curriculum resources avoid lengthy explanations of content, assuming that teachers will reference other sources for such information. *EMPower* does not make that assumption.

Curriculum Development as Research

Whenever possible, *EMPower*'s lessons are based on existing research. Yet with the little research available on adults' numerate thinking, much of what is written is based on instinct, years of informal observation and experimentation, and research on children's mathematical thinking. When the *EMPower* research plan was originally designed, its purpose was to collect data from classes piloting draft lessons in order to inform the scheduled revisions. We want to know how doable the lessons are, what piques people's interest, what kinds of conceptions arise, and suggestions teachers might offer to strengthen the investigations. We also watch for evidence of teacher and learner attitudinal and behavioral change, including how the class comes together as a community. While these goals remain a central focus of our formative evaluation, the pilot and field-test phases of curriculum development are an opportunity to collect data on adults' mathematical understandings. Over time, our findings will dispute or concur with research on children or ethnographic studies of adults using mathematics in authentic contexts.

What has become interesting to us is the variation of mathematical understandings within and across groups. It is clear that at the same time we collect student work and tapes of classes in order to refine the curriculum, we have a golden opportunity to analyse the data to learn more about adults' mathematical ideas. Given the paucity of research on numeracy in adults, *EMPower* is uniquely situated to contribute to the field. We are determined to not to lose this opportunity and therefore our website is intended to do more than market the curriculum. By including examples of what we have collected, we make moments from our research publicly available. By sharing these examples and commenting on their meaning, we will begin to foster a dialogue about the development of mathematical thinking in adults.

Similarly, our curriculum will include conversations in classes and background on how adults conceptualise. We imagine that a set of understandings and strategies, as seen in written work or heard in classroom conversation, might be included in the curricular materials as examples of how lessons have played out. These data also help us craft suggested rubrics for teachers to look more closely at student work. Examples of what we are learning follow.

Geometry, arrays, and multiplicative thinking. One example of how the curriculum development and research intertwine can be seen in the way a geometry investigation uncovered ways of counting. As veteran teachers, we know that people confuse perimeter and area. Whenever we teach it in our GED classes, students do not differentiate between five inches and five square inches, and area and perimeter formulas are interchanged as well. In the *EMPower* geometry unit, we want to find a way to highlight the distinction between linear and square units. We present learners with a rectangle marked off in square centimetres and ask: "How many square centimetres cover the 5 cm. by 10 cm. rectangle?" Phrased that way, rather than "What is the area, in square centimetres?" the problem becomes one of determining quantity. The question seems to focus attention on what we had hoped, namely the square unit. Most learners arrived at the answer of 50, but the strategies for arriving at that number varied. We observed students:

- Counting each square centimeter, one at a time.
- Counting the number in one row (5) and skip counting by 5's (5, 10, 15 … 50), or counting the number in one row (10) and skip counting by 10's (10, 20, 30, 40, 50).
- Combining skip counting with some doubling (5, 10, then by two rows at a time … 20, 30, 40) or (5,10,15, 20, 25) with 25 + 25 making 50.
- Multiplying 5 by 10.

We continued to notice the frequent appearance of this combination of skip counting and doubling. Surprisingly, the array did not signal the operation of multiplication. The strategy of skip counting and doubling persisted as a strategy that made sense. Because we thought that the formula $A = lw$ would make more sense if an l by w array signalled multiplication, we made sure to include opportunities in other investigations to make sense of pictures, such as the number of cars in a neatly arranged parking lot.

From field notes, videotapes, and student work, we are beginning to uncover phenomena that occur during open-ended investigations. We want to pursue and respond to these as we continue to develop the curriculum. We have noticed recurrences such as:

- During investigations on the relationship between the circumferences and diameters of various circles, students examine data suggesting that the circumference measure is about three times the diameter. For some, this sits uncomfortably beside the idea that the diameter is the line that cuts the circle in half, not thirds. This makes us wonder about what it is that people are focusing on when they compare a table of numbers with the geometric (spatial) information.

- Disagreements as to whether seven 1,000,000s is the same as a million 7s. People's sense of multiplicative commutativity is more fragile than we had anticipated.
- Asked to make a graph from a two-column table of values, some people tended to copy the list of values on the axes as a list of numbers rather than as a number line with a scale.

How to best respond to these phenomena is not always so obvious. How careful should we be not to impose a skill hierarchy, such as the one that we implicitly imposed on how people counted the number of square centimetres? If we do rationalise a hierarchy, how careful should we be not to assume that the range and variation of a skill hierarchy in the population does not necessarily imply that the individual must or will develop according to that hierarchy? Adults who have not been to school in a while may feel quite free to use the bits and pieces of their understanding as resources for thinking about new ideas. And those bits and pieces may jump all over the scope and sequence of school-taught skills.

Conclusion

I come from a small and committed community of adult education teachers deeply concerned with the inequities associated with poverty, inadequate schools, and racism. The mathematics that concerns us are the percentages that the students in our adult education classes are part of: the 40% of the adult population with inadequate skills; the 89% of GED graduates who enrol in Community College who do not complete more than one year; and the persistent achievement gaps between black, Hispanic, and white students. We may not talk about those inequities in our daily work, but they are the phenomena which drive us as we strive to change the face of numeracy instruction from one of remedial isolation to inquiry community, from places where school-based arithmetic algorithms reign to places where the ways in which adults mathematise the world are major resources. Folks come to our classes hopeful to pass tests, but we want them to have bigger hopes than that. They are the reasons we have banded together in teams: the Massachusetts ABE Math Team, the Adult Numeracy Network, and the EMPower team. They are the reasons we search internationally, to our counterparts in Australia, the Netherlands, and the United Kingdom, and the Adults Learning Maths- A Research Forum.

Governments and their education and training systems seem more or less motivated to close the gap between the least numerate and the adequately numerate. In most cases, the interventions to remedy the situation are targeted at the school level and at the university level insofar as the preparation of teachers is concerned. I have spent many years in the classroom as a teacher, another number of years working with teachers, working at the state level, and now as a researcher and curriculum developer. I think we do not have a chance of moving toward the vision unless the interventions we propose are interdisciplinary. Hybrids only need apply.

Acknowledgments

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Making a Difference: The Early Numeracy Project

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The Numeracy Development Project (New Zealand), now into its third year of implementation, spans years 1 to 10 of schools. By the conclusion of 2001 the project had involved approximately 3300 teachers and 64000 students. The Number Framework forms the core of the project programme by providing teachers with a knowledge of how students acquire number concepts, an increased understanding of how they can assist students' progress and an effective means to assess students' levels of thinking in number. This paper focuses on the impact that the project has had on the students in the first three years of school and their teachers.

Background

The Numeracy Development Project is positioned within the context of the Ministry of Education's Literacy and Numeracy strategy and reflects the key themes of that strategy: clarifying expectations, improving professional capability and involving the community. The focus of the project is "improving student performance in mathematics through improving the professional capability of teachers" (Ministry of Education, 2002, p i).

In light of the view that professional development programmes can improve outcomes for students, much work has gone into identifying key elements of effective programmes (for example, Clarke & Cheeseman, 2000; Fullan and Hargreaves, 1992; Stephens, 2000). In a review that links professional development programmes with improved teacher capability, Parsons (2001) identifies the following as characteristics of quality programmes:

- the systematic identification of needs;
- a focus on the real world of the classroom, that is, the dynamic of the learner, content knowledge, assessment and pedagogy;
- links to the gathering, analysis and use of high quality assessment data and school development initiatives;
- opportunities for modeling, observation, coaching, critique and reflection, and the use of action research approaches;
- a school based approach over time that fosters collaboration and collegial support;
- effective facilitation, and ongoing guidance and support from within and outside the school;
- the involvement and commitment of school leadership;
- connection with the school and wider community;
- rigorous evaluation based on both quantitative and qualitative data which provides evidence of the impact of professional development on student achievement, teachers' knowledge, beliefs and expectation (Parsons, 2001, p. 15).

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 49-57). Sydney: MERGA. ©2002 MERGA Inc.

These findings are in line with the current professional development policies of the Ministry of Education. The evaluation reports prepared for the Ministry of Education by Higgins (2001) and Thomas and Ward (2001) reinforced the significance of these characteristics in the Numeracy Development Project.

Parallel to the work involved in identifying key factors of effective professional development programmes is a body of research that aims to identify the teaching practices of effective teachers of numeracy. An important project in this area was conducted by King's College London (Askew et al, 1997). It explored the knowledge and beliefs that underpinned the practices of effective teachers and found that what distinguished highly effective teachers from other teachers was a set of beliefs that formed the basis of a particular series of classroom practices. These concerned what it means to be numerate, the relationship between teaching and students' learning of numeracy, and which presentation and intervention strategies are effective. In addition to a well-developed set of beliefs, the "highly effective teachers of numeracy themselves had knowledge and awareness of conceptual connections between the areas which they taught in the primary mathematics curriculum" (Askew et al, 1997, p3).

A feature of the Numeracy Development Project is its evolutionary approach to implementation that reflects an ongoing commitment to develop understanding about how students learn number and what constitutes effective professional development and effective facilitation. In 2001 the Numeracy Development Project was managed as three projects, which spanned years 1 to 10 of schooling. This paper reports on the impact of the Early Numeracy Project (ENP), which focuses on students in the first three years of school, and followed the successful implementation of the Count Me in Too (CMIT) pilot project in 2000. The ENP involved approximately 40 facilitators, 450 schools, 2000 teachers and 39,000 students.

The Number Framework (New Zealand)

Research into children's understanding of number over the last decade suggests that there are identifiable progressions in how children develop number concepts (Carpenter et al, 1999; Clarke et al, 2001; Cobb et al, 1997; Fuson et al, 1997; Jones et al, 1996; Steffe, 1992; Steffe et al, 1983; Wright, 1998; Young-Loveridge, 1999). These progressions have led to the development of models, or frameworks, of early number development that can be viewed as providing useful pedagogical frameworks for teachers. This is based on the assumption that if teachers can identify where a child is on a framework, and can then identify the next step for the child, then teaching should be more effective.

The Number Framework used in the ENP 2001, was developed from the Learning Framework for Number, used in the CMIT Pilot Project. The Number Framework formed the core of the professional development by providing teachers with:

- an effective means to assess students' current levels of thinking in number;
- guidance for instruction;
- knowledge of how children acquire number concepts and an increased understanding of how they can assist children to progress.
(Ministry of Education, February 2001a)

The Number Framework[1] contains two main sections: strategy and knowledge. The strategy section looks at how students solve number problems, focusing on the mental processes they employ. Nine different strategy stages are described, with increasing levels of sophistication. The nine stages fit within two broader bands: counting and part-whole. It is important to note that the stages on the framework are "big" ideas and that much learning goes on in between each one. It has also become evident that the stages on the framework are not equally spaced. The knowledge section outlines the key aspects of knowledge children need to acquire and has been categorised into; numeral identification, number sequence and order, grouping/place value, basic facts, and written recording. The two sections of strategy and knowledge are seen as dependent on one another, with students needing to make progress in both areas simultaneously.

The Early Numeracy Project Assessment[2] (ENPA), the diagnostic tool used on ENP 2001, was designed to give teachers quality information about the knowledge and strategies of their students, as aligned to the Number Framework. Having teachers assess and monitor the development of children through one-to-one interviews was an integral component of the ENP. The ENPA tool enabled the teachers to develop a number profile for each student. This profile had five components:

- The Stages of Early Arithmetic Learning (SEAL). This related to the students' level of sophistication in counting and other strategies to solve addition and subtraction problems.
- Facility with forward number word sequences (FNWS).
- Facility with backward number word sequences (BNWS).
- Ability to identify numerals (NID)
- Understanding of the place value nature of the number system (BTS)

Teachers used the initial and subsequent assessment to make decisions regarding learning experiences necessary for individual children and groups of children.

Feedback from Participating Teachers

The evidence drawn for this section was gathered from questionnaires sent to 246 teachers in 50 randomly selected participating schools. One hundred and forty-eight questionnaires (60%) were returned. The questionnaires had two components: one aimed at collecting demographic data and the other asking a series of open-ended questions. Responses to the open ended questions were analysed for key themes and patterns. In general, responses fell into three broad areas: feelings about the project, perceived impacts of the project and issues with specific elements of the project.

The teachers were very positive about the project and the impact that it had on their teaching of maths. Ninety-six percent of the teachers believed that their knowledge of how children learn maths had been developed as a result of their participation. They link this increase in knowledge to the belief that they were now more effective in their teaching with

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[1] A detailed description of The Number Framework can be viewed at: http://www.nzmaths.co.nz/Numeracy/Numeracy_PDFs/numfwork.pdf

[2] A detailed description of the Numeracy Project Assessment can be viewed at: http://www.nzmaths.co.nz/Numeracy/Numeracy_PDFs/diagint.pdf. This asessment tool has been developed from the ENPA 2001 and the Advanced Numeracy Project Assessment (ANPA) 2001.

92% noting changes in the way they teach number. This belief is summarised in the following quote from one of the teachers.

> I must have been very vague before [the ENP] but now I know how to recognise, promote and teach to strategies. I feel I am much more focused in my teaching objectives…The children are getting a much better deal now in my class and that shows in their progress.

There were several common themes in the changes that teachers reported they had made in their classrooms:

- Increased focus on both number strategies and knowledge linked to The Number Framework.
- More effective assessment and grouping of children.
- Increased emphasis on listening to children and encouraging them to explain their thinking.
- More opportunities for children to share their strategies with others.
- More focused expectations of the progress that children will make.

The most consistent issue that teachers had with the project was related to the resource folder and classroom. Forty-four percent of the teachers felt that too much time was spent making classroom resources and 62% suggested that the layout of the resource folder was confusing.

Student Achievement

Arguably the most important aspect of the Early Numeracy Development Project was its impact on the numerical development of the participating children. As indicated in the previous section most of the teachers believed that their students made significant improvements. This section examines the results of students in Phase One schools. These are the schools which participated in the ENP during terms 1-3 and had submitted final ENPA results by October 12, 2001.

Student progress on the Number Framework was impressive. There was strong positive growth in the five aspects of number learning assessed, irrespective of the students' gender, age, ethnicity or the school's region or decile. Although there was no assigned reference or control group, the profile of students at the start of the project provides a degree of control when compared with the profile of students of the same age at the end of the project. Figure 1 compares the growth that occurred, in each aspect, over the six-month duration of the project with the growth that would have been expected with time alone. As is clearly shown, the gains made on each aspect of number learning during the project exceeded the gains that would have been expected in the students' previous classroom programmes by between 0.7 and 1.2 stages. Overall, the trend was for students to make gains of approximately one stage on each of the five aspects assessed. With the exception of Base 10 strategies, which had a mean gain of 0.8 of a stage, the mean gains for the aspects assessed are between 1.1 and 1.4 stages.

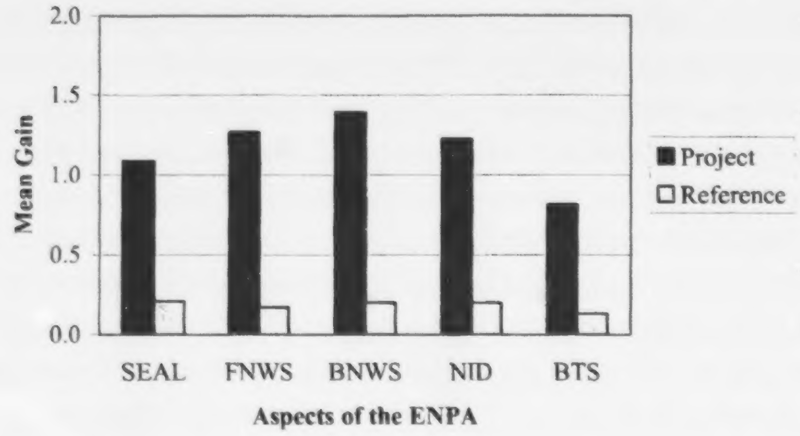

Figure 1. Mean gains by all students.

The size of the gains made was strongly linked to the students' starting points on the Number Framework. Figure 2 links the mean gains made on the SEAL to the initial SEAL stage of the students. As clearly illustrated, the gains are greatest for the lower SEAL stages, which reinforces the notion that the stages are not equally spaced and that the earlier stages are indeed "smaller" or easier to progress through. Students assessed as emergent at the initial assessment made mean gains of 1.56, compared to 0.33 for students assessed initially as early additive. It is interesting to note that older students make greater gains than the younger students regardless of starting stage.

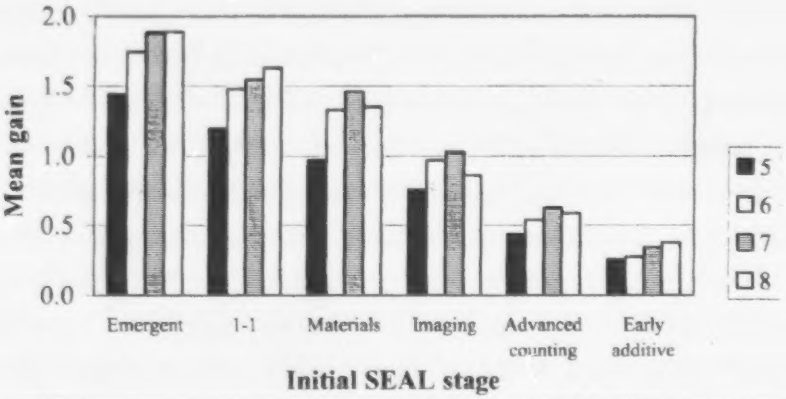

Figure 2. Mean gains on the SEAL linked to initial stage and age.

Figure 3 presents the initial and final breakdown of SEAL scores as a cumulative frequency distribution and clearly shows a decrease in the percentage of students at the lower stages and a corresponding increase in those at the higher stages. At the time of the initial interview, 7% of the students were part-whole (early or advanced additive) in their use of number strategies. By the end of the project this had increased to 23% and, as expected, the majority of these students were the seven and eight-year-olds. Correspondingly, the percentage of students who were assessed as emergent had decreased from 14% to 2% over the duration of the project.

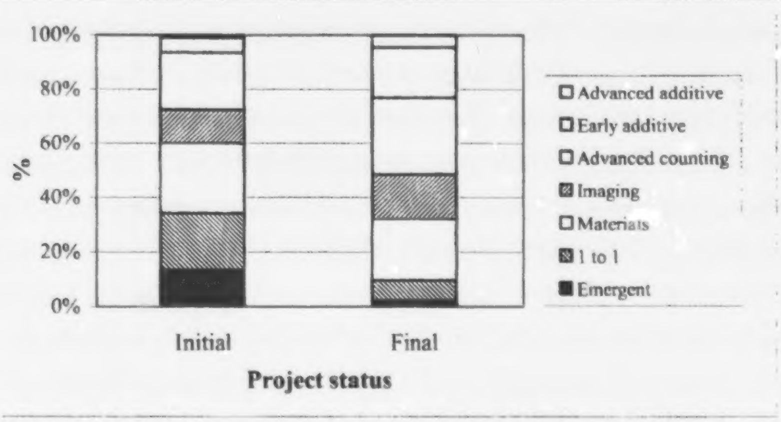

Figure 3. Percentage of students at each SEAL stage by project status.

The remainder of this section examines the performance of students when grouped by ethnicity with a particular focus on the transition from counting to part-whole strategies. The transition to part-whole strategies has been identified by researchers and educators as being critical to success in mathematics (Clarke & Cheeseman, 2000, Wright, 1998; Young-Loveridge, 1999). Figure 4 illustrates the gains made on the SEAL when linked to ethnic group and initial SEAL stage. The gains made by Māori and Pacific Islands students are slightly but consistently lower than the other groups.

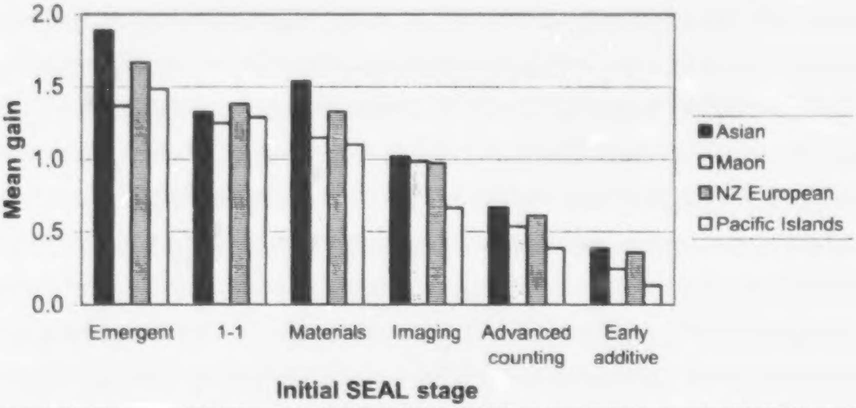

Figure 4. Mean gains on the SEAL linked to initial stage and ethnicity.

The most concerning difference is in the proportions of students who make the transition from advanced counting to part-whole strategies. Figure 5 illustrates the final SEAL stage of students who had an initial SEAL assessment of advanced counting. Just 36% of the Pacific Islands students made the transition to early additive or advanced additive compared to 49% of the Māori students, 57% of the NZ European students, and 58% of the Asian students. A higher proportion of Asian students (9%) moved the two stages from advanced counting to advanced additive than any other ethnicity. Pacific Islands students were again poorly represented with only 2% moving two stages.

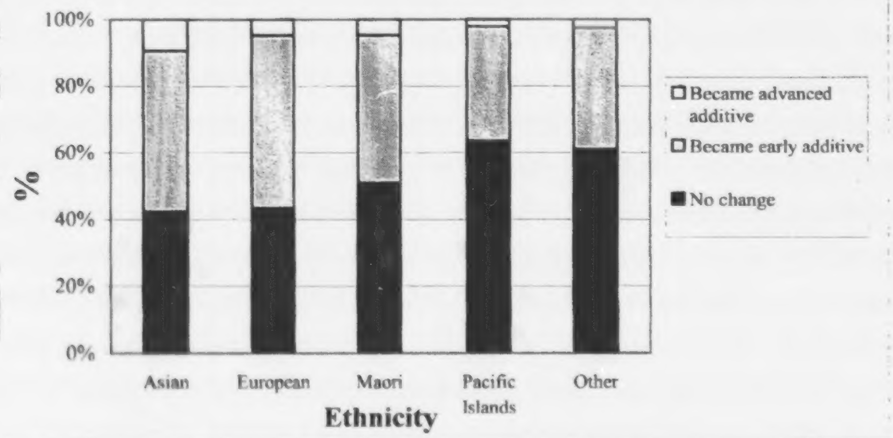

Figure 5. Final SEAL status of students who were initially advanced counters by ethnicity.

Figure 6 compares the mean SEAL scores by ethnicity of the students in the ENP with a sample of 1741 students from 22 schools that were in the CMIT pilot project, and have now participated in the numeracy project for two years.

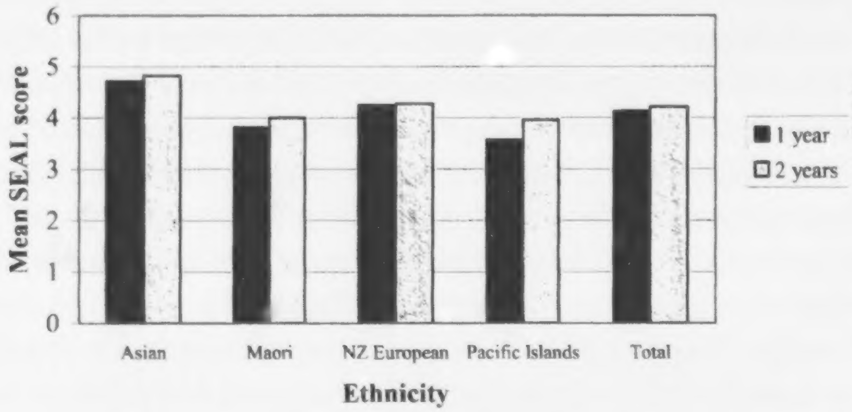

Figure 6. Mean SEAL score by ethnicity and years in project.

Students from schools with two years involvement have slightly higher means across all ethnic groups. The Pacific Islands students have the greatest difference with a mean SEAL score 0.37 of a stage higher in the schools which have participated for two years.

Given the importance of improved education outcomes for all students it is critical that further refinements of the project continue to focus on improving the outcomes for underachieving students.

Concluding Comments

The most consistent view about the Numeracy Project expressed both formally and informally, is that it empowers teachers and benefits students. Consequently there is a feeling of optimism in participating schools about improved mathematical outcomes for students. Positive publicity associated with the project has increased awareness and demand to the extent that there is a long queue forming as other schools wait for their turn to participate.

Acknowledgments

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Invited Paper

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What Does it Mean to Teach Mathematics Differently?

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In the past decade, many countries called for changes that result in students knowing a 'different' kind of school mathematics. This is due to a changed view of learning that requires different forms of teaching which are only beginning to be understood. In this paper, findings from research in primary maths classes with 'reform-oriented' culture are presented; examples drawn from these classrooms illustrate differences in teaching that influence the mathematics children learn. Two dynamic dimensions derived from this research are presented.

Two aspects characterize the current attempts to improve mathematics education in many countries over the past decade. First is the influence of the 'cognitive revolution' in psychology that redefined how individuals learn, and second the attempt to change the perception of the discipline of mathematics in school. Currently there is widespread acceptance of the view that learning is an active constructive process. It is thought that as students learn they impose their own interpretation and use these to make sense of the world (Bruner, 1990, 1996). Additionally, rather than conveying a view of mathematics as narrowly consisting of executing well-established procedures, the shift in goal is to present it as a subject that consists of patterns and relationships that are understandable through mental activity that involves mathematical reasoning and logic (Devlin, 1994).

At the center of this reform in mathematics education at the primary level are two interrelated qualities that are thought to underlie children's learning that, when taken into consideration, designate significant changes in the nature of teaching. The first quality is the social nature of children's learning and the fact that rich social interactions with others substantially contribute to children's opportunities for learning (e.g., Bruner, 1990, 1996). The second aspect is the interplay of children's developing cognition and the unfolding structure that underlies mathematics (e.g., Nunes & Bryant, 1996). These qualities suggest that social situations for learning are needed that are different from those that exist in most primary maths classes (Wood, 2001). It follows from this that a different form of teaching is needed.

Research shows that the quality and nature of students' mathematical learning in the 'reform classes' in the United States are significantly different from pupils' learning in conventional classes (e.g., Carpenter, Fennema, Chiang, & Loef, 1989). Cobb, Wood, Yackel, and McNeal (1992) found that, in particular, these classes could be distinguished by the differences in the quality of children's explanations and justifications they were required to give for their solution processes. Furthermore, these differences were found to affect the quality of children's thinking and reasoning about mathematics.

Along with this, there are similar studies that examined teaching in these classes. The research has ranged from studies examining teacher beliefs to descriptions of changes in practice (Fennema & Nelson, 1998; Wood & Nelson, 2001). Yet, our understanding of these forms of teaching are is still not as clear as our knowledge of children's learning in these settings. For past decade, I have examined this teaching from a perspective grounded in the practice of teachers through the examination of the interaction patterns that exist in

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B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 61-67). Sydney: MERGA. ©2002 MERGA Inc.

conventional and reform classes in an attempt to provide better understanding. Therefore, the purpose of this paper is to present findings from this research, to reconsider what it means to recreate a class culture for learning mathematics differently, and to offer suggestions about how teachers might accomplish this. Empirical examples of teaching drawn from a research project in primary maths classes are used to illustrate the differences in teaching and the ways these influence children's opportunities for learning. In the end, two dynamic dimensions are offered as a way to reflect on and examine mathematics class cultures which teachers might find useful to examine the interaction in their own classes. In order to examine teaching, it is important to take into consideration that teaching by definition is an interactive activity and teaching in school settings is an endeavor that is conducted as children are participating as members of a group (Wood & Turner-Vorbeck, 2001). Thus, it is essential to not only characterize the teaching that is observed but to also examine the ways teachers create a culture for learning in order to fully understand how pedagogy influences students' opportunities for mathematics learning.

Methodology and Analysis

The data for the analysis of primary maths classes was collected as part of a larger project that examined teaching and teacher learning.[1] The data was collected over a two-year period and consisted of approximately 300 videotaped mathematics lessons gathered over a period of a school year in 15 'reform-oriented' second-and third-grade mathematics classes in the United States. From this data resource, six classes were selected and 30 lessons from each were analysed using a coding scheme developed for the analysis. For further details about the lessons, the coding scheme and methodology see Wood and Turner-Vorbeck (2001). In a secondary analysis, this data along with data from conventional classes were analysed for the purpose of examining the nature of children's mathematical reasoning related to teacher questioning during these lessons.[2]

Mathematics Lessons

The 'reform-oriented' lessons frequently consisted of children working in pairs for 25 minutes followed by class discussions of their solutions for approximately 20 minutes. Lessons from conventional classes consisted of whole class discussion followed by individual seatwork. The aspect of the lesson of interest for the investigation of teaching is the class discussion. This is the most complex lesson event for both teachers and students and therefore provides the greatest opportunity to examine demands of teaching in such situations.

Lessons were collected on a daily basis during the first four weeks of school. These *sequential lessons* were primarily used to examine the manner in which each teacher initiated and established the social norms that underlie the interaction that occurs during the mathematics class. Additional lessons were collected twice a month for two consecutive days and thereafter throughout the remainder of the school year. Another set of lessons was drawn from this latter data source that consisted of those discussions in which all the teachers used the same instructional activity. These *anchor lessons* were used as contrast lessons for the purpose of analysing the patterns of interaction and discourse that were established between the teachers and their students. The lessons for the conventional classes were also collected during the first weeks of the school year and contained similar mathematical content.

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[1] This project was sponsored by the National Science Foundation.

[2] This project was sponsored by the Spencer Foundation.

The methodology and analysis follows a qualitative research paradigm in which observations of lessons were the data source for the interpretative procedures used. Initially, the methodology and analysis used were similar to that of Strauss and Corbin (1990) in which categories were developed and coded. In this approach, the process of coding is seen to underlie analysis. That is, interpretation is an integral part of the coding process. Following this analysis, micro analytic interpretative procedures similar to those described by Voigt (1990) and Krummheuer (personal communiqué March 1996) were used to analyse selected lesson discussions.

Each lesson was videotaped and fieldnotes taken. Following the videotaping, a log of each lesson was made which served as a detailed record to be used for later analysis. The technique of logging captured the nature of the interaction that occurred during the lesson but it did not contain the precise detail of a transcript. The logs were recorded in a line-by-line format and coded according to the scheme developed.

A set of coding categories was developed for coding the discussions in a line-by-line manner. Each line of the log was coded individually by three members of the research team and then discussed as a group, and any points of differences in coding were further deliberated among the members until consensus was reached. One set of coding categories was used to analyse the norm statements made by teachers. Another set of coding categories was used to analyse the teachers' questions or statements made during class discussion. The codes then served as the starting point for describing the patterns of interaction and the discourse that occurred during the class discussions. Initially, the codes served as a common notation for reacting to each sentence in the log which allowed not only for the identification of patterns in the interaction and discourse but also for the initial interpretative conjectures used to guide the micro analytic interpretative process.

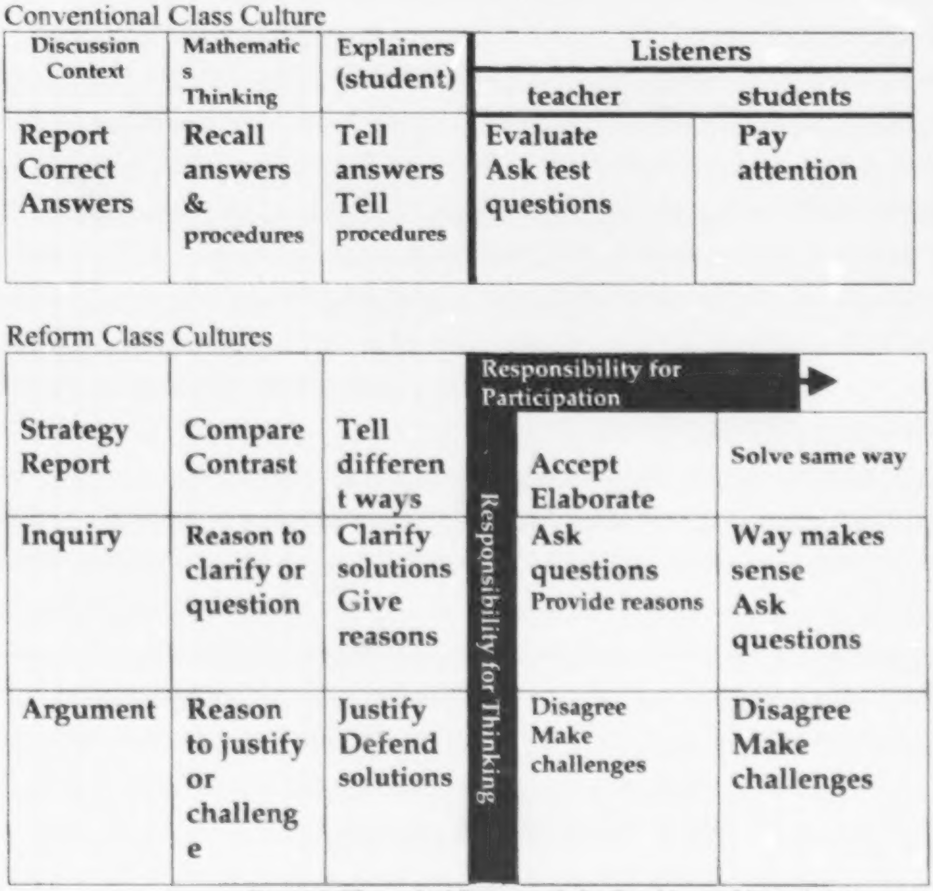
Micro Analytical Interpretation

Following the coding, the typical patterns in the interaction and discourse were identified and used to establish the episodic sections of the discussion. In addition, from the data resource, specific lessons were selected and transcribed. These lessons were analysed using micro analytical interpretive procedures. These procedures continue the line-by-line examination of the dialogues but provide in depth analysis of the events that occurred in the episodic sections. This process allowed interpretations to be made about the meanings held by the teacher and students in each situation. Interpretative memos were written which contained detailed descriptions of each event.

Results

In this work, the nature of teaching is broadly described in relation to opportunities created for students' reflective thinking and reasoning revealed in empirical analysis of the students' talk. The categorical patterns of interaction served as the basis from which to identify the similarities and, more importantly, the differences in the classroom mathematics practices that represent the ways in which the culture of the classrooms differed. The findings from the analyses confirmed that reform-oriented classes are fundamentally different from traditional classes. In addition, the analysis revealed that among reform-oriented classes the classroom mathematics practices do not consist of a singular form of practice, but rather consist of distinctive environments or contexts. The distinctions in classroom environments also illustrated the augmented demands for more sophisticated types of pedagogy for teachers.

Following the analyses of the empirical data, the results were condensed into generalizable patterns of interactive and communicative exchanges from which the theoretical framework was created as shown in Figure 1.



Conventional Class Culture

Discussion Context	Mathematics Thinking	Explainers (student)	Listeners	
			teacher	students
Report Correct Answers	Recall answers & procedures	Tell answers Tell procedures	Evaluate Ask test questions	Pay attention

Reform Class Cultures

				Responsibility for Participation	
Strategy Report	Compare Contrast	Tell different ways	Responsibility for Thinking	Accept Elaborate	Solve same way
Inquiry	Reason to clarify or question	Clarify solutions Give reasons		Ask questions Provide reasons	Way makes sense Ask questions
Argument	Reason to justify or challenge	Justify Defend solutions		Disagree Make challenges	Disagree Make challenges

Figure 1. Theoretical Framework for Teaching and Learning.

The findings revealed differences in students' thinking and reasoning could be attributed to the type of questions teachers asked. In addition, interaction patterns that were created by teachers with their students for participation influenced the nature of participation. The differences that exist in the class cultures, thus, could be attributed to specific variations in the expectations for participation the teacher establishes and the cognitive demand teachers' questions place on students' mathematical reasoning. Two dimensions, participation and student thinking, form the theoretical framework used to conceptualize differences among reform-oriented classroom into three separate contexts, *strategy reporting*, *inquiry*, and *argument*.

Thinking Dimension and Axis

Therefore, in the Theoretical Framework, one axis is the *Responsibility for Thinking Axis* (represented by the vertical black arrow in Figure 1) that represents conjectures made about the relationship of the three types of classroom practices, in particular the questions asked, to the possibilities for students' reflective thinking. It is conjectured that the observed differences in children's thinking that occur in the three types of environments

indicates an increase in the demand for reflective activity. Thus, the inclusion of the Responsibility for Thinking Axis shown in Figure 1 enables connections to be made from the categorical changes in demand for children's thinking to theoretical cognitive explanations of conceptual understanding. Thus, the inclusion of the Responsibility for Thinking Axis enables links to be made from the categorical changes in children's thinking to the shifts in teachers' activity. Finally, the assumption that underlies the Thinking Axis is increased responsibility for student thinking is an increase in student autonomy in learning.

Micro analytical interpretive analysis of the interaction patterns reveals notable differences in opportunities for students thinking. But more importantly the comparison of the micro interaction patterns with students' thinking considered as collective or group thinking reveals differences. Specifically, the analysis reveals the extent to which students are engaged in mathematical reasoning that is the heart of which is abstraction and generalization in the construction of mathematical knowledge. Moreover, it is the demand for *reasoning of justification* that separates these class cultures (Samarapungavan & Wood, 1998).

Participation Dimension and Axis

In the Theoretical Framework the other axis is the *Responsibility for Participation Axis* (represented by the horizontal black arrow in Figure 1) that illustrates the increasing responsibility of students to participate in the ongoing discussion. Micro analytical interpretive analysis of the class discourse reveals some distinction in the behaviour of the explainer. But, the important difference among the class cultures resides with the expectations for what to do if a listener. The origin of these differences is found in teachers' normative statements and the manner by which they establish meanings for these expectations at the beginning of the school year.

Illustrative examples of class cultures. The following dialogue of an example of a Strategy Reporting culture is taken from a second-grade class (7 year olds).

Teacher: How about 36 plus 36? (Teacher writes on the overhead, 36 + 36 = __). Sheila.
Sheila: 66.
Teacher: And how did you get 66?
Sheila: Well, 3 plus 3 is 6.
And then there are two 6's and a plus. Then I just got 66.
Teacher: What did you do with these two 6's?
Sheila: I just added them up.
Teacher: You added them up.
And what did you get when added 6 and 6?
Sheila: Well, I got 12, but it didn't work.
Teacher: 12 didn't work.

Sheila: I think it is 12 (inaudible), I think it is 72.
Teacher: Did anybody do it a different way?

The detail of the children's descriptions varies depending on the extent to which teachers, through their questioning, demand comprehensiveness and clarity in the explanations from the children. This situation can be depicted as one in which students "tell how they solved the mathematics problem."

The following dialogue example of Argument culture is taken from a second-grade class (7 year olds).

Teacher: What did you get for an answer for this problem? [52 – 33 = ___]
Fred: 25.
Sara: 19.
Adam: 21.
Teacher: Any other answers? Okay. Fred tell us how you got 25.
Fred: We used the unifix cubes. 52 and then we took away 33. First I took away the tens. 42, 32, 22. Then I counted back the ones, 21, 20, 19.
Karen: But you said it was 25.
Fred: I know, but now I think it is 19, because I counted it again with the cubes.
Teacher: John what do you want to say? (He has his hand raised).
John: I went back to 52 take away 30 is 22 (points to second problem on the paper). And I took away 3 more and that was 19. So I think it is 19.
Teacher: Okay. But why did you take away 3?
John: Because 52 take away 30 is 22, and 33 is 3 more than 30 so it was 19.
Teacher: How did you know that it was 19?
John: Because if 52 take 30 is 22, 52 take away 33 is 3 more than 30, so then I had to take away 3 more from 22, and that would be 19.
Teacher: That makes sense. Sarah what would you like to say?
Sara: Well if you take 30 from 50, then you would have 20. Then you would have 2 and that 3, so you could take 1 from 20, and that would be 19.
Mark: This is too confusing for me. Sarah, I don't understand why you took the 1 from 20.
Sara: Because you have 2 minus 3 and so you need 1 off the tens.
Mary: But if you took 1 from the 20, what happened to the 2 and the 3?
Sara: I took 1 from the tens and added it to the 2 to make 3. [Then] 3 minus 3 is 0. So then I had to take 1 from the tens-20 and that makes the answer 19.
Ryan: Well if you check it by adding 19 and 33, you get 52, so 19 is the answer.
Karen: I think the answer must be 19, because we did it so many different ways to figure it out. And we got 19.
Class: Agree. It is 19.

Discussion and Implications

As adults, with experience in traditional mathematics pedagogy, it is not easy for us to envision how teaching mathematics could be different. Thus, it is somewhat difficult to transform the calls for change into the reality of our everyday teaching. We might begin to reconsider and recreate a mathematics class culture for learning by examining interactive situations that occur during whole class talk. To do this, the two dynamic dimensions discussed above might provide a means for reflection on the interaction that occurs in whole class talk. By focusing our reflection and examination on the students, what they do to participate, and what they reveal of their thinking, as teachers we can begin to examine the culture created in our mathematics classes in light of the changes proposed for student learning.

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Practical Implications Award Paper

Supporting Beginning Primary Mathematics Teachers through a 'Fellow Worker' Professional Development Model

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A large study tracked four primary teachers during their first year of teaching, with a focus on how a 'fellow worker' professional development support model influenced their pedagogical practices and beliefs concerning mathematics. The case studies used an interpretive research approach for analysis of data from interviews, observations, reflective journals, and group meetings. The findings highlight how the realities and challenges of beginning teachers' classroom experiences prompted adoption of 'traditional', teacher-centred mathematics pedagogy. Support in the form of a 'fellow worker' helped the beginning teachers survive and then begin to teach in ways consistent with current trends in mathematics education.

Current reform efforts in mathematics education have been setting ambitious goals for schools, teachers and students (e.g. Australian Education Council, 1994; Department of Employment, Education and Training, 1989; National Council of Teachers of Mathematics, 2000). In these efforts, mathematics classrooms are envisioned as places where teachers and students engage in rich discourse about important mathematical ideas while exploring problems grounded in meaningful contexts. Such views are a departure from a 'traditional' mathematics classroom in which teachers tell students how to think about concepts and how to perform particular procedures, with students then expected to practise skills and reproduce ideas. These two perspectives for mathematics classrooms are based on different assumptions and beliefs about the nature of mathematics and effective mathematics teaching and learning. Hence, to move forward with a reform agenda requires substantial learning on the part of many teachers because the new perspectives challenge teachers' mathematics knowledge, beliefs and teaching practices. Progress is unlikely to happen without the support and sensitive guidance of appropriate professional development (Borko & Puttnam, 1998).

It was the context of mathematics education reform that motivated the study upon which this report is based, while the more focused concern for appropriate professional development for teachers arose from the recognition that teachers are key in any reform agenda. Their learning needs must be accounted for if beneficial change is to occur. The overall study from which this paper arose (Sparrow, 2000) aimed to examine three main things:

- How a range of personal and contextual factors impact upon beginning primary mathematics teachers' pedagogical practices;
- How these factors impact upon beginning primary mathematics teachers' pedagogical beliefs; and
- The effectiveness of an empowerment model of professional development that incorporates a 'fellow worker' support mechanism.

The focus of this paper is upon key findings that arose from the larger study with regard to the third research area. However, since the 'fellow worker' played an influential role in

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supporting the teachers as they made pedagogical decisions, the first two research foci are integrated into the discussions with further details available in Sparrow and Frid (2001a, 2001b). The research is of significance in reporting on the challenges faced by beginning primary mathematics teachers, the related choices they make, how the foci or nature of their decisions change as they develop professionally, and, most importantly, how a 'fellow worker' can play a valuable role in this development.

The 'Fellow Worker' Professional Development Model

Teacher Professional Development: The General Context

Teacher professional development, as conceived of in this study, aligns with constructivist philosophy in that learning to teach is viewed as a process of personal growth that is influenced by beliefs, commitments, and ways of operating in and interpreting one's world (von Glasersfeld, 1995). Here, teacher professional development is "that natural process of growth in which a teacher gradually acquires confidence, gains new perspectives, increases knowledge, discovers new methods, and takes on new roles" (Jaworski, 1993, pp. 10-11).

It might be expected that beginning teachers in Australia, having completed pre-service education within a mathematics reform context, would be leaders of change. Mathematics teacher education programs today, as indicated by the perspectives adopted by commonly used mathematics education textbooks (eg. Bobis, Mulligan, Lowrie, & Taplin, 1999; Klein, 2000), promote mathematics as more than a collection of facts and skills to be taught in a routine, drill-oriented fashion. They offer, instead, a view of mathematics teaching and learning that sees students engaged in inquiring, problem solving, meaning-oriented, relevant endeavours. However, earlier, *The Discipline Review of Teacher Education in Mathematics and Science* (Department of Employment, Education and Training, 1989) noted a discrepency between what was taught in teacher education courses and what evolved later in teachers' classrooms. This finding highlighted a need to address the professional development of beginning teachers, yet, over a decade later, means by which growth can be facilitated are not fully understood and articulated within the educational literature.

The 'Fellow Worker' Model

An intent of this study was to design a professional development model specifically aimed at supporting beginning primary mathematics teachers to develop skills and confidence to acquire knowledge and competence in their teaching as needs arise. Extensive examination of the research literature related to teacher professional development indicated several factors that would need to be incorporated into the professional development model, including: (i) fostering teachers' development as reflective practitioners (Clarke, 1994; Clarke & Peter, 1993; Prawat, 1991; Pultorak, 1996), (ii) empowering teachers to be decision makers and take actions (Begg, 1996; Clarke, 1994), (iii) the nature and role of adult learning (Knowles, 1984; Feiman-Nemser, 1992), and (iv) the need for support from a more experienced colleague (Feiman-Nemser, 1992, Speck, 1996). The 'fellow worker' professional development model was therefore designed as empowerment via *reflection* and *experimentation*. More specifically, it was designed to use the researcher as a more experienced colleague (a fellow worker) who would support the beginning teachers in the process of reflection, consideration of options related to teaching practices, and small-scale experimentation with options. This role meant the fellow worker was a hybrid of a critical friend, colleague, listener, mentor, resource person, and supportive, interested person. His role was also as a catalyst by which the beginning teachers could reason about and learn from their own teaching experiences. In this context it was recognised that, as Easen (1985) noted:

> You cannot change other people, nor can they change you; people can only change themselves. The best anyone can do is to provide a structure, which helps others to change, if that is what they want to do. (p. 71)

Consequently, the fellow worker had to be more concerned with choice than with change, and be wary of reinforcing his self-image of mathematics teaching irrespective of the needs and wishes of the beginning teachers. At the same time, as a professional, he could not be too 'laissez-faire' and allow an 'anything-will-do' attitude. Key features of the structure that framed the fellow worker professional development model are outlined in Figure 1 along with the literature sources from which these principles were developed.

- The fellow worker will be an experienced teacher (Clarke, 1994 Feiman-Nemser, 1992; Speck, 1996);
- The beginning teacher is empowered to decide which problem or issue will be the focus of attention (Begg, 1996; Clarke, 1994; Knowles, 1984);
- The beginning teacher is empowered to decide what will happen in the classroom or elsewhere to address the problem or issue (Clarke, 1994; Knowles, 1984; Speck, 1996);
- Options for action will be decided between the fellow worker and beginning teacher, but the beginning teacher will choose which option to put in to action (Feiman-Nemser, 1992; Knowles, 1984);
- Reflection on practice and beliefs of the beginning teacher will be emphasised (Clarke, 1994; Clarke & Peter, 1993; Eason, 1985; Prawat, 1991; Pultorak, 1996; Speck, 1996);
- Beliefs and assumptions about mathematics and mathematics teaching and learning will be examined (Thompson, 1984);
- Experimentation in the classroom will be encouraged to provide data and evidence for reflection (Clarke, 1994; Clarke & Peter, 1993);
- Meetings with other beginning teachers will be established (Barnett, 1992; Clarke, 1994).

(Adapted from Sparrow, 2000, p. 72)

Figure 1. Key features of the fellow worker professional development model.

Method

Since its aim was to understand the nature of a professional development learning environment and the ways beginning teachers interacted with it, this study was designed as a naturalistic, interpretive inquiry. Hence, qualitative methods, with their capacity to emphasise contexts, meanings, and individuals' interpretations, were adopted.

More specifically, the research involved case studies of four beginning primary teachers with the researcher (author 1) in the role of 'participant-as-observer' by serving as the fellow worker of the professional development model. The beginning teachers were volunteers for the research and they all had recently completed a one-year Graduate

Diploma in Education in Western Australia after prior completion of a three-year Bachelor's degree. The sample consisted of two males (Harry and Gaz) and two females (Stephanie and Tiffany). This was not typical of the overall enrolment in the diploma course, where generally males are in the minority. However, proximity to the university and the need for the researcher to visit the teachers on a regular basis governed selection of participants. All four teachers were assigned to multi-aged classes for their first teaching position. Stephanie had a Years 5/6 class, Harry had a Years 5/6/7 class, and Tiffany and Gaz each had Years 2/3.

Data were collected from interviews, teacher and researcher journals, group meetings, and classroom observations. Interviews occurred at least twice each term, while group meetings took place at the end of Terms 1, 2, and 3. Interview transcripts were the initial data analysed, with the other data sources used to substantiate and expand themes identified in the interview data. Hence, data analyses proceeded inductively, with NUD*IST as a data handling tool (Qualitative Solutions & Research, 1997). Initial nodes for use in NUD*IST were selected from factors identified from the literature as relevant influences upon pedagogy.

Findings

The following discussion considers the two fundamental features of the model of empowerment: *reflection* and *experimentation*. These overviews are followed by consideration of the overall effectiveness of the model for supporting the beginning teachers and engendering change. The case of Stephanie is used as an exemplar of the overall findings that were in common amongst the four beginning teachers. This provides a complete, coherent, and authentic account of one teacher's experiences without fragmenting the common findings amongst more than one teacher. The discrepancies that arose between the four teachers are touched upon in the section on the study's implications.

Reflection

The beginning teachers found it was difficult to find time to reflect on what was happening in their classrooms. The fact a fellow worker visit had been organised meant a time was actually committed and the reflection process could begin. Without the catalyst of the professional development model it appeared unlikely that time for reflection would have been found, as noted by Stephanie:

> There's so much time just getting through day-to-day, and to do in the day, that you don't have time to really sort of plot and think a lot about what you are doing. (Stephanie)

Initially the fellow worker prompted reflection by asking the teachers to describe successes and failures they had experienced in their mathematics teaching. Questions were sometimes specific to the situation, inviting the beginning teacher to reflect on what had happened since a previous conversation concerning specific teaching endeavours. When possible, the fellow worker asked questions designed to encourage teacher reflection in a way that would remind the teacher that she had prior valuable knowledge related to children and how they learn, or guidelines and ideas in mathematics curriculum documents. In this way, features of changed or developing practice were highlighted to help the beginning teacher realise that she was actually developing into a different teacher.

For Stephanie, this approach helped her to see and articulate the journey she was undertaking. Hence, as a result of the meetings and related reflections, teaching successes and weaknesses were considered. From these considerations positive features were noticed and later built upon. For example, in the following interview excerpt, Stephanie notes how trying a group work activity with her students led to valuable learning for her students:

> And it was quite interesting using paper. I think we had three sessions. Like they started off really well and there were four groups and they did all different things. One group began cutting out triangles and then one group traced around them, and one group actually worked out that one triangle was double the next one, and was double the next one. And did it like that and did it very quickly.

Talking incidents through with another person appeared to clarify aspects of what had happened and provided Stephanie with insights into children's thinking and understandings, as well as the complexity of the task of teaching. For example, the following transcript related to students exploring the concept of area reveals how Stephanie was enabled through the reflection process to 'see' the children in a different way, as individuals rather than as a homogeneous mass:

> Stephanie (S): … but one girl actually worked out that she had gaps and she knew you couldn't have, you couldn't have gaps. You couldn't work out gaps and she actually came up with using plasticine. 'And have you got any plasticine I can use to fill in the gaps?' …
>
> Fellow Worker (FW): What? So she squished the plasticine up?
>
> S: Yeah, … then she realised she could not measure how much plasticine she'd used. And so she actually made her plasticine into two centimetre little worm things. … And she was the only one who explicitly came out and said, 'But there's gaps. And like, no I can't have those. What shall I do about it?' And that was quite good thinking. A pleasant surprise I suppose.
>
> FW: So going back and thinking, and I think our initial conversation was you actually wanted to know where to start with kids, and what they knew. Do you think something like this … do you think something like that is helpful?
>
> S: Yeah, because there were some kids who had that kind of understanding and some kids who didn't. Some kids just put the round counters and then counted how many counters and didn't care if there was bits left over. Yeah, like I would probably say, yes that kid knows space and that kid doesn't. Maybe use two groups in the classroom, [rather] than have one lot.

Stephanie's last comment about the possibility of using two distinct learning groups in the classroom shows how reflection led her to consider other ways of teaching mathematics beyond having all the class complete the same task in the same way. This capacity to see alternatives did not arise until Term 3 of the school the year, as outlined in the next section, but once it did emerge it led to changes in her teaching.

Experimentation

Insights into pedagogical practice and the possibilities of other, maybe more effective, ways became apparent as Stephanie's reflective talk continued into the second half of the year. She began to consider differences in children's achievements, while also considering how she might change her teaching methods to develop other aspects of their learning. For example, she wanted to develop more talk amongst students when they were working in groups, so she thought about how that might be facilitated by the way she designed the task:

> S: … even though they were in small groups they were actually, it was, 'How big is my foot?' Not how big is one [person's] foot, you know, even though they were working in groups. They were still

> all actually measuring [on their own], so maybe if they were, you know, all told to pick one person out of the four and find out how big their foot was, they might have to sort of do more talking to each other, and that might have made [their] writing easier, and the marking easier.

Another of the earlier successes in Stephanie's teaching, which in turn led to a gradual change in her teaching style, was associated with mental mathematics. In the first term and part of the second term, mental mathematics was a 'traditional' exercise of recall and speed. A simple and small option, changing from asking for the answer to supplying the answer and asking for a possible sum, provided a catalyst for further experimentation. For example, she used questions such as: "The answer is 12, What is the sum?" After getting a positive response from the children in relation to their learning she endeavoured to ask more questions aimed at requiring the students to think.

As Stephanie's confidence with her ability to control the class and her success with a teaching style other than teacher-directed and procedural continued to grow, she began to try other ideas. She did this even when she was not convinced they would actually work. This risk taking led to some success, and in turn this prompted her to question her beliefs about mathematics teaching and what the children could achieve:

> S: That lesson on the types of multiplying, I was just blown away. I was just like, 'There's no way they're going to be able to make that connection between the squares and the digits.' But they said, 'Oh yeah, that's this and that's that.' All very easy. Probably that made me see that doing something like that constantly first really does actually get it. … you always knew the theory of it, but having actually seen it like clearly …"

Aspects of Change and the Role of Support in the First Year of Teaching Mathematics

Initially, Stephanie's teaching style was directed to the whole class with everyone doing exactly the same activity and with the same learning objectives (also see Sparrow & Frid, 2001a, 2001b). Later, through the support of the fellow worker and related reflection and experimentation, Stephanie changed her practice. She began to use group work with the children, and at times each group had a different task to complete. There was also a change of emphasis. The children were now sometimes required to think and solve simple problems. This was in contrast to her earlier teaching in which the children merely followed instructions. Talk by children, as part of learning mathematics, also gradually became a feature of her pedagogy over her beginning year. Initially, children worked "on their own" and generally worked in silence. Stephanie had used that initial time to establish control of the children, what they were doing, and what they were (supposedly) learning.

The development towards group work and children discussing mathematics was not an easy transition. Not only did she have to believe it was worthwhile and that the children would learn something from it, she also had to teach the children how to socialise and cooperate with their peers:

> S: Towards the end of that first session they began almost breaking up and making little pairs and things rather than working as a whole group. And so by the next session they were also sort of off and kept going off and working in two's and three's and fighting and arguing and things.

She persevered with group work and children talking because she wished to incorporate aspects of children's writing about mathematics. She saw the use of talk as a necessary rehearsal for writing. This reflected a change in her beliefs about teaching and learning mathematics in that teaching mathematics now involved children talking and using their own ideas. She also developed skill with assessing their level of understanding by listening to their talk and observing what they did:

> S: The assessing, I tried to include some different ways. ... First and second term it was mostly just a test at the end of the unit, and I did a little bit of speed maths. ... I've done a lot more observation things ... 'Yes, they've got this, ... no, they haven't,.' ... checklist sort of thing. I've also tried to include some kind of a practical component as well, so I take things like the measuring perhaps, I get them to read off the container rather than saying, 'Here's a picture of a container.'

Understanding the children's achievement levels and identifying what they knew had been the first issue or problem for investigation identified by Stephanie at the first meeting, but it took her much of the year to reach a point where she successfully implemented changes in her pedagogical style to accommodate her earlier concerns. The empowerment format of the teacher's identification of a need and the fellow worker being able to respond helped to change professional practice in Stephanie's classroom. Stephanie saw the support model, with its built-in need for reflection, as a positive and useful thing for her as a beginning teacher:

> It has certainly got me thinking about what I do, which is good. ... Quite often I think, 'I don't know what I was doing or why I was doing it that way.' ... and that [reflection] has led me to think then ... 'I'll try doing something else.'

Conclusions and Implications

During her beginning year, there was a change in Stephanie's pedagogical practice in mathematics. She changed from a traditionally-oriented teacher to one who was experimenting with a variety of teaching and assessment styles, and who paid explicit attention to the abilities and needs of the individual children in her classroom. Her work in mathematics teaching was approaching more closely the recommendations of recent documents in mathematics education. Whether this development can be attributed solely to the fellow worker professional development model is debatable. However, there is evidence to suggest that is was influential in that with the assistance of the fellow worker instances of 'good' practice were supported and further developed. This did not occur to the same degree or at the same rate for the other three teachers. Tiffany, similarly to Stephanie, became more adept at reflection, and then giving her own suggestions and acting upon them. Gaz also experimented with different teaching strategies as a result of both his principal and the fellow worker acting as support persons and option-providers. In comparison, the professional development model did not appear to influence Harry's teaching practices until very late in the year. For Harry, *very* strongly held 'traditional' beliefs about mathematics and how it should be taught, along with extreme home pressures and many school pressures, formed a situation where reflection was highly restricted.

These findings suggest three main things concerning mathematics teacher education and related research:

- More mentoring or other forms of support are needed by beginning teachers as they 'survive' their initial teaching experiences and the many challenges they face in their classroom management, curriculum planning, teaching and assessment practices, and personal and professional identities.
- The impact of pre-service teacher education programs needs to be examined, particularly with regard to how to bridge the gaps between what is promoted in mathematics education studies and what are the realities of schools and classrooms.

- In pre-service education, teachers need to further develop their skills and confidence as reflective practitioners who can examine the inherent complexities and conflicts of teaching alongside a wide array of potential resolutions and how they might be put into practice.

Practical Implications

The findings of this study point to a strong need to break the cycle of tradition of placing beginning teachers in a 'sink or swim' situation. The beginning teachers faced situations that for them were extremely challenging and demanding upon their professional knowledge, skills, and self-esteem, particularly in relation to their mathematics teaching. They saw their experiences very much as that of 'survival'. Even when they expressed concerns early in the year about aspects of their teaching, they were not able to give them much more than a casual thought. It was only after sustained, longer-term interactions with the fellow worker that the beginning teachers reached a point where they were able to engage in reflection that led to consideration of options and experimentation with different teaching practices.

Thus, if beginning mathematics teachers are to be able to develop professionally they must have a 'mentor' of some sort who supports them in an ongoing way for the full first year of their teaching, if not longer. The successes and weaknesses of implementation of the fellow worker model highlight the following practical implications regarding the form of mentorship or support needed by beginning mathematics teachers:

- A fellow worker needs to be provided for each beginning teacher so that the uniqueness of individual teaching situations are effectively examined and acted upon. The group meetings with all the beginning teachers did not serve to foster depth and breadth in reflection. They were valuable at a more emotional level, but did not engender reflection that led to experimentation and subsequent pedagogical changes. Hence, the traditional form of group professional development 'workshops' are not recommended as means by which to effectively assist beginning teachers. To grow as teachers the beginning teachers needed individual, personally and professionally context-specific support. It was not a case of 'one size fits all' professional development.
- The fellow worker needs to be independent of any form of evaluation of the beginning teacher's 'performance'. That is, he cannot serve as someone who assesses the teacher for school or Education Department records. The freedom to express concerns over personal weaknesses, a key feature of the fellow worker model, would not have been present within the context of performance evaluation pressures. A similar statement can be made for concerns about the school's expectations or ways of doing things. Reflection, which proved to be a prime ingredient in the recipe for professional growth and change, would not have been fostered without the impartial concern of the fellow worker. Hence, to engender success for the fellow worker model, the fellow worker should be someone external to the school.
- Formalised meetings need to be scheduled between the fellow worker and beginning teacher. It was clear that if it had been left to the beginning teacher to contact the fellow worker 'if needed', as is often the case when a teacher at the school is assigned as a support person, then the sessions and reflections would not

have happened. The beginning teachers had opportunity make use of school personnel and resources, but opted not to do so because they did not want to "bother busy people" and did not want to show any weaknesses.

- It is vital that the fellow worker be an 'option-provider' rather than a 'solution-provider'. The beginning teachers' professional growth was dependent upon their engagement with the particular challenges, problems and dilemmas of their teaching contexts. This engagement, or solving of problems, was part of the professional development process. It would not have been successful if it had been a process of merely reproducing the fellow worker's ideas and strategies for mathematics teaching.

The implications of these points for stakeholders in the education of mathematics teachers include:

- Education Departments and other employing authorities need to develop policies that do not abandon beginning teachers. They need to establish 'fellow workers' for at least the first full year of a beginning teacher's employment. Employment of experienced educators as fellow workers, who have professional skills related to particular learning areas (e.g. mathematics) as well as skills in working in mentorship roles, is recommended. The employment contract should also consider, in conjunction with individual school input, how a beginning teacher can have 'time' made available by which to regularly meet with the fellow worker and then follow up with planning for experimentation with new ideas.
- All schools need to make sure they have formal mechanisms in place so that they too do not abandon beginning teachers. Assigning a 'mentor' that does not have a structured role or is merely a contact person is not sufficient. In addition to an external person as a fellow worker, we recommend that all schools develop specific strategies to assist beginning teachers to endeavour to experiment and engage in professional problem solving, perhaps by team teaching activities, group action research endeavours, or regularly scheduled peer-sharing sessions.
- Faculties of Education need to focus more on the major issues identified as in need of attention concerning the gaps between what is promoted in mathematics education studies and the realities of classrooms. Students need more opportunities to examine the inherent complexities and conflicts of mathematics teaching alongside a wide array of resolutions and their related advantages and disadvantages. The beginning teachers in this study did not appear to have much breadth in their professional repertoires for implementing alternatives to teacher-centred mathematics teaching. Instead, they opting for a teacher directed approach that ensured they could control students' behaviours and learning. Hence, pre-service mathematics education needs to include more of a case study, problem-solving approach to mathematics teaching. Alongside basic mathematics teaching knowledge and skills, pre-service teachers need to develop capacities to reflect upon practices and consider options. Only then will the 'cycle of tradition' be broken so that mathematics is not taught in a procedural, replication mode.

The form of mentorship developed for this study, the fellow worker, is a model that has demonstrated it can effectively support beginning teachers in reflection, experimentation, and change in mathematics pedagogical beliefs and practices.

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Research Papers

The Mathematics Enhancement Project: The Pilot Phase

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This is the second report to MERGA of a project working with senior secondary mathematics in low socio-economic schools in Auckland, New Zealand. It is aimed at enhancing the achievement and participation of Year 12 & 13 mathematics students and promoting their transition into tertiary mathematical programmes. This paper outlines the theoretical basis for the project, and describes the 2001 and 2002 Pilot Studies.

In 1999/2000 a project was designed to enhance the achievement and participation of senior secondary mathematics students in low socio-economic schools (Barton & Graham, 2000). The resulting Mathematics Enhancement Project (MEP) is in a pilot phase in 2001 and 2002. The Scoping Study, design of the project, and preliminary results from 2001 were reported in Alangui, Autagavaia, Barton & Poleki (2001). The overall theoretical basis is given in more detail in Barton, Autagavaia, Poleki, Alangui (2002).

The project proposal was for a comprehensive, long-term approach to mathematical enhancement. It includes four components: teacher development, student support, community participation, and research. The community aspect is now regarded as a parallel project and is not reported here. The project team is based in the Mathematics Education Unit in the Department of Mathematics at the University of Auckland. It includes both staff members and teacher/researchers on one-year Study Awards.

In its 2001 Pilot Phase the project focussed on student support. The team worked with three classes in each of two schools: the Yr 13 Mathematics with Calculus class, the Yr 13 Mathematics with Statistics class, and one Yr 12 Mathematics class. All classes had 15-20 students. Two teachers were involved from each school. The three team members each attached themselves to two similar classes – one from each school – visiting for one period per week during most of the year. Each researcher conducted a study at the same time as they helped students and contributed to teaching activities. The studies focussed on language issues, mathematical background, graphics calculators, and indigenous issues. The overall research question, however, was the 'pilot' question of whether this model of research and development was practical and likely to lead to successful outcomes.

In the 2002 Pilot Phase the main studies will focus on teacher development. In addition base-line descriptions of students and teachers, and two smaller studies on student motivation are being made. Two schools have been added, with a total of ten teachers now involved. Five team members will make regular visits during the year.

A Critical Theoretical Approach

The research associated with this project has three distinctive characteristics. One of these, the politico-cultural dimension, is the subject of considerable literature within mathematics education, particularly in Australasia (see, for example, Atweh & Forgasz, 2001). The other two do not have such established theory, and thus provide interesting

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arenas for investigation. We welcome ideas and critiques of our approaches.

Political Considerations

Projects in low socio-economic or particular ethnic communities face issues of power, positioning and politics. On one level these issues are cast aside: a teacher faced with a mathematical query from a student rarely pauses to consider ethnic relationships or socio-economic background. At another level a teacher operates within a set of behaviours, values and attitudes with respect to particular ethnic and socio-economic contexts. This project does not seek to politicise classroom interactions, but to bring about an awareness of ethnic and socio-economic aspects of the habitual behaviour of students, teachers, researchers, the school, and the community alike. The underlying assumption is that awareness will lead to critical reflection, and any changes will be made by individuals themselves.

Issues of power, positioning, and politics also exist in the philosophical approach to the project, its documents, its aims, its methods, and its recommendations. We expect that our responses will evolve throughout the project. The most important of these issues is the positioning of Maori and Pacific Island students, or students from low socio-economic backgrounds, as "the problem". Data showing under-achievement of these groups exists (e.g., Te Puni Kokiri, 2000), but the causes are widely debated. The creation of an under-achieving category immediately sets up expectations that make the statistics self-fulfilling. Zevenbergen (2001) places responsibility for under-achievement on the structural hegemony of educational institutions and their defining documents. In addition there is research which describes (Pacific Island) cultural characteristics which impact on school learning (Mara, Foliaki & Coxon, 1994; Pasikale, 1999; Silipa, 1999), including specific features which are likely to impact negatively on school mathematics (Manu'atu, 2000). Pasikale (1996) reminds us that the cause is not deficiency, but mismatch:

> Success in learning is intricately linked to concepts of self-image and social image. Educational success for Pacific Islanders can only be assured if the cultural assets learners bring to the learning environment are not negated by the goals and practices of educational institutions. (p. ix)

This project turns Bishop's concept of cultural conflict (Bishop, 1994) into a positive stimulus. This involves the steps: recognition, adaptation, and creation of opportunity. These steps need to take place on the community level as well as in the classroom.

Another major issue is the political basis of assessment. Senior mathematics is measured through a national examination. The usefulness and appropriateness of such a measure has long been questioned, but other methods fail to get recognition for vocational purposes or further education. "Whose knowledge?" and "Who sets the gates?" are important questions. The gate-keeping consequences for the participants in this project need to be addressed. Furthermore national examinations measure short-term learning, whereas this project is concerned with long-term participation and achievement.

A third issue concerns teachers. This project regards teachers as professionals (not service providers). Our model of teacher development includes: building on the good from the past; gradual implementation tailored to the resources available; and depends on the active, voluntary participation of those involved. It is theory and research driven and assumes that there is no 'ideal' situation which can be implemented in isolation.

Using Critical Theory

Two uncommon aspects of this project are the linking of research and development, and the clustering of diverse research studies. When research and development go together the researchers become familiar with the classroom and the progress of the students. However, they have an investment in the effects of their research and hence objectivity in evaluation becomes problematic. A project of many studies must integrate several factors which are not simply additive, but interrelate in complex ways. Practically there needs to be focus on individual aspects, analytically there needs to be a holistic approach.

A framework has been adapted from the theory of Skovsmose and Borba (2000). Their triad of Current Situation, Imagined Situation and Arranged Situation can be enlarged from a single research study to a project of many studies taking place alongside development.

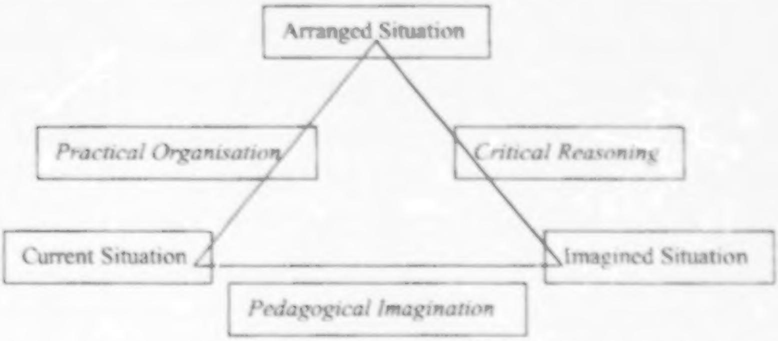
Arranged Situation
Practical Organisation
Critical Reasoning
Current Situation
Imagined Situation
Pedagogical Imagination

Figure 1. Triangular model of research in the classroom (adapted from Skovsmose & Borba, 2000, p. 12).

Skovsmose and Borba conceive development as the trajectories followed by each of the vertices of this triangle as it is repeated. To use this for a multi-faceted project, the triangle is conceived in three dimensions. Thus the Arranged Situation becomes the reality of what happens in the classroom as a result of all project activities. The reality of the classroom situation is the result of several Imagined Situations. Thus the vertices Current Situation and Arranged Situation should be taken to be a single trajectory (see Figure 2).

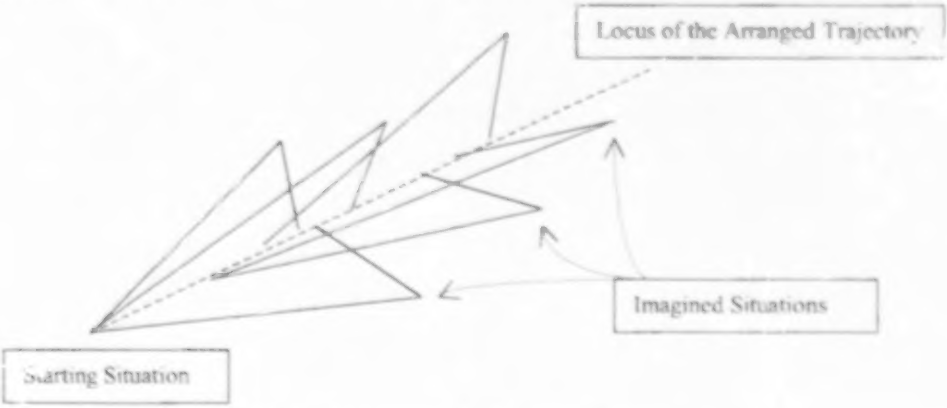
Locus of the Arranged Trajectory
Imagined Situations
Starting Situation

Figure 2. Model for a research & development project.

To this model we have added an evaluative component. The locus of the Arranged

Trajectory is not controlled by any one research study nor by any developmental initiative, but must be evaluated as a whole. The final adaptation of this model then, is that it must take place within a Cone of Improvement. It is hoped that this construct will initiate discussion on what dimensions might be suitable as defining characteristics of this cone.

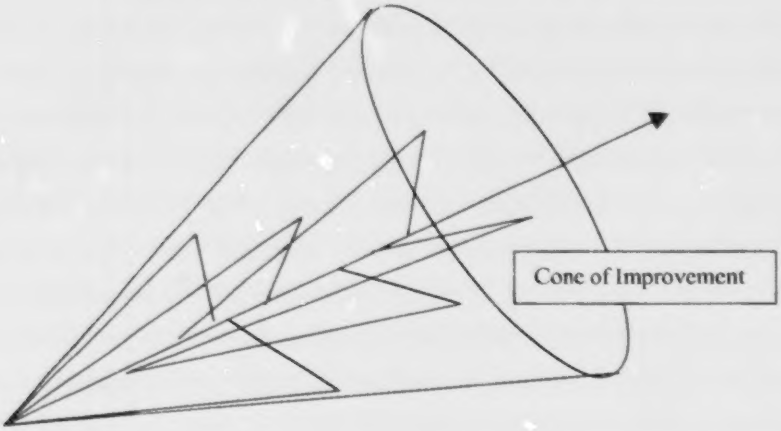

Figure 3. Evaluating the tTrajectory.

The Pilot Process 2001

The prime questions of a pilot process are whether the project is worthwhile, practical and likely to be successful. The pilot has given an affirmative answer to these questions, with a few qualifications, and guidance as to how to continue.

Pilot Questions

The 2001 pilot confirmed that there is an issue to be addressed. There was no evidence that the problem had been overestimated in the Scoping Study. Two of the team were teachers and they reported that the organisation and quantity of mathematical work done in class was significantly below their experience. Although this pilot focussed on student learning, the team remained convinced that teacher professional development was likely to be the most important factor in initiating change in student achievement and participation, and noted that teachers were under more than normal stress. Testing this conviction has become the focus of the second pilot year. Thus the conclusions of the Scoping Study remain unchanged, and the project remains worthwhile.

Is it practicable? The model of using teacher-researchers on one-year study awards revealed some drawbacks and some unexpected positive spin-offs. The overall plan of conducting research in a classroom during regular visits, at the same time assisting with teaching and student support, proved both workable and satisfying. The budget proved realistic, and progress was made on an instrument to observe student learning in the classroom within the parameters of these visits. Treating students as successful mathematics students (having made it to Yrs 12 & 13) having further potential for success in mathematics was well-received. On the negative side, the lead-in time required for teachers inexperienced in classroom research was not properly anticipated, and future projects will be redesigned so that they are better supported as researchers early on in the process. It was also harder than anticipated to maintain team meetings when everybody had significant other studies, teaching or research as well as MEP work. An unexpected outcome was the notable growth in the interest and research abilities of the teacher-researchers, and hence their potential to be used in future stages of the project.

Is it likely to be successful? Each of the studies below demonstrated that a difference can be made relatively easily. We are confident this conclusion will extend to other areas.

Factor Study 1: Language Issues

(It will be assumed that the reader is familiar with Alangui et al. (2001) from MERGA 2001 which details the aims and processes of each study. We note that the four studies in this pilot are on-going, so we remain tentative with respect to their conclusions.).

The study aimed to describe the language environment in the classroom and develop strategies to use available language skills and overcome language difficulties.

The overwhelming features of the observation phase were the richness of the students' language, and the paucity of the classroom language environment. Over 80% of students are bilingual, with more than half of those able to understand at least three languages. Nevertheless only English was acknowledged in the classroom. Nearly 90% of student time was linguistically passive. Over 60% of students had difficulties explaining in English the meaning of common mathematical symbols. Some classrooms display little mathematical language, and the seating organisation does not promote communication. Another observation was that problems are as much a matter of language confidence and speaking conventions as they are of English language ability.

Two main interventions were undertaken with the aim of creating opportunities for students to express themselves mathematically: situations for group and class mathematical discussions were created; and journal writing was instituted.

Students responded to language-use situations well, both participating in group discussions and being willing to give explanations to the whole class. Group work in particular was liked and valued by students. Students found journal writing to be a difficult mode, only half reporting it favourably, however, with teacher guidance, it became productive both mathematically and as way to communicate about the classroom. Further research needs to be done to evaluate the gains against the time consumed.

Factor Study 2: Mathematical Background

The study aimed to describe the students' mathematical background and investigate strategies for using the knowledge they have and improving areas of need.

A recording mechanism was developed that allowed individual student and whole class profiles to be made. It was found difficult to do this comprehensively within the class time available to the researcher. The mathematical background of most students was adequate in the basic areas of number and algebraic manipulation, but poor in concepts such as Normal Distribution which should be understood at this level. What was known was of an isolated nature, requiring extensive prompting to retrieve. There were few connecting links between what was known, and most students experienced difficulties handling unfamiliar problems.

Three strategies were attempted to enhance background knowledge: the use of worksheets; explicit teacher instruction; and homework revision and testing. The worksheet intervention was not successful, possibly due to practical and affective causes rather than cognitive ones. Students failed to complete the worksheets and feedback opportunities were restricted. The explicit teacher instruction during class time was successful as measured in post-tests. This was attributed to the very narrow domain targeted in the intervention. Whether this is practical in terms of class time for all areas needing attention, is still an open question. The revision and testing strategy targeted the problem of links between mathematical concepts, and no significant change was measured. Considerable difficulty was experienced in knowing how to approach this area, however, as it is regarded as crucial in the literature and was identified as a need with these students, it must be researched further. Overall students were positive about the interventions, the preferred mode being short tests, liked for their motivational effect and indication of progress.

Overall conclusions included the general finding that, in these classes, controlled in-class interventions are liable to be more successful than out-of-class activities needing independent student work. It was noted that the measuring instruments were probably inadequate and needed more work in future research.

Factor Study 3: Graphics Calculator Technology

Texas Instruments TI-83s were made available to the Year 13 students and this study investigated the resulting community of practice.

Unsurprisingly, most students embraced the technology, and developed a better knowledge of it than their teachers, using features not mentioned in the mathematics classroom. The calculators quickly became part of their classroom lives, over 60% having them on their desks routinely, and virtually all students having them available every lesson. They discuss their use, and challenge each other with new uses. Only one student had any previous experience, but 75% of students used them in classes other than mathematics. This has happened despite the fact that teacher use is minimal. Over the year students developed from instrumental attitudes towards the calculators to an appreciation of their transforming capabilities. A large majority attributed improved performance to their use.

Factor Study 4: Indigenous Students

The aims of this study were to describe issues to do with indigenous people through the eyes of the Maori students in the study. There were five such students who were interviewed about their views on: the relevance of mathematics to Maori; stereotyping; choice of school; language; their mathematics learning experiences and self-assessment; their expectations and aspirations; and their reflections on multi-ethnic classrooms.

These students were interested in mathematics and felt at home at these schools. They had so far enjoyed mathematics but were finding it difficult this year. They felt high expectations upon them, but most of them still believed in their ability to succeed. They particularly appreciated one-on-one learning opportunities, and teachers who made them work hard. They found the multi-ethnic classroom a learning experience, and expressed apprehension about being treated differently because they were Maori.

The Pilot Process 2002

Some of this year's activities will be reported in more depth at the July conference.

The focus of this year's pilot is to understand more about teacher development in the project schools. We plan to trial five professional development strategies, and to initiate a sixth in more depth, however the main aim is to understand more about the parameters of professional development in the environment exhibited in these schools. There are four objectives for the year in the area of teacher development.

The first is to report on the existing state of teachers' well-being and attitudes, and the factors in these schools environments which impinge upon their teaching. It will involve teacher questionnaires and interviews, both of which will use classroom observations as stimuli for some questions. The teachers will participate as researchers and subjects.

The second objective is to establish a monitoring process to measure change. It will use the base-line data from the first, and will develop of instruments to measure classroom variables, in particular the three criteria that have always been part of the project proposal: on-task time, mathematical quality of tasks, and classroom relationships.

The third objective is to evaluate teacher development strategies. It will compare five strategies with consideration for the particular situation of these teachers and the extensive literature on mathematics professional development. They are: organising teachers' meetings on mathematics education issues; organising mutual peer observations and reflection; mentoring teachers in their own classroom; supporting teachers doing action research; and providing readable digests of mathematics education research literature.

The strategies will be evaluated with: pre- and post-interviews and questionnaires; teacher reports of classroom behaviour changes attributed to these initiatives; teacher attitude changes as measured by standard instruments; observed classroom learning behaviour changes attributable to these initiatives; and analysis with respect to theories and other research reported in the literature.

The final objective is to begin to investigate in depth the use of mathematics itself as a stimulus in teacher professional development. This particular strategy is indicated in the literature, and by our experience, to be particularly effective as a means to initiate professional development amongst teachers who have little time or motivation. It is hypothesised that teachers' own learning of mathematics will be effective in engaging them in discussion of learning and teaching, and thereby in reflecting on their classes. There is evidence that mathematics teachers in particular are deeply motivated by their subject matter. Two Saturday workshops will be mounted and evaluated, one focussing on an advanced mathematical topic, the other on extensions of a Yr 13 mathematical topic.

A second theme for the 2002 pilot is student motivation. Two studies will be mounted, one aimed at researching a strategy to increase classroom motivation, and one aimed at investigating the underlying causes of the phenomenon of mathematical "turn-off".

In the context of final year national assessments, finding the time and appropriate resources for classroom interventions to motivate students to study and to continue mathematical studies is always difficult. One researcher will initiate a motivational programme with one class. This will be evaluated by: gathering student responses through questionnaires; observing student reactions in the classroom; and gathering teacher responses through interviews. As well as evaluating this programme, we expect to draw general conclusions about the potential for increasing these students' motivation.

A second study will investigate the phenomenon of students who have shown early promise in mathematics and who have suddenly abandoned all effort in the subject. It will coordinate recent research on Pacific Island and Maori students with mathematics at Yrs 12 and 13. Students who have not fulfilled their potential in mathematics will be identified by comparing results from national examinations in Yr 11 and Yr 13, and will be interviewed. We hope to comment on relationships or discrepancies between these two sources of data.

Also this year we aim to develop the theoretical base. We expect to work with Ole

Skovsmose and colleagues in Denmark to develop the model described above. In addition, further theoretical work needs to be done in collaboration with others working in the area of Pacific Island and Maori perspectives.

Conclusion

The Mathematics Enhancement Project is aimed at developing a model intervention for implementation in low decile schools at reasonable cost. The Pilot programme presently underway has already given us good information about the planned project, and some information as a result of particular studies. The present focus on teacher development does not imply a change in our view that development is multi-faceted. We seek critiques and comment from other researchers.

Acknowledgment

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Swaps and Switches: Students' Understandings of Commutativity

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This paper engages current thinking about the links between students' early experience with number properties and their future understandings of algebra. It reports on a study of students' understandings and explanations of commutativity. Central to the analysis are both the forms of representation students draw upon and the language they employ to make their understandings explicit. The investigation reveals that despite the rhetoric of educators and the intention of curriculum developers, student learning continues to remain at the procedural level, constraining rather than enabling algebraic reasoning.

"The power of mathematics", according to Deborah Ball (1993), "lies in part in its capacity to represent important relationships and patterns in ways that enable the knower to generalise, abstract, analyse, understand" (p. 163). Students who are successful in mathematics have moved beyond definitions, concepts and skills to an engagement with the underlying structure and regularities of mathematical operations, and are able to express their insights about mathematical relationships. The capacity to generalise, to construct ways of representation, and to offer justifying arguments and explanations, is a defining feature of a sophisticated mathematical experience.

Algebraic thinking relies upon insightful expression of the nuances between numbers and operations; these very same principles that govern symbolic representation and structure in algebra build on students' extensive experience with numbers and their properties. Indeed, as Schifter (1999) argues, implicit to the strategising efforts which students employ for calculating and for solving problems is an engagement with algebraic reasoning. For example, most students make implicit use of algebraic reasoning when they change the order of the numbers in their addition calculation. However, although students draw on the fundamental properties of addition, most are not aware of the underlying structure and properties of commutativity and hence are not able to make connections between the structure of algebra and the structural properties of systems of numbers (Carpenter, Franke, &, Levi, in press). Kaput (1999) and Schifter (1999) have both argued that many students fail to recognise that the preliminary ideas of algebra are merely the generalisation of arithmetic. Recognition of this link is crucial for "breaching the cognitive gap" (Warren & English, 2000, p. 624) and takes the student beyond competence with the traditional procedural missing-value problem into some of the most important algebraic understandings which have to do with equivalence and the transformation of variables.

Current thinking among mathematics educators and curriculum developers shifts the mathematical experiences of students from a procedural focus to active engagement with mathematical ideas, promoting a vision of teaching and learning opportunities which encourage reflection and argumentation (Yackel, 2001). Integral to our research are those forms of argumentation central to students' understanding of commutativity. We asked: What understandings do students have about the commutative property in relation to addition, subtraction and multiplication? Our theoretical framework is derived from Krummheuer's (1995) standpoint on argumentation. Krummheuer offers a reflexive perspective on learning, maintaining that a student's active involvement in a process of argumentation influences learning, and in turn, the student's process of argumentation is

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coursed through his or her cognitive capabilities. Through exploration of their own and other students' conjectures, students who engage in forms of mathematical argument evolve an understanding of what constitutes an appropriate explanation or justification. More importantly, they are better positioned to make explicit connections between arithmetic experiences and algebraic reasoning.

Research Study

This paper reports on data drawn from New Zealand's National Education Monitoring Project (NEMP). The latest available data, involving 3000 Year 4 (ages 8-9) and Year 8 (ages 12-13) students, assessed content knowledge and process skills in mathematics (Flockton & Crooks, 1997). The reported project focuses on students' knowledge of and articulation of ideas about commutativity .

The researchers viewed videotapes of 50 Year 4 students and 50 Year 8 students randomly selected from the NEMP bank of student responses. They investigated and analysed the differences in students' ability to recognise the commutative law and in their capacity to explain this law in everyday language. The videotapes provided a means to paint a national portrait of mathematics understandings presented verbally by students—the plain talk, the tangled persuasion and the resounding eloquence—at two distinct year levels. But the videos did more than this: they provided a visual image rich in descriptive power which allowed us to capture the processes involved in the demonstrated rationales offered by the students.

Each student was introduced to the assessment task by the interviewer's words: "Let's imagine that someone in your class needs some help with maths and that you are going to try to explain the answers. I'll ask the questions and you try to explain the answers. You will need to say more than 'yes' or 'no' because you want them to understand. Use the cubes to help show what you mean." The students were then asked the following questions:

- Is 4 plus 3 the same as 3 plus 4?
- What about 4 minus 3 and 3 minus 4? Are they are same?
- Does 2 times 5 give the same answer as 5 times 2?

Numeric representations on cards were placed on the table for each question. For example, the card 3 + 4 and the card 4 + 3 were both presented to the student for question one.

The researchers transcribed half of the tapes each then carried out a reliability check of the transcriptions of the other fifty. In this way both researchers viewed all 100 responses. Initial codes, listing all explanations and/or demonstrations which were explicit contrasts from other criteria, were refined as the researchers re-applied the system to the actual study tapes. The final choice of the best-fitting category for any response was by mutual agreement, arrived at through re-viewing the tape segment and/or further discussion.

Results and Discussion

The focus here on how students articulate their conjectures goes beyond a simplistic interest in communication engagement. If students are learning with understanding, they are engaged in justification. It was found that students *can* engage productively in justifying conjectures, yet many struggled to appeal to a model or use a language with the rigour and precision required to convey meaningful learning.

A Question of Addition

All but two students in the study volunteered that 4+3 is the same as 3+4. The arguments of these dissenting students (both from Y4) were based on differences that the students observed in the placement of the numbers on the two cards:

> No, this one here has 4+3, and this one here has 3+4.

All other students responded positively and were able to offer some explanation in support, or were able to justify by example. Responses were coded according to the following indicators:

- 1a The numbers are 'switched around'. (15 Y4 : 13 Y8)
- 1b Both sides add to seven or the same amount (11 Y4 : 11 Y8)
- 1c The same numbers *and* they both add to 7 or to the same amount. (19 Y4 : 25 Y8)
- 1d Other. (5 Y4: 1 Y8)

Explanations for 1a included the responses of students who either (i) switched around a model of *one* set of three and four cubes to represent the second card, or (ii) demonstrated the equivalence of the *two* spatial arrangements of cubes that modelled each respective card.

> Yes, you have 3 cubes and 4 cubes and it's just the same as if you have these cubes [refers to the 4 cube set] and put 4 cubes in front of the 3. (Y8)

Number switches provide a powerful visual image to the student. Schifter's (1999) research class coined the expression "turn-around" to convey the numeral switch in number. In the current study "turn-around" came in many other linguistic guises: "going that way, and the other is going the other way", "switched around", "put backwards", "in a different way", "the opposite way", "changed around", "upside down", "swapped around" and "changed sides".

It is possible within 1a coding that some students' justifications were based on an appeal to authority, involving the repetition of someone else's claim rather than on any meaningful understanding of commutativity. We make this suggestion knowing that a significant proportion (21%) of students justified the commutative property for subtraction on the basis of the same 'number switching' argument.

Likewise, it was unclear as to whether all students who provided explanations of type 1b were aware of commutative law. It is possible that in cases where students appeared to count each conjoined set separately, that the equivalence conjecture was backed by a similar warrant as might be applied to the equation 3+6 = 2+7.

> Yes, because they make the right answer. If you get 4 cubes and add 3 onto them they make 7 [counts all the cubes]. And if you get 3 cubes and add 4 onto them, that makes 7 [counts all the cubes]. (Y8)

> I've put the equals sign in both of them and they add up to the same one. (Y4)

The videos offered a wealth of information concerning differential levels of confidence and verbosity between Year 4 and Year 8 students. Implicit to the Year 4 students' characteristic long pauses and inflections in tone was an appeal for reassurance and affirmation. Their responses were often marked by unsophisticated and imprecise language:

> Um, yes, it is because that one [points to the card 4+3] is going that way and the other one is going the other way. So they both equal the same number. That equals 7 [points to 4+3] and so does that cos I counted up [counted up 3 and 4 cubes]. (Y4)

In comparison, a Year 8 student demonstrated the equality of the two addition statements in relation to the equal sets of cubes:

> Yes, because they both equal 7. 4 plus 3 is seven [joins 4 cubes and 3 cubes]. And that doesn't matter if you have them the other way [switches the cube sets] because they'll still be the same amount of cubes.

Analysis of the 1c coding responses revealed that more Year 8 students supported the notion of 'switching' by providing an additional warrant of a form similar in intent to: "you get the same answer", or "they both equal 7".

> Well, it can be because 4+3, you just get 4 cubes and 3 cubes and then you just swap around. And you just get the same answer because that there, you've got 3 plus 4 and [swaps the order of the cubes], you've got 4 plus 3. (Y8)

One student attempted to take the argument to a higher level of generalisation. However, in essence while wanting to provide an argument that is general, the student falls short, in that her argument simply restates the conjecture in slightly different words:

> Yes they are the same because they've got the same numbers. And in pluses it doesn't really matter how you have the numbers. They'll equal the same if the numbers are the same. (Y8)

While answering the given question was not conditional on supplying a generalisation, it was surprising, given the centrality of commutativity to the development of early number understanding, that generalised statements were not offered by more students. When we consider alongside this observation that some Year 4 students could not provide a model or explanation to support their conjecture, we are led to wonder if discussions about properties of numbers and operations have been commonplace in the New Zealand classroom.

A Question of Subtraction

Coding indicators for subtraction:

• 2a Yes they are the same.	(14 Y4 : 3 Y8)
• 2b 4-3=4, and 3-4=3	(6 Y4 : 0 Y8)
• 2c Double model	(2 Y4 : 10 Y8)
• 2d Correct modelling & not explained; or no model & explained	(9 Y4 : 5 Y8)
• 2e Comparison	(1 Y4 : 2 Y8)
• 2f Correct modelling and uses 'minus' or 'you can't'	(18 Y4 : 30 Y8)

Students in the 2a category backed their argument by making reference to the identical numbers on the two cards. A few students appeared to be unsure of how to read a subtraction problem, claiming that the problems would have to be the same because on the second card the numbers were *"Just swapped round"*.

> Yes, you've got 4 there and 4 there. And if you take away. And you get the same answer. (Y4)

> Yes, they both have the same, but swapped around and equal to one because, like, if you had 4 minus 3 it would be 4 blocks and take away 3 and you've only got one left. And 3 minus 4 is just the same as 4 minus 3 but it's round the other way. You've got 4 blocks and you just take away 3 blocks again. (Y8)

Some students provided a further warrant by arguing "*It's just like the other one*" [the addition problem].

> They're the same as well. Except they're minus so they both equal the same number. Exactly like the other one but they're minus. And they're the same, by two 3's and the same by two 4's. (Y4)

Analysis revealed that 13 of the 38 students, who initially supported the conjecture that 4-3 was the same as 3-4, altered their response as they proceeded with an explanation or demonstration with the cubes. Proportionately more Year 8 students revised their assessment of commutativity for subtraction when prompted to provide a justification.

A consideration of the responses of the students who disagreed with the conjecture that 3-4 was the same as 4-3 (code 2b) raised an important issue concerning forms of representation which students gave to the problems. A large number of students (both Year 4 and 8) modelled, with the cubes, the numbers on the cards and then became confused about what exactly to subtract.

> You've got 4 then you take away 3 and that just leaves 4 [i.e., □□□□ – □□□ = □□□]. And then you have 3 and you take away 4, you'd leave yourself 3 [i.e., □□□ – □□□□ = □□□]. (Y4)

Others (code 2c) who modelled the numbers on the card [4-3], self-corrected by taking away 3 cubes twice to offer '1' as the answer. The frequency of these mis-representations suggests that modelling basic subtraction problems was not firmly established in the students' experiences. The inability to directly model such elementary problems with concrete materials will invariably hinder the student's propensity for visualisation and abstraction deemed necessary for algebraic thinking. In earlier studies, children's reluctance or inability to translate between the written and concrete representations using a model depicting 'separation' led Hughes (1986) to conclude that "some of the children's responses revealed a major gap between their concrete and written understanding" (p. 132)—a conclusion that appears to be valid even today.

Not surprisingly, given no context for the subtraction problem, very few students chose to use a compare model to demonstrate their conjecture. Included here is one of the few demonstrated comparison examples, recorded in the 2e category:

> 3, you have to put one on. That equals 4 [constructs a column of 3 cubes alongside a column of 4 cubes]. So that's one left over. And if you do 3-4, 4 doesn't go and if you go, count backwards, minus 1 from zero, so it's minus one. (Y8)

One Year 4 student couched her compare explanation within a context meaningful to her:

> Yes, well pretend you've got 4 apples and you had 3 visitors come and one of them was you. You would give, um, and you had, you would give the person one apple each. And then 4 take away 3 would be 1. Because there's not enough for it to equal. See, there's a person and there's an apple and that's a person and there's an apple and you can see that's really 2 apples going into one more person. And you gave them the apples. And that's for you. And you've got one more apple left.

For those students (2f) who provided a clear demonstration of their understanding of the subtraction process, most stated that 4 minus three "can't be done"; sometimes offering an additional warrant that there were not enough cubes to take away:

> You can't do it because 4 is higher than 3. (Y4)

When confronted with a numerical situation that could not be contained within their number experience, some students volunteered that the answer to 3-4 would be zero or nothing:

> You can't have 3 and take away 4 cos there's only one there. Take away 3 there. Take away 3. Nothing. (Y4)

Other students offered some explanation related to a negative answer for 4-3.

> [For 3-4] Imagine you had 3 apples and somebody took, somebody came one night and took 4 out, out of 3. And then you would have, you'd have those 3 taken, and you'd have no more left. And that would be minus one. (Y4)

> …but if you had 3 cubes and tried to take away 4 you would take away all your 3 and you won't have another one to give away so you would be minus 1. (Y8)

A Question of Multiplication

Coding indicators for Multiplication:

- 3a Models numbers on cards and says '7'. (9 Y4 : 2 Y8)
- 3b Models numbers on cards and says '10'. (5 Y4 : 2 Y8)
- 3c Known fact with no demonstration. (2 Y4 : 0 Y8)
- 3d Models correctly one set only, with explanation. (8 Y4 : 7 Y8)
- 3e Forms 2 sets of 5 cubes then rearranges to form 5 sets of 2. (11 Y4 : 28 Y8)
- 3f Forms 2 unique sets; one representing 5x2, the other 2x5. (8 Y4 : 6 Y8)
- 3g Other, including reverse modelling (7 Y4 : 5 Y8)

Demonstrations of the 3a kind based on a direct model of the numbers on the given cards do not constitute a legitimate representation of the commutativity of multiplication.

> [Pointing to the card 2 x 5] You've got 2 on this one and you add 5 and you get 7. [Referring to the card 5x2] You've got 5 and you just add on 2 and it equals the same. (Y4)

Likewise, students' warrants categorised under 3b ignore the array structure of multiplication. Instead, students use a known fact to justify the 'sameness' of 2 x 5 and 5x2.

> I: You've made a set of 5 and a set of 2. [Y4 modelling the cubes □□□□□ □□]
>
> S: [reaffirming by counting] 1,2,3,4,5. And a set of 2.
>
> I: How much does it come to?
>
> S: 10. The same as that one [pointing to the card 2x5]. 10.

Analysis revealed that 28% of Year 4 students recited a known fact and modelled unconventionally. 8% of Year 8 students began by modelling the numbers on the cards then set aside this initial representation and started afresh:

> They're the same. It's like the other one [referring to addition problem; models the numbers 2, 5 □□ □□□□□]. There's 2 5s □□□□□ □□□□□ [changing arrangement of cubes]. There's 5 plus 5 there. Two 5s equal 10.Then 5 pairs of 2 [□□ □□ □□ □□ □□] So 2, 4, 6, 8,10. Same thing. (Y8)

Student responses denoted as 3c begin with an appeal to authority and when pushed for further clarification are unable (or unwilling) to provide a model, offering instead an approach incorporating repeated addition:

> Yes it will [equal the same]. The order is just changed around. You get charts. If you don't know the times table. You get charts to name them. It's just 5 times 2 is 10. All you're doing is just doubling, tripling, 5 times 2 is 10. It's just the same as plus. You're going up 2 or three numbers [pause]. 5

> ones are 5 and you just add on another 5 and that makes 10. You go up 5 each time. 5 twos are 10. Just the same, It's like 2 times tables. You use the 2 and you go up 5 times. (Y8)

Additive reasoning develops intuitively from everyday additive encounters (Lamon, 1995; Simon & Blume, 1994). Multiplicative reasoning, on the other hand, is a more elusive development which requires progressive pedagogical exposure to multiplicative structures. Many researchers (e.g., Sowder et al., 1998; Thompson & Thompson, 1994) have noted the conceptual difficulties associated with multiplicative reasoning but insist that such experience is important not only for students' conceptual understanding of rational numbers but crucial to the development of the ideas fundamental to algebra and calculus. Whilst additive thinking, in contrast, is "a familiar and comfortable means of solving multiplicative problems" (Jacob & Willis, 2001, p. 306), it is also limiting.

In the following examples a progressive development from additive reasoning and tending towards multiplicative thinking is made explicit:

> Yes [they are the same] because 2 5s [□□□□□ □□□□□] is the same as 5 times 2 because you put 2 [□□], 2[□□ ,] 2[□□], 2[□□], 2[□□]. It's 10. Then either one is the same. 10. (Y4)

> You get 5 times 2 is 10. And 2 times 5 is [counting out 10, □□□□□□□□□□] So that's 2 times 5. This is a hard one, 2 times 5 is 10. So 5 times 2 is 10 as well. …if you had 5 times 1, it would just be 5. So 5 times 2 is adding another one on for every one. That's 10. That's the same. (Y4)

> Well, you've got…you start off with the 5 times 2. You'd need to do 2 sets of 5. So you'd have [□□□□□ □□□□□]. 5 sets of 2 with other blocks [□□ □□ □□ □□ □□]. Once I'd done that I'd say, OK, count how many blocks there are. And they'd say it's the same because once again, it's the same numbers but just a different method of showing it. And what it is…you've got 2 sets of 5 that add up to 10 and 5 sets of 2. And you're doing the same thing except that you're doing 2 of these [pointing to the card 2x5] and 5 of these [pointing to the card 5x2]. (Y8)

Responses classified in the 2f category either exhibited confused thinking or reversed modelling. Imaging by reversal appeared to make a good deal of sense to some students:

> Yes, cos you've got going 2x5 [□□ □□ □□ □□ □□]. It's got, you've got 2, and you've got times 5. It's like. Um, it's the same cos you've got to go 2 times 5. You've got all these [pointing to cubes] and you just, so you've got 2, you've got 5 lots of 2, so you can either add them together or if you know your tables you can go 2x5, and it's the same with the other one. It's just, you're going, you've got 5 lots of 2 [□□□□□ □□□□□].

It is our contention that few of the research participants exhibited sophisticated multiplicative reasoning and commutative understanding. A student with multiplicative thinking and commutative understanding operates at a higher level of abstraction and would probably reveal that thinking by *implicitly* reorganising the array structure. Hiebert and Behr (1988) argue that, whereas addition operates on singleton units, multiplication deals with composites. An array enables the student to see, simultaneously, the singleton units and the aggregation of those units. None of the sampled students appeared to construct and coordinate composite units through an array—visualising the array in two unique orientations—that would be suggestive of reasoning multiplicatively and thinking in commutative terms.

Conclusion

Justification is central to mathematics. In order to make sense of and genuinely understand mathematical concepts and move beyond solutions at the surface procedural level students need to be able to explain and justify and convince themselves and others.

Reasoning through argumentation works on many levels of sophistication from appeal to authority, justifying by example, analogising and generalising. In the process of learning those forms of argumentation which are persuasive and useful to the problem in hand students acquire a firm foundation for the learning of algebra.

Descriptions from this study indicate that, despite the intent of curriculum developers, many students have learned arithmetic in a way which is not conducive to the enrichment of structural understanding. We maintain that the focus on procedural learning which persists has little flow-on effects for the development of algebraic reasoning. To foster this development we suggest that students need more productive modes of instruction than are currently in operation. Forms of conjectorising and generalising about the properties of numbers need to be encouraged. Students need opportunities to make explicit their understanding of *why* number properties such as commutativity 'hold good'. And they need to be able to apply these procedures in a variety of contexts. In subscribing to these approaches teachers can ensure that students will learn to make the connections between the students' own understandings of number properties and the study of algebra.

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Some Problematics in International Collaboration in Mathematics Education

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This paper discusses data from interviews with mathematics educators from Australasia and Colombia on their views of, experiences in and expectations from the internationalisation and globalisation of their discipline. International collaboration is essential for moving the discipline forward in a globalised world and avoiding the colonialism of the past - and allowing the discipline to play its role in bridging the increasing gap between developing and developed countries. The aim of this analysis is to examine some problematical aspects of this collaboration between countries of unequal "power" and resources.

In a previous publication (Atweh & Clarkson, 2001b), we argued that mathematics education is perhaps the most internationalised subject in higher education. This is evidenced by both the similarities on curricula around the world and the number of international organizations, conferences, journals and handbooks in the discipline. However, there has been very little research which examines their causes and effects on the different players in the global scene. In particular we drew attention to some voices from developing countries calling for increased collaboration with developed countries, even to the extent of "a global minimum curriculum below which no continent should be allowed to drift, however under-developed" (Kuku, 1995, p.407). On the other hand, few voices from developed countries have expressed great concern about the effect of international exchanges of the past and their effect on developing countries. In the words of a leading mathematics educator in Australia with a notable reputation and international experience:

> Over the past 20 years I have often had cause to reflect that it is Western educators who were responsible not only for getting their own mathematics teacher education equation wrong, but also for passing on their errors to education systems around the world. (Clements, 1995, p.3)

Both these views were expressed in an ICME regional conference on regional collaboration in mathematics education. Arguably, these specific views may be contentious and perhaps not widely shared within the mathematics education community in their respective regions. However, as we argue in the previous publication, these differences of voice from developed and developing countries do point to the need for further research and dialogue around the world between educators from developing and developed countries about the aims, processes and outcomes of international exchanges in the field.

In another conference paper (Atweh & Clarkson, 2001a) we discussed the issue of a "global curriculum vs. global collaboration" in mathematics education. We have argued that while a global curriculum may be an anathema to many mathematics educators, global collaboration is essential for moving the discipline forward in this globalised world - at the same time avoiding the colonialism of the past - and allowing the discipline to play its role in bridging the ever-increasing gap between developing and developed countries.

This paper discusses data from two focus group interviews with leading mathematics educators - one conducted with educators from Australia and New Zealand and the other conducted in Colombia in Latin America. This is part of a project funded by the Australian

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Research Council on Internationalisation and Globalisation of Mathematics Education. The aim here is not to compare and generalize the understandings, concerns or experiences of educators from the two regions. Rather, the discussion of the two interviews is intended here to highlight some problems and issues regarding international collaborations between privileged and less privileged countries.

Methodology

This part of a two-year study consisted of the conduct of focus groups (Morgan, 1997; Vaughn, Schumm, & Sinagub, 1996). Later stages of the project will involve surveys and case studies. The focus groups were conducted in Australia, Brazil, Mexico and Colombia. In each case the local organizers of the focus groups were requested to invite leading mathematics educators in their countries with substantive international contacts and experiences to participate in the groups. The focus group discussions lasted between one and two hours each and comprised between five and ten educators. Prior to the focus group, the participants received a short summary consisting of some definitions of terms used and some issues that they may want to address. A major characteristic of focus groups is that they allow participants to raise issues that are important to them rather than address the questions posited by the researchers. From time to time, the researchers asked some clarifying questions and directed the discussion to move on to other topics. The languages of the participants were used in the discussion paper and in the focus groups. The discussions were transcribed, translated into English where applicable, and the translation confirmed by a second native speaker of the language. The transcripts will be sent back to the participants to check its accuracy and to suggest any alterations and additions that they would like to make. The analysis used here will be restricted to the interviews conducted in Australia and Colombia. The Australian interview consisted of Australian and New Zealand participants. Hence we will refer to the interview as the Australian interview and to the participants as Australasian educators

Findings

Colombian Context

In the background report of a World Bank project targeting higher education in Colombia (World Bank, 2002), the report states that the country is experiencing its worst economic performance in over 60 years, and this has been compounded by ongoing internal conflicts. Growth in GDP decreased from an average of 4.5% during the first half of the 1990's to -4.5% in 1999. In 1998 the GDP was AUS$205b (UNSD, 2002). In spite of showing signs of economic recovery, the increase in unemployment and poverty arising from the economic crisis and high structural unemployment continue to pose serious threats to the welfare of the poor and vulnerable. With a current population of just under 43 million people, unemployment reached an all time high in 2001 of just over 20%, almost double the traditional rate of 11 percent. The National Planning Department estimates that enrolment of 7-11 year old children from the poorest families declined from 87.3% in 1996 to 83.2% in 1998 and the enrolment rates of 12-17 year olds fell from 66.3% to 64%. Higher education, which had been expanding throughout the 1990's, also has seen a reversal in its growth. Beginning in 1998, and continuing until the present, the number of new entrants to tertiary education is declining. The coverage rate[1] currently stands at just 15% which compares unfavourably to other countries in the region and to the OECD country average of 54 percent. Public institutions have increasingly shifted their revenue base towards cost-recovery where 49% of revenues came from students as of 2000. As a consequence, the number of entrants into tertiary education declined by 19%. Private providers enrol more than 2 out of 3 students. This makes the higher education sector in Colombia far less accessible, and hence, far more inequitable than ever before. Only 192 students were enrolled in Doctoral level studies in the country. The quality of the teaching in higher education was also highlighted as a matter of concern. While there has been a gradual phasing out of academic staff with less than 4 years of tertiary education, the share of faculty members with doctoral degrees still remains at 2 percent. Finally, the report concludes that the quality of research is at risk, and generally, the quality of programs are insufficient to support the internationalisation of Colombian higher education.

Selected Views of Colombian Mathematics Educators

While many Colombian universities have developed policies expecting that lecturers be involved in research as well as teaching, the reality is that resources at universities are extremely limited for the conduct of research. With the country in economic crisis, education is not high priority for governmental spending. Similarly, with declining funding, universities do not dedicate resources for research. Extended teaching hours for staff allow little time for research. Research and publications are not seen by the academics as rewarding when it comes to employment and promotion. Research activities are often the initiative of individual researchers who, in most cases, use their own private resources to conduct research. Of the nine lecturers involved, there was a single doctorate holder. In the 1990s a few academics from the country were successful in obtaining scholarships for undertaking doctoral programs at overseas universities (Colombian focus group, p.19). In the mid 1990s, as a result of collaboration between five private and public universities in the capital Bogotá and some regional cities, a national doctoral program in science and mathematics education was commenced in the country. By the end of the decade, such a program was seen to be too expensive for the universities involved and was put under review putting its future under threat (Cardenas, 2001, private communication). Private universities in the country tend to be more entrepreneurial and they raise some of their own funds for research. Obviously in countries such as Colombia this is not an easy task to say the least (Colombian focus group, p.20). When asked what their expectations were from international contacts, the participants in the focus group were very direct and candid in their reply. They aspired for more internationally financed research or at least co-financed research projects. One academic gave an example of a research project between Argentina and Mexico and Colombia. The project was planned and financed from outside and the data was gathered from the three countries. Not only did it provide funds for the conduct of the research but also it provided some professional development in research for the involved Colombian educators.

The major themes that arose from this interview were a sense of isolation, a feeling of difference from other countries, and a lack of reciprocity in their international contacts. The country has very limited up to date resources in mathematics education. Internet access is rather recent, very limited and not always reliable. However, its introduction was seen as highly rewarding for staff for giving a chance to realize that "we are not alone" (Colombian focus group, p.3) in facing educational problems in the mathematics classroom.

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[1] Total enrolment in a specific level of education, regardless of age, expressed as a percentage of the official school-age population corresponding to the same level of education in give school year.

The country was not seen by the participants as ready for full participation and competition on the global scene. This was attributed partly to government policies that do not reflect a great value in higher education or a commitment for the use of education as means of achieving social justice (Colombian focus group, p.6). The lack of recognition of the country's degrees overseas implied to them that they were not preparing their students for a globalised world (Colombian focus group, p.7).

Colombia mathematics educators operate in a globalised world with a sense of lack of reciprocity and a limited ability to "exchange" with overseas countries on equal terms. Hence, few educators hesitated to use the terms internationalisation of mathematics education in their context (Colombian focus group, p.11). One academic made the distinction between "copying" and "appropriating" ideas from outside the country. The former means of international exchange was seen as more dominant in their situation and as a form of colonialism. To be fair however, these views were not shared by everybody in the group. Another academic referred to the past reforms in mathematics education that occurred in the country as the labour of local educators working autonomously.

At times, ideas from outside constituted "fashions" that were uncritically accepted locally. These mathematics educators did not feel that they have developed "counter arguments or proposals " (Colombian focus group, p.16) for balancing these fashions. At times, the changes were introduced so rapidly with insufficient time and resources to evaluate their appropriateness for the local context. Further, the exposure to outside ideas was seen to be highly selective. With limited direct access to international publications and conferences, exposure to overseas ideas tended to be limited by the overseas destinations of doctoral candidates and/or by the language of publication - in this case Spanish.

For some of the participating academics, internationalisation is associated with the exchange of "scientific knowledge and technological advances that knows no geographic boundaries" (Colombian focus group, p.5). Perhaps better saying the only boundaries knowledge knows are the economic capacity of the different countries for its generation and the enjoyment of its benefits. Hence, the similarity of mathematics programs around the world was seen by some as emerging from the very nature of mathematics itself (one educator calls it universal mathematics, Colombian focus group, p.12). One educator put it this way: "definitely it is impossible to have a different program in arithmetic for the first years of school. The differences [between curricula around the world] may be the methodological approaches or [sequencing of topics]. I think this [is true] because of the character of mathematics" (Colombian focus group, p.9).

However, at least one educator is not convinced that internationalisation of mathematics curricula is a result of the nature of mathematics itself. He gives the example of the introduction of calculus in the education system in the United States based on the need to create and maintain a technological society. However, calculus was also introduced in the Colombia that does not have the same needs because it imports technology rather than produce it. Hence, he argued that there must have been other factors that determined its adoption in Colombia (Colombian focus group, p.13). Others attributed the global patterns in the curriculum to the patterns of colonialisation of Latin America and Colombia in particular in the past hundred years (Colombian focus group, p.10).

The Australian Context

In 2001, the Australian population of 19.5 million was nearly half of that of Colombia. The current GDP index is increasing at a rate of 4.5% per annum from AUS$745b in 1998 (roughly 4 times the Colombian figure) (United Nations Statistical Division, 2002). In the 1990s, funding in higher education has significantly changed in Australian universities with the introduction of user-pay schemes and a reliance on self funded overseas students. The table below illustrates a decrease in the percentage of public spending on higher education and an increase in fees and charges and in Higher Education Contribution Scheme (HECS) (based on data from DEET, 1999). The increase in fees and charges is due mainly to overseas students and full fee paying students studying at higher degrees. The increase in percentage in HECS is due to an increase in student population and higher rates paid by the students. In a relatively affluent country such as Australia, these increases in rates of contribution are not translated into a decrease in number of students as is the case in Colombia. For example, data from the Department of Education, Science and Training (DEST, 2002) show that between 1991 and 2000 the total number of domestic higher education students rose by 20%. In 2000 there were 695,485 students enrolled in universities (3.6 of total population representing 76% of the specific age group). Of those 20% are enrolled in higher degrees including 5% as postgraduate research students. Of these, 13.7% are overseas students. In 2001 there were some 95,607 overseas students enrolled in Australian universities who come from 100 different countries. Approximately two thirds of these students come from the Asian region. Between 1991 and 2000 the total number of international students in higher education students rose by 220% (DEST, 2002).

	1995	1999
Commonwealth grants	57.2	48.8
HECS	12.0	19.0
Fees and Charges	11.7	17.7

Selected Views of Australasian Mathematics Educators

In the Australian focus group the participating academics were well aware of the role of their country as a middle player in the international scene. On the one hand, Australian educational and research thinking is significantly affected by ideas received from overseas – mainly from the United States and United Kingdom. On the other hand, Australian mathematics educators were highly active in professional development and in teaching overseas students from a variety of contexts in particular in the Asian region.

These Australasian academics had significant involvement in international scene in mathematics education. The vast majority of them regularly, if not annually, attended international conferences and had easy access to the Internet and many (mainly English speaking) journals in mathematics education. By no means were they passive participants in these international contacts. One academic was on the International Committee of a highly influential international research group in mathematics education. Another was the editor of an international journal of research in mathematics education as well as at least two international books in the field. Three of the others were writers of chapters in international handbooks. Some were organizers of discussion groups and presenters at ICME meetings. At least four spoke of their experiences in consultancies overseas that involved staff and curriculum development in a variety of Asian countries. Almost all would had been involved in teaching international students in Australia and abroad.

The Australasian educators pointed to the role of economic factors in international exchanges both for developing and developed countries. Commenting on the needs of Laos, one Australian mathematics educator asserted that developing nations need assistance from more developed countries because "they want to be part of the world economy ... part of the international scene" (Australian focus group, p. 14-15). After a considerable time of isolation due to internal politics, Laos is slowly opening to international trends and markets. Another Australian educator pointed out that

> the critical part is that there is an ideology out there, that if you take a Western view, whatever that may be, and I think that even within the Western view, it's a narrow American view, that if they take that view of the world then that's what is going to give them access to power ... That is the way they're going to get out of their, in a sense, oppressed state, by [adopting] the American curriculum. (Australian focus group, p.20-21)

These economic factors act differently in developing countries and in developed countries. The Australian focus group discussed the increased reliance of many Australian universities on funds obtained from overseas students. Mathematics educators in both Australia and New Zealand have shown considerable concern about problems that such emphasis on the marketisation of higher education can cause. One educator who has worked with Canadian teacher training students coming to Australia to obtain their qualifications, points out the conflict between the requirements for local registration of teachers in Australia and the needs expressed by the students themselves who aspire to teach in their own home country (Australian focus group, p. 23).

Economic factors behind international contacts are not the only ones identified by the Australasian educators. In contrast to the educational marketisation policies of the government and of universities, mathematics educators often expressed more sincere humane and ethical reasons for being involved in international projects of development and research. An Australian educator who is the editor of an international journal in mathematics education as well as an international handbook on mathematics education discussed her efforts to solicit articles and chapters from a wide range of developing countries. Documents from non-English speaking countries often needed more assistance and editing to reach the "standards" set by the publications. She adds

> one of my greatest rewards I get from the Journal is working with people who present papers of potential, and working with them over a period of time to get their papers published, and they are eternally grateful for this and they come from many many countries. To me, that is [very] rewarding. (Australian focus group, p.11).

Another educator with extensive work in Papua New Guinea and Laos commented that many mathematics educators are less colonial in dealing with colleagues from developing countries than in the past (Australian focus group, p.17). She discussed the need to be not only more sensitive to the local culture but to really know it and understand it. This she had achieved by living in the country with the people she was working with, learning their language and history and designing programs that were based on what they already had and designed to suit their expressed needs (Australian focus group pp.14-16).

Discussion and Conclusions

While one can argue that international contacts and exchanges in mathematics and mathematics education have existed since the early developments of both disciplines, undoubtedly they have increased in the new age of globalisation and will continue to exponentially increase in the future with further developments in technology, ease of travel and population movements. We do not construct such contacts as necessarily either good or bad. Our research is based on the need to scrutinise the effects of such interactions on the different players. This can only be achieved through deliberate and targeted research, reflection and debate. Further, we argue that such actions need to be done in collaboration between mathematics educators from around the world.

The term collaboration however needs further examination. Hargreaves (1994) argues that "one of the emergent and most promising metaparadigms of the post-modern age is that of *collaboration* as an articulating and integrating principle of action, planning, culture, development, organisation and research" (p.245). In the context of school change which Hargreaves was discussing, he points out the benefits of collaboration, as well some of the dangers the term carries and certain conditions under which it can be most effective and ethical. Here, we make three main points about international collaboration in mathematics education.

First, collaboration between mathematics educators from around the world is particularly problematic when it occurs between players with different agendas and differing access to resources. As we have seen from the two cases examined above, the conditions in some countries around the world make their contribution to international exchanges highly problematic. Their participation in the usual arenas where international contacts and exchanges occur is very limited. A history of colonialism and economic oppression often leads to the silencing of their voices in international debates. Further, one should be aware of the differing economic interests of the different countries in the race for globalisation and international markets. While developing countries may aspire to maintain and improve their standing in the race, developing countries are struggling even to reach the starting line! Arguably, the very metaphor a "race" and the rhetoric of "competitiveness" require careful critique and scrutiny. Not only are they contrary to the metaparadigm of collaboration but also their adoption would lead to further widening of the gap between the rich and the poor making genuine collaboration even more problematic.

Second, one characteristic of the post-modern and globalised times we face is the lack of certainty and an awareness of the complexity of issues. While it may be neither desirable nor possible to establish guidelines for ethical international contacts that apply to all situations, one needs to be aware of the effects and limitations of such contacts both in the near and long term futures. At the same time that we are becoming aware of the politics of difference and local interests, internationalisation is also making us aware of our similarities and global interests. We argue that these should not be constructed as either/or constructs in the traditional positivist logic. What is needed is that all international contacts be self reflective and critical of their processes and effects. Not only should they reflect on the benefits and gains in knowledge by the different parties involved but also on how different parties can be actively involved in developing their own voice and taking increasing control in managing their own mathematics education to achieve their interests. International collaborations between mathematics educators should be transparent, reflective and accountable in examining their own rationale, aims, processes and outcomes. Questions of voice and power should always be upfront.

Third, genuine collaboration must aim to balance the tension between voice and vision. Hargreaves (1994) argues, "[v]oices need to be not only heard, but engaged, reconciled and argued with. It is important to attend not only to the aesthetic of articulating…voices, but also to the ethics of what it is those voices articulate!" (p.251). International collaboration should aim at developing a shared vision between the different players and realise that the contribution of the different players with differing access to power is problematic. Similarly, these international exchanges should aim to balance the tension between changes in structures and changes in cultures that allow for genuine collaboration. Not only do questions of costs and processes of international exchanges need scrutiny but also the assumptions behind them. Exchanges that are simply based on "helping" developed countries (to become like us?) are often based on paternal colonial assumptions and do not contribute to genuine collaboration. Collaborations should be based on mutual respect and trust in the ability of the different partners to contribute different types of learning to the collaborative enterprise.

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Effectiveness and Efficiency of Calculator Use for Computation in Years 8-10

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A sample of 141 Year 8-10 students in a suburban secondary school was given a computation test to complete by using a four-function calculator. Students were required to record all their key presses for each item in order that they could be assessed on their efficiency of calculator use. There was a small increase in effectiveness of calculator use over the three year levels, but efficiency actually fell away in Year 10. Overall, only 62 percent of students used their calculators highly efficiently.

For more than 20 years, policy and curriculum documents have promoted the use of calculators at all year levels of school mathematics (NCTM, 1980; Cockcroft, 1982; Australian Education Council, 1991; AAMT, 1996; Education Department of Western Australia, 1998; NCTM, 2000). Meta-analyses of earlier research in the 1970s and 1980s showed a tendency to focus on comparisons between paper and pencil skills with or without calculator use (Suydam, 1982; Hembree & Dessart, 1986, 1992). Gradually there was a shift towards investigating the role of calculators *per se* in school mathematics programs, particularly in the primary school sector, with an increasing focus on the calculator as a tool for learning mathematics (Groves & Cheeseman, 1992; Groves & Stacey 1998).

Scientific calculators have long been used in mathematics programs in the final years of secondary schools, and now these are being replaced by graphing calculators. However, despite this extensive use over about two decades, the use of calculators in the first high school years has been slower to eventuate. It would be true to say that the use of the basic four-function calculator in primary schools is still very sporadic. Sparrow and Swan (1997), in an extensive survey of primary teachers in Western Australia, found that while teachers generally favoured the use of calculators, this support was heavily qualified. Thus their use was very limited indeed. The four-function calculator is regarded as an essential tool for everyday living, yet how well are people able to use them? If they are not used much in primary school, then will this handicap students at the high school level in using scientific and graphing calculators? Little is known about how well students use calculators. It is probably true to say that for many students, they have largely had to fend for themselves in learning to use a calculator.

The purpose of this study was to determine the effectiveness and efficiency of use of a basic four-function calculator by students in the first three years of high school (Years 8-10) in a Perth suburban secondary school. Effective calculator use was defined in terms of the percentage of correct calculations. Efficient calculator use, as suggested by Fielker, "can be measured by the number of keys one must press"(1992, p. 33). Efficiency of calculator use has thus been defined for this study in terms of the number of key presses required to achieve the correct solution to a calculation.

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Methodology

Subjects

The sample consisted of six classes, ranging from 25-35 students per class, with two classes from each of Years 8, 9 and 10. The total sample was 141 students, comprising 64 females and 77 males. The sample was drawn from a suburban senior high school and the classes were from the middle streams in each year level in that both the top and bottom stream classes in the three cohorts were not used in the study. The reason for this was that it was considered that more useful information could be obtained from students in the middle groups as there was a broad cross-section of abilities among these cohorts. If students from the very top stream in each year level had been chosen, the majority of questions would have been answered correctly and little information would result. Similarly, if students from the bottom stream had been chosen it was expected that a significant number of questions would have been answered incorrectly.

Instruments

A Calculator Computation Test (CCT) was developed, piloted and then modified for the study. It consisted of 16 items, half of which were non-contextual, and the remainder were items presented in context. All items involved computations which could be carried out with a four-function calculator. An answer sheet was developed for the test, including a grid for each item to enable students to record their individual calculator key presses for that item. The same test was used for all three year levels in order to more readily determine the changes in calculator effectiveness and efficiency across Years 8-10. A semi-structured interview schedule was developed for use in a follow-up to the CCT with a sample of 12 students (four from each year level). Modifications to this schedule were made as appropriate for individuals, based on their CCT results and on their responses during interview.

Procedure

All students were given the same type of four-function calculator to use. The CCT was administered to each of the six classes over the course of one week. During the CCT students were required to use the calculators for every computation. This was obviously a contrived situation since, in normal circumstances students would choose between mental, calculator and paper-and-pencil computation strategies; or some combination of these (Swan & Bana, 2000). Thus, students were forced to use the calculator for every item in order to display a measure of their calculator effectiveness and efficiency in each case. Students were given several practice examples to familiarise them with the calculator provided, and to clarify the requirements for recording all key presses for each computation item. The answer sheet used by students had the structure shown in Figure 1 for all 16 items. They were required to record the key presses in the grid, then check that the recorded sequence did provide the answer stated. If not, the process had to be repeated until a match was obtained. This requirement was implemented to maximise the likelihood of a *bona fide* record of key presses for each item.

Question No. _____ Answer: _________ Checked: _____

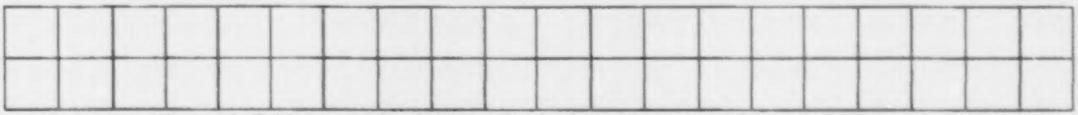

Figure 1. Grid to record solution and calculator key presses for each CCT item.

Students were selected for interview according to their performance in the Calculator Computation Test (CCT). A male and female from each year level who scored above the mean for that year level and a male and female who scored below the mean were interviewed. Thus four students from each year level – twelve students in all – inclusive of both genders and a range of ability in calculator use were individually interviewed, and the interviews were audio-taped. Interviews were conducted within two days of administering the CCT in order to maximise students' retention of how they went about the CCT. Students brought their own calculator to the interview in order to compare how they would normally have used their calculator for particular items, as opposed to how they used the one supplied. Students were asked to re-do questions directly from the CCT as they did in the class test, to do questions with their own calculator, and to explain if they would normally have done the question differently in class. The interviewees were questioned in detail re the processes used, and their understanding of the calculator and of the computations carried out.

Results and Discussion

The results of the Calculator Computation Test (CCT) were analysed for effectiveness of calculator use (correct responses) and to determine how efficiently the students used the calculator. Only the correct responses were considered when determining efficiency of calculator use. The level was rated as 'highly efficient' (0-2 key presses more than the minimum required), 'reasonably efficient' (3-10 key presses more than the minimum required), or 'inefficient' (more than 10 key presses above the minimum required to answer the question effectively).

The percentages of students correct on the CCT – thus indicating the effectiveness of calculator use – are shown in Table 1. The performances are only at a moderate level, and increases across the three year levels are small, considering that all students were given the same items. It was not possible to determine to what extent this was due to inadequate calculator skills. However, it was clear that performance was largely due to difficulties with the arithmetic. For example, many students had problems with items involving the rule of order of operations. This was confirmed in the interviews where some students were also unaware that their scientific calculator took account of this rule of order. Items involving fractions also proved difficult, with one student freely admitting at interview that he had "never been very good at fractions", and others making similar statements. It was also evident from the interviews that most students did not make use of estimates to check the reasonableness of results, but put blind faith in what the calculator displayed. They were obviously not in the habit of estimating before calculating, which is a key mathematical process, especially when using calculators or computers for computation.

Table 1

Summary of Percentages Correct on the CCT for Students in Years 8, 9 and 10

Year Level	N	Mean	SD	Minimum	Maximum
8	46	45	14.9	19	88
9	51	53	14.1	19	88
10	44	56	12.4	38	81
Total	141	51	14.5	19	88

The percentages of the three defined efficiency ratings for all students in the sample are shown in Table 2. Only correct responses were used when determining efficiency levels. The percentage of students using highly efficient methods climbed from 62 per cent in Year 8 to 69 percent in Year 9, but then fell away to 55 per cent in Year 10. These students typically use a scientific calculator rather than the four-function calculator supplied. It may be that as student use of scientific calculators increases over the three year levels, they become less adept at using a four-function calculator such as the one supplied for the CCT. Certainly there was some indication of this possibility in the interviews. Another factor may be that that the Year 10 students have limited practice with basic arithmetic calculations in their school mathematics program. The overall results show that little more than three-fifths of the Year 8-12 students used a four-function calculator in a highly efficient manner. It is quite likely that students are expected to have skills in calculator use when they enter secondary school and thus probably receive little further tuition in calculator use. The most common inefficient actions were as follows:

Table 2

Percentages of Efficiency of Calculator Use for Students in Years 8, 9 and 10

Year Level	N	Highly Efficient	Reasonably Efficient	Inefficient
8	46	62	25	13
9	51	69	22	9
10	44	55	33	12
Total	141	62	27	11

- The "=" symbol was keyed in unnecessarily;
- Zero was keyed in when not required; and
- Expressions were re-keyed into the calculator.

Of the 12 students interviewed, most had limited knowledge of how to use the memory on the four-function calculator or on their own scientific calculator. This could also be seen in the patterns of key presses for the CCT items. They remembered a result and keyed it in again later. Also, many were unaware that their scientific calculator took account of the rule of order of operations. Another aspect of concern was that some of the students interviewed did not know what the square root key did. Few knew how to enter a fraction into a calculator. Of those who had a fraction key on their own calculator, one half of them did not know it was there and few knew how to use it. The following extract from one interview with a Year 10 student illustrates this problem, as well as exemplifying student difficulties with fractions.

> I. How would you do $25^3/_7 + 18$ on your calculator?
>
> S. Well . . .
>
> I. Do you have a fraction button on your calculator?
>
> S. No, oh, I don't know. I don't know how to use that fraction button, but . . .
>
> I. OK. So how would you do it.?
>
> S. Well, um . . .
>
> I. Do you just leave out the fraction questions?
>
> S. No, I don't just leave out the fraction questions. I just, I always have to ask how to do them first, because I don't.

It seems likely that even if students were using their own more familiar scientific calculator, they would probably have made the same errors. Of the students interviewed, most stated that they would generally not write anything down when completing the given items, but would do the computation in their head, with their calculator, or not at all. The key-press patterns and the follow-up interviews indicated that many students had little or no knowledge of some of the fundamental aspects of the calculator, such as the continuity of operations in the register, or the constant function attached to the "=" key. Also, as already stated, far too many students had limited knowledge of how the calculator's memory function could be used.

It is beyond the scope of this paper to examine the results for each of the 16 items in the CCT. However, the results for two items – one non-contextual and one presented in context are given in Table 3 and Table 4 respectively. They exemplify the types of items used in the CCT. Table 3 show that students had very limited success with this item, and this was chiefly due to the rule of order of operations being ignored. Loss of efficiency of calculator use for students with correct responses was generally caused by a lack of appropriate use of the calculator's memory functions. Results for the contextual item in Table 4 indicate that the majority of successful students were highly efficient in their calculator use, while others tended to perform unnecessary extra calculations. It was considered that both contextual and non-contextual items should be used in the study. However, no valid comparisons could be made since the two sets did not have matched items.

Table 3

CCT Item 3 Percentages of Students Correct and their Calculator Efficiency

Item 3: 63 × 3 – 19 × 4 = ________

	Year 8	Year 9	Year 10	Females	Males	Total
Number of students	46	51	44	64	77	141
Percentage of students correct	20	14	18	14	19	17
Highly efficient (%)	56	57	0	22	47	38
Reasonably efficient (%)	44	43	88	78	47	58
Inefficient (%)	0	0	12	0	7	4

Table 4

CCT Item 16 Percentages of Students Correct and their Calculator Efficiency

Item 16: Your mother is going to keep 21¢ of every dollar of pocket money you receive to pay for your pet ferret's food. If you earn $43 pocket money, how much will your mother keep to pay for ferret food?

	Year 8	Year 9	Year 10	Females	Males	Total
Number of students	46	51	44	64	77	141
Percentage of students correct	15	43	59	38	40	39
Highly efficient (%)	71	91	73	75	84	80
Reasonably efficient (%)	29	9	23	21	16	18
Inefficient (%)	0	0	4	4	0	2

Conclusions and Implications

The results show that there was limited growth in calculator effectiveness over the Year 8-10 levels. In particular, students continued to have significant problems with fractions and with the rules of order of operations. The arithmetic content tested in the CCT would normally have been well covered by Year 8, but students still had significant deficiencies through to Year 10. Thus, it is essential that a regular review of the relevant knowledge and skills be maintained over years 8-10. This is vital for effective everyday usage, as well as to enhance the students' overall mathematical ability, especially in the algebra strand. Particular attention needs to be paid to the process of estimation, since this seemed to be lacking in students' calculator use, yet is so essential for effective and meaningful computation.

With only 62 percent of the Year 8-10 students judged to be highly efficient in their calculator use, it is clear that far many students lacked the necessary skills in this area. It is probably the case that mathematical ability is one contributing factor. However, the major cause is clearly a lack of knowledge of the tool itself. There seem to be two main related reasons for this. Firstly, there is very limited calculator use by students in primary schools before they enter secondary school in Year 8. And secondly, students have been given insufficient instruction on the use of calculators. Calculators should be used much more extensively at the primary school level as a learning tool, rather than as a calculating tool.

At the same time, there needs to be a shift from paper and pencil algorithms to calculator algorithms. In this way, students should improve their skills in using calculators more effectively and efficiently.

It is recognised that the calculator computation test used in this study was somewhat contrived in that students were forced to use their calculators whether they wished to or not. Also, the requirement to record all the key presses was also an abnormal situation. Thus it would be useful to follow this up with more individual interviews where students have computational choice, and to check out their calculator use by direct observation. This should also be extended to scientific and graphing calculators to assess the levels of efficiency of calculator use with these tools.

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Secondary Mathematics Teachers' Beliefs About Teaching and Learning: Some Significant Factors

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The espoused beliefs of 465 secondary mathematics teachers in Greek State secondary schools regarding the teaching and learning of mathematics were the focus of this study. The data for this investigation were collected using a 34 items questionnaire. There is evidence from this study that there are teachers who may be assumed to espouse a 'constructivist or contemporary orientation', teachers who may be assumed to espouse a 'dynamic problem-driven orientation', teachers who may be assumed to espouse 'traditional-static' and 'traditional-mechanistic' beliefs, and teachers who espouse a 'cooperative orientation' to mathematics learning and mathematics teaching. It was also found that teacher characteristics, such as, gender, teaching experience, position held and postgraduate qualifications possessed, significantly influence teachers' beliefs about mathematics and the teaching-learning process.

Teachers' beliefs influence their classroom practices, the beliefs are formed early and beliefs about teaching are well established by the time a prospective teacher starts attending University classes. It is therefore instrumental to mathematics educators and researchers, as well as mathematics education administrators and curriculum designers, to understand the impact teachers' beliefs have on their everyday cognitions and classroom practices.

In this study we have endeavoured in an attempt to explore and chart the complex topology of mathematics teachers' beliefs regarding the teaching and learning of mathematics. Attitudes and beliefs have attracted a considerable interest in recent research studies (McLeod, 1989, 1992; Lester and Garofalo, 1987; Leder, 1993). Pajares (1992) argued that knowledge and beliefs are inextricably intertwined and that individuals' beliefs strongly affect their behaviour and Hollingworth (1989) reported that the way teachers implement new methods or programs in their classrooms relates to whether their beliefs are congruent with the proposed new methods or programs. Pajares (1992) argued that beliefs are prioritised according to their connections to other cognitive and affective structures. In this study the complex topology of mathematics teachers' beliefs regarding mathematics learning and teaching has been explored and documented.

Ernest (1989) noted that the term 'beliefs' is not an isolated construct but it consists of the mathematics teachers' beliefs system, values and ideology. He claimed that beliefs should rather be thought of as 'implicitly held philosophies' (p. 20) and distinguished three philosophies of mathematics among the many variations possible (a) *The dynamic - problem solving view*: Mathematics is a continually expanding field of inquiry and its results are open to revision. It is also a field of creativity and invention and a cultural product, (b) *The Platonist view:* Mathematics is a set of monolithic and immutable structures and truths, and a static but unified body of expert knowledge. Mathematics is not created but discovered, and (c) *The instrumentalist view:* Mathematics is consisting of an accumulation of useful but unrelated facts, rules and skills to be mastered in the pursuit of some not clearly defined end.

Thompson (1992) reported that the findings of studies investigating the relationship

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between mathematics teachers' espoused beliefs and enacted beliefs have not been consistent. Some studies have found a high degree of agreement between mathematics teachers' espoused beliefs and their instructional practice whereas some other studies have reported 'sharp contrasts' (Thompson, 1992, p.137).

Attitudes and beliefs have attracted a considerable interest in recent research studies (McLeod, 1989, 1992; Leder, 1993). Various researchers have put a number of definitions forward over the past two decades. "Beliefs" has been a particularly difficult term to define in the educational and psychological literature and a number of researchers have offered definitions. The definition McLeod (1992) put forward was considered sufficient for this research study:

> Beliefs are largely cognitive in nature, and are developed over a relatively long period of time. Emotions, on the other hand, may involve little cognitive appraisal and may appear and disappear rather quickly…Therefore we can think of beliefs, attitudes and emotions as representing increasing levels of affective involvement, decreasing levels of cognitive involvement, increasing levels intensity of response, and decreasing levels of response stability. (p. 579)

A number of studies conducted by Australian researchers have investigated various aspects of teachers' beliefs, e.g. how teachers' beliefs impact on teachers' daily practices (Buzeika, 1996); encouraging reflection on their teaching practices (Malone, 1995); providing alternative models for mathematics teaching (Malone, 1995; Malone et.al., 1997); espoused primary and secondary teachers' beliefs about mathematics and the learning and teaching of mathematics (Howard, Perry & Lindsay, 1997; Perry, Howard & Conroy, 1996; Perry, et. al., 1999; Nisbet & Warren, 2000).

Methodology

The Instrument

A 34 items researcher - designed questionnaire was the instrument for this investigation. In developing the questionnaire items, we drew on previous research findings about teacher beliefs issues in mathematics education (Van Zoest, 1994; Howard, Perry & Lindsay, 1997). The investigation examined the espoused and enacted beliefs of secondary mathematics teachers, working during 1999-2000 in State High schools in Greece and covered the following areas: subject demographics such as gender, age, length of teaching experience, position held, postgraduate studies, beliefs about mathematics, mathematics learning and mathematics teaching. On the survey, a Likert-type scoring format was used - teachers were asked to indicate the extent to which they agreed (or disagreed) with each statement presented. A five point scoring system was used – strongly disagree (SD) to strongly agree (SA). A score of 1 was assigned to the SA response and a score of 5 to SD. The survey contained three subsets of items: Beliefs about Mathematics (6 items), Beliefs about Mathematics Learning (7 items) and Beliefs about Mathematics Teaching (21 items). A space was also provided for teachers to comment on any aspect (of each part) of the instrument and its items.

Participants

Six hundred survey forms were sent to a random selection of grade 7-12 mathematics teachers in Greece. The return rate was 78% and the resulting sample comprised 465 (276 males, 145 females, 44?) mathematics teachers in 39 Greek State High Schools. The returned surveys reflected a reasonably well-balanced distribution of grade level experience.

Data Analysis

Data from the questionnaire responses regarding beliefs about mathematics, mathematics teaching and mathematics learning were analysed using SPSS_{win}. A Principal Component Analysis (PCA) was used in order to interrogate the 34 questionnaire items for a typology of teachers' espoused beliefs. The significance level was set at .05.

Results

Principal Component Analysis

It could reasonably be expected that teachers' beliefs about mathematics, mathematics learning and mathematics teaching would be linked in various ways. A Principal Component Analysis (PCA) was used in order to interrogate the 34 questionnaire items investigating: *Beliefs about Mathematics (BM: 6 items), Beliefs about Mathematics Learning (BML: 7 items) and Beliefs about Mathematics Teaching (BMT: 21 items),* for a typology of teachers' espoused beliefs. Given the exploratory nature of the study and guided the scree plot and the interpretability of the factors, a five factor, orthogonal solution was accepted after the extraction of principal components and a Varimax rotation. The solution accounted for 35% of the variance, and 22 of the 34 items were used to delineate the factors. A final confirmatory factor analysis (Table 1) was carried out, following the elimination of psychometrically "poor" items. Variables loading on more than one factor have been eliminated. The extraction method used was Principal Component Analysis and the rotation method was a Varimax with Kaiser Normalization. The rotation converged in 7 iterations.

Six items loaded on Factor 1, 6 items loaded on Factor 2, 3 items on factor 3, 3 items on factor 4 and 4 items loaded on Factor 5 (Table 1). The naming of factors was guided by the nature of the items associated with each factor. The factors are as follows:

Teachers, whose beliefs are expressed by *factor 1*, can be assumed to espouse *a socio-constructivist or contemporary orientation* to mathematics, mathematics learning and mathematics teaching. They believe that they should create problematic situations for learners, that mathematics learning is enhanced by activities, which build upon students' experiences, that students are rational decision makers and that mathematics learning is enhanced by challenging activities within a supportive environment. Teachers, whose beliefs are expressed by *factor 2*, can be assumed to espouse *a dynamic problem-driven orientation* to mathematics, mathematics learning and mathematics teaching. They believe that challenging activities within a supportive environment enhances mathematics learning and that mathematics is the dynamic searching for models and problems and their results are open to review.

Teachers, whose beliefs are expressed *by factor 3*, can be assumed to espouse *static - transmission orientation* to mathematics, mathematics learning and mathematics teaching. Teachers in this category believe that mathematics is computation, that mathematics is a

Table 1

Factors related to views of mathematics, mathematics teaching and mathematics learning

Item	Item description	Loading
	Factor 1: A contemporary socio-constructivist orientation to mathematics, mathematics teaching and mathematics learning	
24	It is important for students to be provided with opportunities to reflect on and evaluate their own mathematical understanding	.594
14	Ignoring the mathematical ideas generated by the students can seriously limit their learning	.594
32	The education system should be preparing critically thinking citizens who are able to utilise their mathematical skills	.583
17	Mathematics teachers should be fascinated with how students think and intrigued by alternative ideas	.482
16	Teachers should encourage their students to strive for elegant solutions when they solve problems	.385
1	Justifying the mathematical statements that a person makes is an extremely important part of mathematics	.306
	Factor 2: A dynamic problem-driven orientation to mathematics, mathematics teaching and mathematics learning	
13	Mathematics learning is enhanced by challenging activities within a supportive environment	.650
30	The comprehension of mathematical concepts by students should correspond to their cognitive development and it should be a decisive factor in the content sequence to be taught	.608
34	Teachers should respect the mathematical knowledge of their students, which is consisting of a nexus of experiences, beliefs, attitudes, representations, concepts, strategies, connections, values, judgements and emotions	.558
6	Mathematics is a beautiful, creative and useful human endeavour that is both a way of knowing and a way of thinking	.463
19	Teachers always need to hear students' mathematical explanations before correcting their errors.358 problems and their results are open to review	.447
	Factor 3: A static-transmission orientation to mathematics, mathematics teaching and mathematics learning	
7	The most effective way to learn mathematics is by listening carefully to the teacher explaining a mathematics lesson	.718
2	Mathematics is a static and immutable knowledge with objective truth	.675
28	The role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received that knowledge	.637
	Factor 4: A mechanistic-transmission orientation to mathematics, mathematics teaching and mathematics learning	
12	The memorisation of mathematical facts is important in mathematics learning	.623
27	Teachers or the textbook-not the student-are the authorities for what is right or wrong	.602
10	Mathematics knowledge is the result of the learner interpreting and organising the information gained from experiences	.532
	Factor 5: A collaborative orientation to mathematics, mathematics teaching and mathematics learning	
20	An effective way to teach mathematics is to provide students with interesting problems to investigate in small groups	.551
11	Students are rational decision makers capable of determining for themselves what is right and what is wrong	.536
33	All students are able to be creative and do original work in mathematics	.469
8	Students can learn more mathematics together than by themselves	.447

static and immutable knowledge with objective truth, and that the role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received that knowledge. Teachers whose beliefs are expressed by factor 4, can be assumed to espouse *a mechanistic - transmission orientation* to mathematics, mathematics learning and mathematics teaching. They believe that the memorisation of mathematical facts is important in mathematics learning, that teachers and the textbooks are the authorities for what is right or wrong. Teachers, whose beliefs are expressed by factor 5, can be assumed to espouse *a cooperative orientation* to mathematics learning and mathematics teaching. They believe that an effective way to teach mathematics is to provide students with interesting problems to investigate in small groups and that students are rational decision makers capable of determining for themselves what is right and what is wrong.

Teacher Characteristics and Their Influence on Teachers' Beliefs

Data were collected on five teacher characteristics, namely, gender, years of teaching experience at Junior High school, years of teaching experience at Senior High school, position held and postgraduate qualifications possessed. One-way analyses of variance and linear contrasts were performed in order to examine if the five beliefs factors relating to mathematics, mathematics teaching and mathematics learning, varied according to these characteristics.

Gender

The independent samples t-test showed that gender was significant for one of the five factors – Factor 1, *a contemporary socio-constructivist orientation to mathematics, mathematics teaching and mathematics learning* (t_{421}=-3.3, p=.001). The mean factor score for female teachers (0.23) was significantly higher than for the male teachers (-0.10). It could be argued that female teachers in this study placed more emphasis on a socio-constructivist view of mathematics, mathematics teaching and mathematics learning than did the male teachers.

Years of Experience at Junior High School

The ANOVA summary table showed that years of experience at Junior High school was significant for one of the five factors - Factor 1 ($F_{(2, 421)}$=3.514, p=.031). It could be argued that teachers with experience at Junior High school (Years 7-9), placed more emphasis on a socio-constructivist view of mathematics, mathematics teaching and mathematics learning than did teachers with experience at other levels. By examining the linear term, which is also significant across the years of experience at Junior High school, it can be concluded that the contemporary socio-constructivist orientation to mathematics, mathematics teaching and mathematics learning increases consistently between the inexperienced teachers category (0-6 years of experience) and the experienced teachers category (7-15 years of experience). It could be argued (by using the means plot) that the contemporary socio-constructivist view of mathematics, mathematics teaching and learning (factor 1) was more prevalent among experienced teachers and veteran teachers (16+ years of experience) than among inexperienced teachers.

Years of Experience at Senior High School

Years of experience at Senior High school was not a significant variable for any of the factors, suggesting that teachers' beliefs about mathematics, mathematics teaching and mathematics learning were not significantly influenced by their experience at Senior High school level (Years 10-12). By examining the linear term, which is significant for Factor 2 across the years of experience at Senior High school, it can be concluded that the dynamic problem-driven orientation to mathematics, mathematics teaching and mathematics learning increases consistently across teachers' experience categories. It could also be argued (Figure 1) that the dynamic problem-driven orientation to mathematics, mathematics teaching and mathematics (Factor 2) was more prevalent among veteran teachers than among inexperienced teachers and experienced teachers.

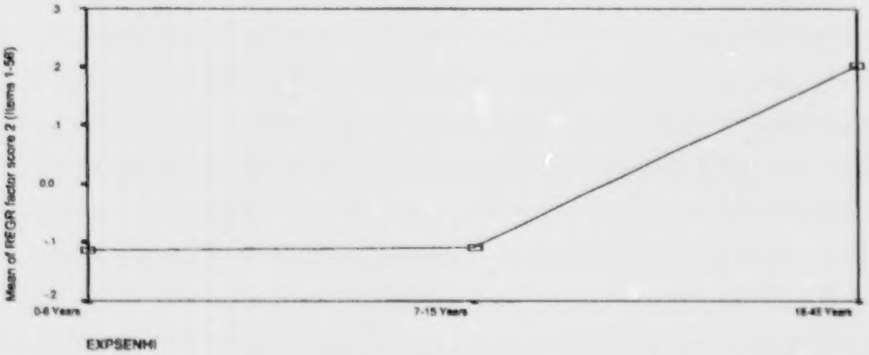

Figure 1. Means Plot (Experience at Senior High school).

Position Held

The ANOVA summary table showed that a teacher's position (teacher, principal, consultant) was significant for two of the five factors - Factor 1, A contemporary socio-constructivist orientation to mathematics, mathematics teaching and mathematics learning (F(2, 412)=3.171, p=.043) and Factor 3, A static-transmission orientation to mathematics, mathematics teaching and mathematics learning (F(2, 412)=4.919, p=.008. By examining the Linear term which is also significant across the years of experience at Junior High school, it can be concluded that the contemporary socio-constructivist orientation to mathematics, mathematics teaching and mathematics learning decreases consistently between the principal and consultant categories of position held. It could be argued (by using the means plot) that the contemporary socio-constructivist view of mathematics, mathematics teaching and learning was more prevalent among teachers and principals than among consultants, and that the static-transmission orientation to mathematics, mathematics teaching and mathematics learning was more prevalent among consultants than among teachers and principals.

Conclusion

The results of the factor analysis can be summarised as follows. There appears to be *two orientations* that are characteristic of secondary mathematics teachers' beliefs:

- A *contemporary - constructivist orientation*, consisting of the following (not mutually exclusive but complementary) views:
  - ☐ The socio-constructivist view (factor 1)
  - ☐ The dynamic problem driven view (factor 2)
  - ☐ The cooperative view (factor 5)
- A *traditional - transmission - information processing orientation*, consisting of the following (not mutually exclusive but complementary) views:
  - ☐ The static view (factor 3)
  - ☐ The mechanistic view (factor 4)

Forty-three percent of the teacher participants in the study fell into the first of these categories and 57% into the second. This information suggests that contemporary ideas on teaching and learning, especially those with a constructivist orientation, have yet to be generally accepted and adopted among the Greek mathematics teacher community. Undoubtedly though, professional development efforts in Greece are taking hold – there can be no doubt that the above percentages would have been reversed if the data had been collected just a few years ago. It also appears that mathematics teachers' beliefs about mathematics cannot be separated from their beliefs about teaching and learning mathematics. These outcomes obviously have implications for professional development programs for mathematics teachers.

Another interesting feature of the outcomes of this study concerned teachers' characteristics and their influence on those teachers' beliefs. It was found that female teachers in this study placed more emphasis on a socio-constructivist view of mathematics, mathematics teaching and mathematics learning than did the male teachers. From data on teacher experience factors, it was concluded that a contemporary socio-constructivist orientation to mathematics, mathematics teaching and mathematics learning ability increases consistently between the inexperienced teacher and the experienced teacher categories. Linked with this finding was another indicating that the contemporary socio-constructivist view of mathematics, mathematics teaching and learning was more prevalent among experienced teachers and veteran teachers than among inexperienced teachers. The dynamic problem-driven orientation to mathematics, mathematics teaching and mathematics learning increases consistently across teachers' experience categories. We also argued that according to our data, the *dynamic problem-driven orientation to mathematics, mathematics teaching and mathematics* was more prevalent among veteran teachers, than between inexperienced teacher and experienced teachers. The position teachers held was a significant factor that was found to influence teachers' beliefs. It can be concluded that the contemporary socio-constructivist orientation to mathematics, mathematics teaching and mathematics learning decreases consistently between the principal and consultant categories. The contemporary socio-constructivist view of mathematics, mathematics teaching and learning was more prevalent among teachers and principals than among consultants.

While teachers' beliefs are difficult to change, making teachers aware of the impact of those beliefs is a first step in assisting them to transform those counter-productive beliefs into a positive force to enhance mathematics teaching and learning.

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Rethinking Curriculum: A Developer's Perspective

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For many years the research-development-dissemination (RDD) model has been used in education. From my experience and scholarship this model seems unsatisfactory and I have been theorizing about alternative models that recognize the complexity of change. In offering a model with eight activities occurring simultaneously at three levels, and with these activities being interrelated, I have replaced the simple RDD model with a complex model. This model emphasizes involvement by teachers and offers an inclusive and empowering approach to change. The challenge is now to trial and improve this model.

Introduction

There are many ways to think about curriculum and my interest is from a developer's perspective. My doctorate was about professional development, as curriculum development did not seem to result in the expected changes. Since then I have developed this new model to include other influences on change that impact on mathematics teachers.

Historical perspective

For over thirty years the research-development-dissemination (RDD) model has been used in curriculum development in many countries. It is a linear model as in Figure 1.

Research —> Development —> Dissemination

Figure 1. RDD model for development.

This model's three discrete stages are usually undertaken by separate groups of people. This, together with its linearity, made it a simple model to plan, budget for, and control. The model fits with new-right political notion of "no provider capture" as people involved in one stage are easily excluded from later stages. However this model does not consider all the influences that impact on development, and in keeping it simple we may have ensured its lack of success. In improving the model three well-documented factors that needed consideration have been the establishment and maintenance of a culture of change, teacher ownership, and the provision of adequate time for the process. In addition it seems important to identify the other influences that impact on teacher change.

While seeking an improved model I was involved in the use of 'exploratory studies' in mathematics. These were small informal research projects undertaken by self-nominated groups of teachers supported by minimal seeding funding that investigated areas of concern that they had identified. The studies were excellent professional development for the participants, and the reports and resources produced were valuable input for curriculum projects. This involvement with research, resources, professional development, and curriculum development seemed to integrate more of the aspects than usual, but I saw it only as a first step. Two other integrating were the 'rich learning activities'[1] of Ahmed (1987) and the idea of 'curriculum anticipating' (Davis, 1996), but more of those later.

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 123-130) Sydney: MERGA. ©2002 MERGA Inc.

Influences on Development, or Co-Emerging Activities?

For me, differences between development, change and learning have disappeared. With an emergent[2] view of learning the question is not 'what influences impact on curriculum and professional development?' but rather, 'what co-emerging activities impact on each other?' I see eight activities—teacher development, teacher practice, research, theory, policy development, assessment development, curriculum development, and resource development. I have replaced nouns with verbs to emphasize process rather than product. In figure 2 these activities are connected with dotted lines to show the two-way relationships between them, these being lines of influence rather than causal relationships.

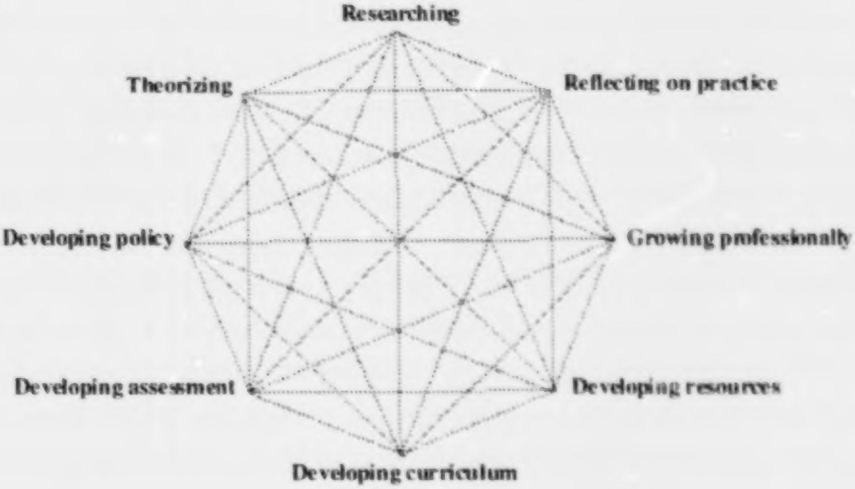

Figure 2. The eight co-emerging activities in the educational change process

This model has 28 dimensions because of the links between activities. It is complex, and becomes more so if one considers 'ideas' flowing to the activity nodes from outside. Other activities may also exist, but these may be implicitly included. While the model may suggest that teachers grow and reflect, academics research and theorize, and others develop policy, assessment, curriculum, and resources, this is not intended. I see teachers involved in the eight activities at three levels (teacher, school, and regional), and this adds to the complexity because development is not top-down, nor bottom-up, but both-ways, and enhanced by rich dialogue. In addition, this model is not dependent on who initiates change. Teacher involvement in the activities is part of their professionalism, and one of my assumptions is that teachers are (or would like to be) professionals, in spite of pressures that exist to deprofessionalize them and replace them with educational technicians.

Growing Professionally

Professional development is not changing other people. One changes oneself, and one only helps others if and when they want to change. Whether teachers plan change individually, as a school project, or as a regional activity, the growth varies with individuals. In addition, the focus for change varies between participants and may concentrate on the personal, social or professional dimensions, or a combination of these.

Reflecting on Practice

The centrality of teachers' existing practice needs to be recognized because, just as the good teacher starts where the learner 'is', the teacher's practice is where the teacher 'is'. When teachers are busy they find little time to reflect, yet until they are aware of what they do, what they could do, and why they might want to consider a change, it is unlikely that they will change. Reflecting-on-practice can be done individually, or with colleagues, and it may include feedback from students, parents, or the community. Reflecting-on practice is not restricted to teachers, all participants in change activities need to be doing it.

For some, thinking about reflecting-on-practice is thinking about what has happened. This is descriptive reflection. For me reflecting-on-practice involves this descriptive level, but also working at an interpretive level thinking about why things happened, and delving deeper into assumptions and alternatives. This deeper aspect of is the 'critical' level, its aim is to critique assumptions made or acted upon, and to empower oneself to act differently if one wishes to do so in the future; that is to anticipate possibilities.

Researching

Research is often thought of as academic. I believe this concept should be expanded. The list in table 1 adds some new categories to the traditional ones, and I see these as research (regardless of pressures that exist not to do so).

Table 1
Some Research Categories and Descriptors

Basic research	to acquire new knowledge
Strategic research	to generate new knowledge in areas which have not advanced to the point where specific applications can be identified
Applied research	to develop knowledge for specific practical objectives (e.g., needs analysis)
Evaluatory research	to evaluate policies, programmes, or practices (e.g., curriculum evaluation)
Scholarship	to expand the boundaries of knowledge across disciplines by the analysis, synthesis and interpretation of ideas
Scholarship of teaching	to transform knowledge by bridging gaps in educational settings, (e.g., with exploratory studies)
Creative work	to generate ideas, hypotheses, images, performances or artifacts, leading to the development of new knowledge, understanding or expertise
Action research	to apply existing knowledge in the resolution of problems
Consultancy	to work with clients in professional contexts in problem solving
Developmental research	to develop, trial and improve artifacts/resources for professional use
Hermeneutic reflection	to develop understandings, interpretations, and applications in professional situations
Professional practice	to theorize about professional practice and make such practice more effective

If one accepts these categories then one may subsume some activities in the model under research. While the boundaries between the activities are intended to be fuzzy (or non-existent), I have retained them all as my view of research may not be accepted by all.

Research, like other activities associated with change, occurs at many levels. Individual teachers do research in many ways—exploratory studies, action research, informal enquiry, development research, and traditional academic research. While research can be viewed as an input to the change/development process, I would also argue that it is of most benefit to the researcher, and fits within growing professionally.

Unfortunately few teachers research policy or curriculum development. Without their input we use politically motivated research that is not always neutral, and international research (cultural imperialism) that may assume that education is to serve capitalism and consumerism, and ignore 'non-market values of humanitarianism, equity, and ecology' (Ilon, 2000).

Theorizing

An example of theorizing is what I am doing here. Based on experience, research, and the work of others (in particular enactivist[2] theory), I have conceptualized a different development model. The result is not right or wrong, nor has it been tested, but for me it makes sense of what I see happening. I assume people will critique it, and if it is useful it will be adopted (and adapted), then improved upon, and finally replaced. When theorizing I started with personal theories and assumptions about mathematics, and education. After these were made explicit I considered alternatives and compared my ideas with those of others. Such explication and discussions is desirable at school and regional level. There needs to be an understanding of the ideas that are generally accepted by teachers, and of new emerging ones that may influence things in the future.

Applicable theories may focus on learning, teaching, assessment and curriculum, but others may also need consideration. For example, theories about mathematics—is it a body of knowledge to be known, a human problem-solving endeavor, or an alternative way of making sense of one's world? Another theory might focus on contexts and applications, and this might depend on one's theory of learning and on what one thinks of as school mathematics. I see this as to do with what one thinks of as 'understanding' and Aoki (1987) summarized this for me when he discussed some ideas from Gadamer (1982, cited in Aoki, 1987).

> (Gadamer) confronts squarely the hermeneutic problem of application in the context of understanding, interpretation, and application, which to him, are all moments of the hermeneutic act. He states that "understanding always involves something like the application of the text to be understood to the present situation of the interpreter" and that application is an "integral or part of the hermeneutical act as are understanding and interpretation" ... The question concerning application raises the hermeneutic problem of the relationship between the general and the particular. At the heart of the problem is the notion that the general must be understood in a different way in each new situation. Understanding is, then, a particular case of the application of something general to a particular situation.

Theorizing helps when considering different perspectives. It enables assumptions to be questioned, theories made explicit, and possible conflicts identified. For example, behaviourism is concerned with *analysis* of subject matter into objectives for teaching and assessment, and constructivism emphasizes *synthesis* which requires us to look at the big ideas[3] within mathematics. With learning based on constructivism and assessment on behaviourism we have conflict. Another example involves *progression*. This underpins curriculum structure and assessment. Progression emerged as behaviourists analyzed subjects for teaching-now it is assumed to be the order in which things are learnt, and this is reinforced by assessment. If we consider how learners construct schemas our ideas of progression might be different. When it is used by the assessment industry it becomes a self-fulfilling prophesy.

An important ethical question related to theories is, does anyone have the right to impose any particular theories? I think not, but if a new theory is imposed many teachers will resist it, and having only one theory may limit the exploration of new or alternative theories and progress may be retarded.

Developing Policy

While some policy is developed regionally, schools determine aims and rules, and teachers with students determine classroom rules—policy development occurs at all levels. When policies have desirable results their influence spreads. School and classroom aims and the ways that they are put into operation are important, but they reflect the values of decision-makers in particular contexts. I am concerned about imposing policies regardless of whether they are acceptable to people in other contexts. It is interesting to look at the interplay between national policies, espoused aims of education, school aims, classroom policies, mathematics education aims, and the enacted values of teachers. Often the result is a lowest common denominator rather than a celebration of difference.

Regional policies related to curriculum, assessment and resources have been used in various ways to control what is done in classrooms. Such policies have influenced the related development activities but the lack of wholehearted endorsement of these policies by schools supports the notion that links within the model are influential but not causal.

Developing Assessment

Assessment occurs at many levels. Region-wide assessment (or sampling to moderate examinations or standards) is common. In schools formal assessment occurs for records and reports, and in class informal assessment helps teachers planning. Self-assessment by students is often neglected yet it is needed if we are developing autonomous learners.

Schools are influenced by national (and international) assessment that often causes teachers to teach to the test. This means emphasizing things likely to be tested though 'not everything that can be measured, counts, and not everything that counts, can be measured'. In-school common assessment has a similar but somewhat lesser influence.

It would be good to have assessment policies and strategies that have a positive impact on learning and provide the required information. This is most likely to come from teachers who think of the desirable aspects of learning rather than from 'experts' with other agendas, and this is another example of an area needing teacher-driven development.

Developing Curriculum

For me curriculum means 'all planned activity for the classroom'. Curriculum documents are only part of the picture—assessment practices, rules and regulations from bureaucracies, teachers guides, and resources also influence the curriculum.

Regional curriculum development usually involves a number of teachers directly and more indirectly depending on the model used. If one assumes my complex model, then many more would be involved. When a regional curriculum is interpreted into school 'schemes' to suit particular situations, and then into lesson plans for particular classes with specific resources, we see every teacher really is a curriculum developer.

Numerous questions arise when one is developing curriculum—what should be in the document, who decides, and how do developers find out? Even in terms of mathematics there are questions—some countries do not teach statistics, and some universities do not teach geometry. Other questions relate to curriculum purpose—is it what students must know, what teachers must teach, or both? Linked to this is the issue of who is the curriculum for, if it is mainly for teachers then what structure best suits them?

Just as my model suggests a complex connected model rather than a linear one for development, perhaps the curriculum needs to reflect complexity. I prefer a focus on the 'big ideas' of mathematics, and a list of successfully trialled 'rich learning (mathematical) activities'[1] that could be used with the big ideas. The key role for teachers would be selecting appropriate and inclusive activities for students regardless of gender, culture, and ability, and then 'curriculum anticipating'. This would involve analyzing possible ways that a lesson might develop, working out the likely paths students will take, finding ways to extend the activity, and considering possible blind alleys.

Developing Resources

The first resource that comes to mind is the textbook. In some countries (e.g., USA) developing textbooks is thought of as developing curriculum. Elsewhere resources follow curriculum development. Regardless, resources influence lesson planning and teaching, that is the implemented curriculum. While resource production is usually initiated by small groups of authors, a research team, or a bureaucracy, teachers become resource developers as they adapt these to suit their classes, and as they produce work sheets to supplement texts or to use ideas from other sources. When such adaptations are discussed and shared with colleagues the influence moves beyond the single teacher level.

Resources include calculators and computers and these are important as they influence mathematics teaching. The influence of the technology is summarized by Engelbrecht & Harding (2001) in endnote 4. The use of these resources is a significant aspect of change and provides another example of the relationship between the eight activities and the three levels in my model.

An Example, Explore-Conjecture-Prove

Consider the notion *explore-conjecture-prove*. It may be desirable to introduce this to schools because it integrates processes (problem solving, reasoning, making connections and communicating) with content strands. If so, what could we do? It would be difficult to work in all eight activities at three levels, but there are things to do.

In terms of research we might find out what has been done in other countries, and survey those teachers to see if they found the notion relevant. We could encourage small research projects within present curriculum constraints to see if it was a viable approach. The results could then be shared so that others would be aware of the findings.

In terms of theory we could look to theories about 'what mathematicians do', we might consider theories about learning and how views of what mathematics is emerge, or we might see how explore-conjecture-prove fits with theories of schema construction. These theories would need to be discussed so that people at all levels become aware of them.

To influence policy we would communicate with policy makers—teachers, heads of departments, advisors, and regional developers. Communication is two-way so we might talk of our concerns, discuss possible solutions, suggest possible ways forward, and allow the decision-makers to think about our ideas. We could make submissions on draft curriculum documents, and suggest how the notion could work in practice.

Because assessment is important in development, we would communicate with assessment people, provide exemplars to show how such a notion could be assessed, and justify why it might not be able to be used in some ways. This might require teachers testing exemplars that could be done with exploratory studies.

To change curriculum we might begin at class level with activities for teachers, and then at school scheme level show how curriculum strands might be integrated. At national level we might use a similar process. In each case we could focus on 'curriculum anticipating' to see how such activities can be used and extended to make more connections. We may not be able to do all of this by ourselves and nor would we see such a process as being separate from resource and professional development, so we would be trying all the time to widen the group of interested people to achieve our goal.

In terms of resource development we would need many 'rich learning activities' for all class levels. We could design some resources ourselves and encourage others to contribute ideas. We could share these with writers of resources through journals and workshops so that more people would see how the ideas might be used in the classroom.

While we would be reflecting on this process as we did it, we would be encouraging teachers to be reflecting-on-action as they tried the activities. We would not be suggesting that what we had was an end product, instead we would be seeking further ways forward.

For the professional development component we could attend conferences and present papers on the idea, we could prepare kits for advisors to use in schools, and offer ourselves as resource people for workshops. If we were teaching in-service courses we would build this topic into our programmes as something to be discussed and researched.

While these eight activities have been presented as disconnected, we would attempt to integrate them because of the relationships between them. We would assume that it would take considerable time for people to take ownership of the idea and implement it with classes because we are not asking teachers to change the topics that they teach (content), but rather change the way they teach (identity). We would assume that modifications would occur, and would not be surprised whether the breakthrough occurred because of teachers pushing for change or with policy makers taking the initiative.

Conclusion-Complexity/Completion/Confusion/Challenge

Complexity—Educational systems at regional, local, school, and classroom levels form a nested group of complex, dynamic and self-organizing systems. Therefore, models for change need to recognize complexity and consider all the influences on development rather than think that some specific input will cause the desired change to be implemented. Presenting this model with eight activities and 28 links at each of at least three levels, is an attempt to do this.

Completion—The model offered is not complete. It does not explore how the links between activities operate, nor does it look at each activity and discuss how each might best be carried out. I feel sure that each of the eight activities are themselves complex and that the people involved will form a self-organising system in the development process.

Confusion—I imagine that my model has people perplexed by the lack of a straightforward way of going about things. If so, then this paper has been worthwhile. There seems to me that there is no one right way, and coping with complexity can appear overwhelming. I take heart in the saying 'think globally but act locally'. Our professional responsibility seems to be to do the best we can in the change activities we are involved in, implement change in our own immediate environments, accept the complexity, and don't loose heart when others are not willing to do the same.

Challenge—There is much to be done in improving the model:

- in identifying other activities in the development process,
- in finding ways to make the links between activities productive,
- in ensuring that interactions between the activities occur at and between levels, and

- in ensuring the model encourages continual change.

We need to be able to cope with multi-dimensional models for change and the uncertainty that is implicit with them.

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Endnotes

1 What is a Rich Learning Activity?
- It must be accessible to everyone at the start.
- It needs to allow further challenges and be extendible.
- It should invite children to make decisions.
- It should involve children in speculating, hypothesis making and testing, proving or explaining, reflecting, interpreting.
- It should not restrict pupils from searching in other directions.
- It should promote discussion and communication.
- It should encourage originality/invention.
- It should encourage 'what if' and 'what if not' questions.
- It should have an element of surprise.
- It should be enjoyable (Ahmed, 1987).

2 Enactivism is an emerging theory about learning that draws from a number of discourses, among them phenomenology, constructivism, ecology, and systems and complexity theories. Enactivism might be considered as an elaboration of constructivist epistemologies. It views learning and knowing, as complex, emergent processes by which dynamic agents maintain fitness with one another and within dynamic contexts. Two of the key concepts within this shift in thinking are:
(i) an enlargement of the notion of cognitive (or learning) systems, and (ii) the combining together of knowledge, activity, and identity. On the first, a learning system is seen as any complex form that can adapt itself to changing circumstances. For the most part, such systems are dynamic and robust, able to change and adapt efficiently. Inherent in this notion is the broader definition of cognition as 'coming to know' which includes traditional rational thinking and other forms of learning. From such a perspective learning refers to transformations, those that expand the learner's potential range of action-and it is here that the second major concept fits into place. The suggestion that learning is a transformation is a reference to the physical character of a learning system. Upon learning, a systems' patterns of activity and its associations-internal and external, with and in other systems, undergo physical change. Put differently, learning affects the entire web of being, and it follows that what one knows, what one does, and who or what one is cannot be separated. (Begg, Davis, & Bramald, in press)

3 The 'big ideas' of mathematics would include notions such as:
- number (numbers in everyday life)
- algebra (relations and functions)
- statistics (variation)
- measurement (comparing with 'standard' units)
- geometry (invariance)
- doing mathematics (explore-conjecture-prove)
- problem solving (problem solving cycle, and problem solving strategies)

4 The influence of technology is nicely summed up by the following:
- Some mathematics becomes more important because technology requires it
- Some mathematics becomes less important because technology replaces it
- Some mathematics becomes possible because technology allows it
- Some mathematics can be taught using technology (Engelbrecht & Harding, 2001)

Development of Understanding of Place Value

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This paper describes a set of cognitive research tasks used to assess the understanding of place value of 21 year 3/4 students. Seven tasks provided enough information about the conceptual development of the students so that a determination of their progress towards understanding could be made. Re-administration of the tasks occurred after six months and then twelve months. The results supported the ability of defined levels of response to the tasks to describe a developmental pathway. Mediation of the tasks during administration identified some validity and interpretation issues. The study confirmed that students of this age are grappling with the development of an understanding of place value and therefore are in need of explicit teaching in this domain.

Background

Student difficulties in developing understanding of the nature of the number system have been highlighted in recent years (Kamii & DeClark, 1985; Ross, 1986; Thomas & Mulligan, 1999) and much research has attempted to plot the path of growth of understanding of place value. Although it has been shown that it can take many years to develop conceptual understanding of place value (Cockcroft, 1987; Sinclair & Scheuer, 1993; Thomas & Donaldson, 1995), it has also been shown that students are beginning to deal with base ten concepts in the early years of school (Carraher & Schliemann, 1990; Wright, 1991; Young-Loveridge, 1997) and in particular, during the middle primary years (Pengelly, 1990; Thomas, 1992).

From cognitive research, definitions of the nature of understanding of place-value numeration are available, and are situated within models of development (Denvir & Brown, 1986a, 1986b; Jones, Thornton, Putt, Hill, Mogill, Rich & Van Zoest, 1996; Kamii, M., 1982 as cited in Kamii & DeClark, 1985; Resnick, 1983; Ross, 1986; Wright, 1998; Wright & Stewart, 1999). One framework for assessing and describing children's understanding of place value of two-digit numbers, developed by Ross (1986) and applied in Australia by Sierink (1989), demonstrated that "meaningful differentiation of performance can be made in the two [Australian and US] settings" (Sierink & Watson, 1991). This framework describes a pathway of development leading to understanding. There are three stages that are defined in Table 1.

As within the various models, specific concepts that contribute to understanding have also been delineated in this model and are reflected in the tasks used. Specifically, Sierink and Watson (1991) reported the ability of seven tasks to provide evidence of understanding of place-value numeration of two-digit numbers as defined by Ross (1986). The seven tasks included six tasks derived from previous research, as well as a task referring to numbers specifying use-by-dates, that tapped an understanding of the place-holding role of 0 (zero). This study tracked development over twelve months and investigated reasons underlying student responses to the tasks. It was able to further test the tasks and the model of development through a longitudinal perspective and through dynamic assessment, i.e., the incorporation of teaching within the assessment.

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 131-138). Sydney: MERGA. ©2002 MERGA Inc.

Table 1
Stages of Development of Understanding of Two-Digit Place Value

Stage	Characteristics
Emergent Stage (3 early stages)	Can read and write two-digit numbers. Associates the whole numeral with the number it represents. No meaning is attached to the individual digits. OR Can state that the digits are in the tens place or the ones place but do not match this to the quantities. OR Can state that each digit represents the number indicated by its face value, but the tens objects are different objects to the ones objects and do not combine to form the total quantity of the number.
Construction Stage	Can explain that the left digit in a two-digit numeral represents sets of ten objects and the right digit represents the remaining single objects but this knowledge is tentative and characterised by unreliable task performances.
Understanding	Can explain that the individual digits in a two-digit numeral represent a partitioning of the whole quantity into a tens part and a ones part. The quantity of objects corresponding to each digit can be determined even for collections which have been non-canonically partitioned.

(adapted from Ross, 1986)

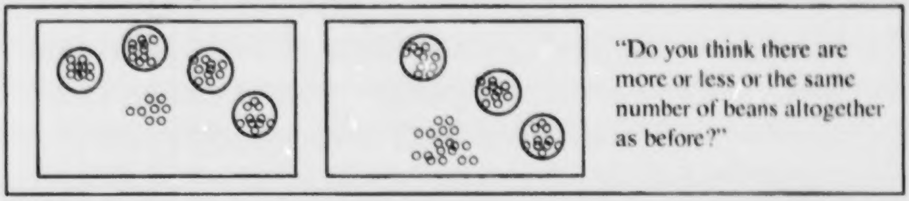
Method

The seven tasks compiled by Sierink and Watson (1991) were administered three times over twelve months to a group of 21 students in a large NSW government country school. The students were all members of a multi-ability year 3 class initially. All assessment was carried out by the researcher on occasions in February, August and then February again, corresponding to early Year 3, late Year 3 and early Year 4 for the students. The assessment occurred in a small room separate from the classroom at the school.

The year 3 program during the first ten months of the study was focused on Mathematics K-6 (NSW Department of Education, 1989) Numeration Units 8-11 reflecting place value numeration of two-digit numbers, while the class programs for Year 4 extended to three- and four-digit numbers. Apart from informal feedback about the sessions to the class teacher, and discussion of the curriculum domain, no specific intervention occurred within the classroom teaching for these students.

The administration procedures for the tasks were taken directly from the work of Sierink (1989; Sierink & Watson, 1991). This included the form of stimulus materials and the wording of the questions and instructions. The qualitatively different levels of response defined for each task were those described by Sierink (1989; Sierink & Watson, 1991). The tasks and their levels of response are described here.

Task 1 Conservation (Ross, 1986)

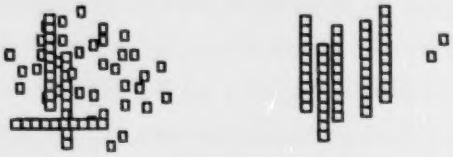


"Do you think there are more or less or the same number of beans altogether as before?"

1. Unable to conserve at all.
2. Counting of the beans to confirm that there were the same number.
3. Respond appropriately, without counting the number of beans remaining.

Task 2 Standard place value partitioning and digit correspondence (Ross, 1986)

"Use these counting blocks to build 52." "How do you know that is 52?"



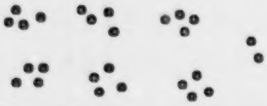
1. Unable to build 52 using the blocks.
2. Attempted use of unit blocks to make 52, before using the tens blocks.
3. Construction of 52 using the blocks, but without explanation.
4. Construction of 52 using the blocks and an explanation of the tens and ones composition.

Task 3 Non-standard partitioning and digit correspondence (Ross, 1989)

"Count these and write down how many."

"Sort the counters into groups of 4."

26



"Does this part of your 26 have anything to do with how many counters you have?"

1. Numeral written incorrectly, statement that they do not know what the digits mean, or that the digits "make it 26."
2. Responses that related the digit '2' to two objects and '6' to six groups, or invented some other meaning for the digits.
3. Stated that '2' was twenty and '6' was six.
4. '2' was interpreted as two tens and '6' was six ones.

Task 4 Counting a large number of items (Kamii, 1986)

"How many beans do you think there are?"

"Count them."

"Count them again by tens"

1. No idea, or counted each bean as ten.
2. Counting the beans into groups of ten without conservation of the whole, i.e., when asked how many beans there were, the student's answer referred to the number of piles, rather than the number of beans.
3. Grouped each ten but "in reality counted by ones" despite the groupings.
4. Counting using the tens and ones systems simultaneously.

Task 5 Explaining digit correspondence with zero holding the place for the tens (06) (Sierink & Watson, 1991)

USE BY: 06 APR

"What number is this?"

"Why?"

1. Stated that '06' was 'sixty' ignoring the order of digits.
2. Stated that the '06' was 'not sixty' but were unsure what it was.

3. Stated that the number was 'six', and that '0' was a place-holder and meant the number of tens.

Tasks 6 & 7 Writing and reading multi-digit numbers (Described by Sierink & Watson, 1991)

"Write these numbers 3, 6, 19, 83, 109, 172, 1607, 3045, 6572."

"Tell me what these numbers are."

5 9 13 40 67 158 603 1004 2407 6251

Classification of performance was related directly to the number of digits that the student could read or write correctly. The lowest level of performance was writing correctly single-digit numbers, the second level was two-digit numbers, the third level was three-digit numbers, and the highest level involved four-digit numbers..

Each task was administered within a dynamic assessment procedure (Berman, 2001); following the administration of each task and the recording of an initial level of response, the researcher carried out a mediation (teaching) session aimed at developing the concepts involved in the task. The mediation was guided by the levels of response, involved the exploration of the reasons behind the response and aimed to meet the individual student's perceived learning needs. The length of the sessions ranged from 5 minutes, for one student who needed no teaching, to a maximum of 36 minutes. The sessions were audio-taped and analysed in order to identify the students' perceived learning needs, any problems with the wording or format of the tasks and the nature of the teaching provided.

Analysis of Results

Profiles of Response

The pattern of responses across the tasks were compiled on the three occasions. The percentages of students providing responses at each level are shown for each of the three assessment occasions in Table 2. The responses of the class moved towards the higher level responses over the three assessments. This occurred for each of the tasks.

*Task 1: Conservation.*From just over half the class (57%) on the first assessment to 91% of the class demonstrated conservation. Because of the dynamic assessment context that allowed exploration of the reasons behind the low level responses, it was found that the wording and format of this task contaminated the assessment of conservation. Of the 43% at levels 1 and 2 on the first assessment, nearly half of them were able to demonstrate conservation once the task was clarified, as were half of the 24% on the second assessment and all of the 10% on the third occasion. These students were found to have been confused by the actions or language in the task.

*Tasks 2, 3, & 5: Digit correspondence.*The first assessment results indicated that all the students in the class could construct the number fifty-two using base-ten blocks and explain the meaning of the two digits in 52 in reference to the blocks. One student (5%) did not demonstrate that during the second assessment, however she joined all the class again on the third assessment.

Table 2

Task Distribution of Levels of Response

	Assessment 1				Assessment 2				Assessment 3			
Task 1		*Level 1*	*Level 2*	*Level 3*		*Level 1*	*Level 2*	*Level 3*		*Level 1*	*Level 2*	*Level 3*
Conservation		38%	5%	57%		24%		76%		10%		91%
Task 2	*Level 1*	*Level 2*	*Level 3*	*Level 4*	*Level 1*	*Level 2*	*Level 3*	*Level 4*	*Level 1*	*Level 2*	*Level 3*	*Level 4*
Standard partitioning				100%	5%			95%				100%
Task 3	*Level 1*	*Level 2*	*Level 3*	*Level 4*	*Level 1*	*Level 2*	*Level 3*	*Level 4*	*Level 1*	*Level 2*	*Level 3*	*Level 4*
Non-standard partitioning	14%	62%	10%	14%	19%	52%	14%	14%	5%	48%	10%	38%
Task 4	*Level 1*	*Level 2*	*Level 3*	*Level 4*	*Level 1*	*Level 2*	*Level 3*	*Level 4*	*Level 1*	*Level 2*	*Level 3*	*Level 4*
Counting by tens	29%		19%	52%	14%		29%	57%	10%		19%	71%
Task 5		*Level 1*	*Level 2*	*Level 3*		*Level 1*	*Level 2*	*Level 3*		*Level 1*	*Level 2*	*Level 3*
06		19%	76%	5%			71%	29%			48%	52%
Task 6	*1 digit*	*2 digit*	*3 digit*	*4 digit*	*1 digit*	*2 digit*	*3 digit*	*4 digit*	*1 digit*	*2 digit*	*3 digit*	*4 digit*
Writing multi-digit numbers		10%	67%	24%			57%	43%			43%	57%
Task 7	*1 digit*	*2 digit*	*3 digit*	*4 digit*	*1 digit*	*2 digit*	*3 digit*	*4 digit*	*1 digit*	*2 digit*	*3 digit*	*4 digit*
Reading multi-digit numbers		14%	38%	48%			24%	76%			19%	81%

(NB: The figures do not all exactly total 100% since each was rounded to the nearest whole number)

The class was spread across all levels for Task 3 on all three occasions. The percentage of students demonstrating highest level response to the task moved from 14% on the first and second assessments to 38% on the third assessment. There was movement up and down the levels on the three assessments. The results of this task contrast with the previous task which was also a digit correspondence task. The reliance on face value was evident for many students in this task, who were focused on the concrete groupings in front of them to find meaning in the digits. While this strategy supported the response for Task 2 it did not for Task 3.

The stimulus items for this third digit correspondence task, Task 5, were grocery items which were altered on each occasion to reduce memory support. On the first assessment one student (5%) provided the highest level response to this task while 19% of the class were at the lowest level, and showed no understanding of the role of zero in this two-digit number. The students responding at the highest level on this task increased to 29% and then 52% on the subsequent occasions, with no students at Level 1.

These three different contexts for digit correspondence provided differing patterns of response across the class. The base ten block task was significantly easier than the other two tasks, showing that an explanation about tens and ones using base ten blocks does not necessarily indicate understanding of place value. From this it is clear that a number of different contexts are required in order to ensure that the evidence of understanding of digit correspondence and place value is valid. The base ten blocks task was so much easier than the other tasks, and elicited many procedural responses. It was not able to indicate whether a student really had an understanding of tens and ones. False assumptions about cognitive development can be drawn from the use of such materials, and need to be guarded against through the use of less familiar materials and contexts. This is a caution for all tasks used in assessment. It is possible to teach or train students to respond to certain types of tasks in certain ways using particular language. It is therefore possible for overestimates of understanding to be made. All assessment should guard against this problem.

Task 4: Counting by ten. Responses to Task 4 on the first assessment resulted in eleven students (52%) who counted efficiently by tens. Six students (29%) gave each bean the value of ten. On the second occasion, more than half of the class (55%) provided responses at the highest level which increased to fifteen students (71%) on the third assessment. However, two students (10%) gave each bean the value of ten.

Tasks *6 & 7: Reading and Writing multi-digit numbers.* During the first assessment 29% of the class wrote all one- to four-digit numbers correctly, 62% wrote one- to three-digit numbers correctly but made errors on four-digit numbers, and 10% wrote one- and two-digit numbers correctly but made errors on three- and four-digit numbers. On the second occasion, of the twenty-one students, 43% wrote all numbers correctly, while 57% made errors writing four-digit numbers. On the third assessment more than half of the twenty-one students (57%) wrote all the one-, two-, three-, and four-digit numbers correctly. The other 43% of the class made errors on the four-digit numbers only. As the focus of the assessment was two-digit numbers the skills of reading and writing numbers were not necessarily indicative of understanding of the structure of the place value system. The reading and writing tasks showed the low level of difficulty and therefore early skills required. The skills of reading and writing multi-digit numbers do not necessarily depend on an understanding of place value and cannot be used as indicators alone. Explanations about the value of digits are required.

Stage of Development of Understanding

From interpretation of the responses as described above, each student was classified as functioning within: the Emergent stage; the Construction stage; or established Understanding. The percentage of students allocated to each of the stages of understanding were compared to those from the previous studies and are compiled in Table 3. The figures show growth of understanding within the class over the twelve months, but indicate that there are still fifteen students (71%) who have not yet achieved understanding of place-value of two-digit numbers in early Year 4. The comparison with the previous studies reinforces the growth of understanding highlighted in the present study. Although the previous studies had small samples and did not revisit the learning of the students at different times, the snapshots provided a picture that suggested development of understanding within these middle primary years of schooling.

Table 3
Stage of Development in Years 3 and 4 Across Studies

		Emergent	Construction	Understanding
Ross (1986)	Year 3	47%	40%	13%
(15 students)	Year 4	47%	6%	47%
Sierink & Watson (1991) (10 students)	Year 3	40%	20%	40%
	Year 4	30%	10%	60%
Berman (2001)	Early Year 3	71%	24%	5%
(21 students)	Late Year 3	67%	24%	10%
	Early Year 4	33%	38%	29%

Apart from one student, all students either remained within the same stage or moved to a higher one over the period of the study. Another student moved through all three stages over the three assessments, however, the other students moved up one stage over the twelve months.

Conclusions

As suggested by Sierink and Watson (1991) a collection of seven cognitive research tasks was able to discriminate levels of understanding of place value among a class of students. The responses to the seven tasks provided enough information from which to determine stage of understanding: as emergent, construction or established. They also provided a profile of skills and knowledge concerning conservation, counting, digit correspondence, and reading and writing numbers that could be used for informing instruction. The patterns of response across the three assessments in twelve months supported the structure of responses provided by previous research and confirmed that there seems to be a general pathway of development mapped out by these qualitatively different responses to the tasks. It was possible, using this framework to plot the development of understanding during this time. This work complements strategy-based frameworks such as those provided by Jones et al. (1996) and Wright (1998).

The mediation and the investigation of reasons behind responses in this study allowed the evaluation of the tasks and a validity check on them. In particular the conservation task was shown to have contamination of its validity due to the format and language of the task. The three digit correspondence tasks clearly highlighted problems in making assumptions about cognitive understanding from tasks that incorporate familiar material, such as the base ten blocks, for which procedural responses can be provided. This presents a caution for all assessment, that interpretation of cognitive understanding from tasks must do more than require behavioural responses.

This study confirms the assertion that students in mid primary years are grappling with the concepts involved in place-value numeration, particularly in respect to two-digit numbers. All the students could be classified within the stages of understanding; they moved between and within stages over the twelve month period. Although only 5% demonstrated understanding at the beginning of the study, 67% showed they had either developed or were constructing understanding by the end of the study. Of the students still seen to be in the emergent stage, development of early concepts was evident over the period and the zone of potential development was reduced for all students over that time. The pace of learning during the study, and the extent of learning evident during the dynamic assessment sessions, suggest that with explicit teaching of place-value concepts, the other students will be able to develop understanding during the middle primary years. The crucial time of this development is reinforced by this study. An implication from the study is that explicit teaching of place-value understanding at this time is appropriate and necessary. New curricula must explicitly address this domain of mathematics in cognitive terms as well as in terms of related skills and strategies.

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Teacher Beliefs: Probing the Complexities

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This paper describes a part of a study of the beliefs about mathematics, its teaching and its learning, held by secondary mathematics teachers. The case of one particular teacher is used to illustrate the difficulties inherent in the study of beliefs, and to highlight both the importance of data triangulation and the caution required in making inferences about a teacher's beliefs.

The importance of teacher beliefs has been acknowledged for some time and recently given prominence in policy documents and various numeracy projects. For example the Commonwealth Numeracy Policies for Australian Schools (DETYA, 2000) states that, "Teachers beliefs, knowledge, understandings and skills are critical factors in students' numeracy development." (p. 23), and highlights the importance attached to teachers' beliefs in both the Early Literacy Research Project, in Australia, and the Effective Teachers of Numeracy project in the U.K.

Difficulties in Researching Beliefs

Despite their importance, discerning an individual's beliefs is fundamentally difficult (Munby, 1982). Studies of secondary mathematics teachers' beliefs have tended to be either large-scale survey based comparative studies of the beliefs of various groups of teachers (e.g., Howard, Perry, & Lindsay, 1997; Tracey, Perry, & Howard, 1998) or case studies involving necessarily small numbers of teachers (e.g., Munby, 1982; Thompson, 1984). While most case studies have used at least two data collection techniques, principally interviews and observations, these have tended to be used in a circular fashion with one informing and guiding the other (e.g., Thompson, 1984). This has the advantage of allowing pertinent issues to emerge and to be explored at greater depth, but it also diminishes triangulation and hence reduces the internal validity of the findings (Cohen, Manion, & Morrison, 2000).

Conflicting Beliefs of Secondary Mathematics Teachers?

The relationship between teachers' beliefs and their classroom practice is controversial with many researchers reporting inconsistencies between the professed beliefs of teachers and their observed classroom practices (Cooney, 1985; Shaw, 1989, & Thompson, 1982 both cited in Thompson, 1992). Others such as Grant (1984, cited in Thompson 1992) and Thompson (1984) have reported a high degree of consistency. While Ernest (1989) and McLeod (1992) stress the relevance of contextual variables on the translation of beliefs into practice, Munby (1982) goes further, attributing apparent inconsistencies to an incomplete understanding of the belief system of the subject. Pajares (1992) also stresses that beliefs must be considered in terms of their connections with other beliefs rather than as isolated entities. In spite of this, studies to date seem not to have identified any specific beliefs that may be of particular relevance to teachers' practice.

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 139-146). Sydney: MERGA. ©2002 MERGA Inc.

Implications for this Study

This study was designed to explore the beliefs held by secondary mathematics teachers regarding mathematics, its teaching and its learning using a methodology that would maximise the validity of the findings. Specifically it employed not only multiple data collection techniques but also multiple informants, and was designed to identify beliefs that were centrally held by the subjects and hence most relevant to their practice. Crucially, classroom observations were not simply used to check the extent to which the teacher translated their beliefs into practice but were regarded as an additional data source that could contribute to a fuller understanding of the teacher's beliefs.

The Study

Subjects

The subjects of the study were 25 experienced secondary mathematics teachers teaching in rural 7-10 high schools in Tasmania. Eighty percent of the teachers were male.

Instruments

Beliefs survey. This required teachers to respond, on a five-point Likert scale, to twenty six items relating to their beliefs about mathematics, its teaching and its learning. The items were taken from similar instruments devised by Howard, Perry, and Lindsay (1997) and Van Zoest, Jones, and Thornton (1994) and were originally part of a forty-item survey that was shortened after use in a pilot study.

Constructivist Learning Environment Survey (CLES). Teacher and student versions of this instrument described in Taylor, Fraser, and Fisher (1993) required respondents to indicate on a five-point Likert scale, the frequency of various teaching/learning practices in their mathematics classroom.

Interviews and observations. Semi-structured interviews lasting from 45-60 minutes required subjects to: reflect upon their own experiences of learning mathematics; describe an ideal mathematics classroom and compare this with the reality of their own mathematics classes; respond to 12 statements about the nature of mathematics based upon the findings of Thompson's (1984) case studies of secondary mathematics teachers; and respond to a further 12 statements about the teaching and learning of mathematics derived from the same source. The 12 statements regarding the nature of mathematics consisted of 4 each that represented Problem Solving, Platonic, and Instrumentalist views of mathematics as defined by Ernest (1989), and the 12 statements relating to the teaching and learning of mathematics were similarly representative of three corresponding views of mathematics teaching and learning. Interviews were audio-taped. Classroom observations of approximately 12 lessons per teacher and further informal interviews were conducted over a period of several weeks.

Procedure

Teachers completed and returned the beliefs survey during the first few weeks of the school year. After a gap of several weeks they completed the teacher version of the CLES. They then gave the student version of the survey to students in at least two of their mathematics classes. Several teachers, including the subject of this paper, felt that their classrooms were different for different classes that they taught and so opted to complete two of the teacher version of this survey; one in relation to each of the two classes to which they gave the student version.

Preliminary analysis of the data from the surveys was conducted during the winter school term. Eight teachers were selected from those who had indicated their willingness, to be interviewed and possibly observed. The teachers chosen represented a range of positions with respect to their espoused beliefs as reported in the surveys.

Interviews were conducted in early October and observations of mathematics lessons of six of the interviewed teachers occurred throughout November and December. While the interviews included responses to sets of statements that represented particular perspectives with respect to mathematics, its teaching and its learning that have been described in the literature, it was not the aim in presenting these to classify the teachers, but rather to provide prompts to which they could respond freely in the hope of obtaining an authentic understanding of their beliefs. To this end also, the researcher did not consult the survey data of the interviewees either prior to or during the interview and observation phases of the study. Further informal interviews were conducted throughout the observation period.

Following completion of the data gathering process interviews were transcribed and analysed, at first independently of the survey data. Issues that emerged as important to each respondent were listed and beliefs about mathematics, its teaching and learning inferred from these. The transcripts were then analysed again in order to find statements supporting, contradicting or clarifying subjects' belief survey responses. Detailed notes made during and immediately following each observation period and the teacher and student CLES data were similarly analysed.

Results and Discussion

The results for one teacher, Nick, are presented and discussed here. Nick had been teaching secondary mathematics and science for 14 years, in two high schools. He had been in his present school for 5 years and was teaching 4 mathematics classes and 2 science classes. He had studied mathematics at tertiary level for 2 years as part of his B.Ed.

Beliefs Survey

Nick's survey score indicated that he, more than any other of the teachers surveyed, had a view of mathematics characterized by Ernest (1989) as problem solving, and held beliefs about mathematics teaching that could broadly be described as learner-focussed (Van Zoest et.al., 1994). His beliefs about mathematics learning were inclined towards constructivism. Consistent with this, he indicated strong agreement or agreement with statements such as:

1. Ignoring the mathematical ideas that children generate themselves can seriously limit their learning;
2. A vital task of the teacher is motivating children to resolve their own mathematical problems;
3. Providing children with interesting problems to investigate in small groups is an effective way to teach mathematics;
4. Mathematics is a beautiful, creative and useful human endeavour that is both a way of knowing and a way of thinking;

5. It is important for children to be given opportunities to reflect on and evaluate their own mathematical understanding;
6. Knowing how to solve a mathematics problem is as important as getting the correct answer;

And he disagreed, or strongly disagreed, with statements such as:

7. It is not necessary for teachers to understand the source of children's errors; follow-up instruction will correct their difficulties;
8. Listening carefully to the teacher explain a mathematics lesson is the most effective way to learn mathematics;

However, he also indicated agreement with:

9. Telling children the answer is an efficient way of facilitating their mathematics learning;
10. It is the teacher's responsibility to provide children with clear and concise solution methods for mathematical problems;
11. It is important that mathematics content be presented to children in the correct sequence.

And agreed strongly with:

12. It is important for teachers to understand the structured way in which mathematics concepts and skills relate to each other.

It is possible that with respect to item 9, above, that Nick focussed on the time aspect of efficiency but did not believe this strategy to be particularly effective. Items 11 and 12 suggest Platonist leanings and, along with item 10, which contradicts item 2, would place him closer to this perspective regarding the nature of mathematics than would be the case had he disagreed with these items. It is also possible that his agreement with both item 2 and item 10 indicates some tension between the way in which Nick would like to teach and the way in which he believes he should, or is expected to, teach. This hypothesis is also supported by the fact that he was undecided about the following items:

13. There is an established amount of mathematics content that should be covered at each grade level;
14. Mathematics material is best presented in an expository style: demonstrating, explaining and describing concepts and skills;

The survey, of course, provides no way of knowing if, or to what extent, Nick is aware of these potential inconsistencies.

CLES

There was considerable agreement between both of the surveys completed by Nick and those completed by his grade 10 class and grade 8 class. In fact Nick's two responses to each item and the average responses for each of the two classes vary by only 1 point on the Likert scale for every item except one. This exception was in regard to the statement "I don't pay attention to other students' ideas". The average grade 10 response was "Often", while the average grade 8 response was "Seldom". Nick agreed with the grade 8 class, but chose "Sometimes" in relation to the grade 10 class.

The consistency of this data suggests that Nick's perceptions of what is taking place in his classes are relatively reliable and that, in spite of his initial assumption that his grade 8 and grade 10 classes were significantly different (which lead him to complete two surveys rather than one), they were in fact quite similar even in his own assessment.

Interview

In reflecting on his own experiences of learning mathematics Nick claimed to have found mathematics, at least up to grade 8, challenging but enjoyable. Beyond this it became more difficult and less enjoyable. Nick attributed his reduced enjoyment to the increasing irrelevance of the content. In relation to his secondary school experience he said:

> … obviously most of the stuff that we learned past grade 8 was theoretical stuff… so I probably didn't tie it up as being really relevant, um, because I mean most of the maths that we do past grade 8 isn't that relevant, …

In relation to his studies of mathematics at university he said:

> … particularly the number theory, just showed how irrelevant that kind of stuff is to real life, talking about imaginary numbers and those sorts of things, just doesn't have any use at all, so, it just drove home to me that people learn all this stuff for no real point, ...

Interestingly, Nick did not attribute his earlier enjoyment of mathematics to its greater relevance at that time, but rather to the sense of achievement he felt at being able to reach a final definite answer.

Nick made many other references to the irrelevance of most of mathematics, often mentioning grade 8 as a pivotal year, and linking it to the problems he experienced in motivating classes in higher grades, and his frustration at the lack of progress many of his students appeared to make in mathematics. He also made further references to mathematics being an exact discipline with definite answers, describing mathematicians as narrow-minded in trying to "always get a particular answer, or try and zone it down."

However, when responding to the statement, "The content of mathematics is 'cut and dried': Mathematics offers few opportunities for creative work." Nick disagreed, saying:

> … obviously maths is involved with many other … disciplines, so for example music, so music can be seen as a form of maths, so therefore it is creative, and it's involved with all sorts of other things that aren't strictly maths but they use mathematical ideas …

In discussing his teaching Nick referred to difficulties in motivating students, particularly those in higher grades and those he described as lower ability. He seemed to accept that such students would not like mathematics or see it as in any way relevant, and described attempts to motivate them by telling them that good marks in mathematics are important for finding jobs or accessing future courses or that it "trains your mind to think logically" as "a bit of a con" that the students could readily see through. In the course of this discussion he described his own perception of his job as a teacher as developing relationships with students that would result in them staying on task in order to please the teacher. He felt that this particularly applied to lower ability students.

Nick described an ideal mathematics classroom as one in which the students could see the relevance of the work they were doing and hence were motivated and on task. Time would be unlimited so the teacher would rarely need to intervene and the students could learn from their mistakes rather than being told the answer. He spoke at length about the need for very large classrooms, at least three times the size of a standard classroom, so that students had space to spread out resulting in a lower noise level, and there would be room to store materials like cardboard and scissors. He also attached importance to the room being a place where students wanted to be and could feel a sense of ownership.

Although Nick attributed many of the differences between the actual and ideal situations to physical factors, he saw the most significant obstacle as the intellectual limitations of the students and related this to their lack of motivation, again referring to grade 8 as a pivotal year:

> … the huge problem of kids having reached grade eight and, and probably having reached the cognitive level that they're going to be for the rest of their life and still force feeding them maths for two years, even though they'd much rather be outside doing … and you know, it's much more inviting, so you've basically got a huge problem in terms of trying to motivate kids that just don't want to be there and probably aren't going to be able to learn that much more than they already have … that would be the biggest problem, for maths.

Nick also attributed many of the difficulties he experienced to societal factors including the relative economic depression of the region; the high proportion of aboriginal students enrolled in the school; poverty; alcohol and drug abuse; and the devaluing of education in many families and the community generally. Having spoken at considerable length about these issues and what he believed to be their increasing prevalence, Nick offered the following, unprompted, perhaps indicating a degree of tension or even guilt in locating the source of all the problems outside of himself and his own practice:

> … because, I mean, I don't necessarily not blame myself. I think as you get older you see things definitely differently, but I definitely think that it, it's mostly to do with kids changing, social things changing rather than myself.

Despite the negativity of much of what he said, Nick seemed quite relaxed and cheerful, and did not hesitate to agree to having his classes observed.

Observations

Nick taught all of his mathematics lessons in the same classroom. A colourful number line spanned the wall above the blackboard, and a number of cardboard polyhedra hung by strings from the ceiling. A number of student-made games including a dart board and a marble maze were either hanging or leaning against the walls at various points.

There were no formal starts or finishes to any of the observed lessons, but students simply entered the room, with their school bags, sat down and started their work as they were ready. Occasionally Nick reminded them to go on with what they were doing last time, but in most cases they began relatively promptly without instruction. Nick's grade 9 and grade 10 classes were all working from the same worksheets copied from a text, on statistics. Nick moved around the classroom assisting students individually or in pairs, often for 5 minutes or more, and largely ignoring those who were off task, provided the noise level was relatively low. He often used humour, occasionally sarcasm, to cajole students into doing the work. Occasionally he raised his voice in response to student misbehaviour and on one occasion sent a student to time out. The overall atmosphere of his lessons was friendly and very relaxed.

In the first lessons observed the grade 8 class were working in groups of two or three, designing and making games based on chance, that they would show and explain to a grade 7 class in a subsequent lesson. These were the games evident around the room. Some of the students who had finished constructing their game were trialling them and collecting data regarding the frequency of various possible outcomes. In later lessons they were working individually or in pairs to draw floor plans of their dream home given certain constraints on size and functionality. They would then construct three-dimensional models of the houses. In one lesson, two boys whom Nick described as "very bright" were also working on planning the wiring for their house as they were studying electricity in science (also with Nick).

Nick sometimes told students the answers to their questions but at least as often attempted to help them to understand what they were doing. Several cases were observed in which Nick began by attempting to help students to solve difficulties for themselves but ended up telling them the answer when other students demanded his attention or he became frustrated with the student's apparent disinterest in understanding the mathematics. On several occasions he asked students who had not sought help how they got the answer to a previous question. The responses of students in these situations suggested that this was a common though not necessarily welcome occurrence.

In conversations during and after lessons Nick reiterated all of the concerns and attributions that he expressed in the interview. He also revealed that both the number line and the polyhedra were produced by his grade 8 class, and that he was quite an accomplished pianist.

Nick's Beliefs

The interview and observations provided considerable evidence that he does indeed hold the beliefs he claimed to in the initial survey, however it is clear that the survey data do not provide the complete picture. For example, it seems likely that in strongly agreeing with the statement numbered 4, above, he had in mind fields such as music. Nick also seems to hold at least two beliefs that were not tapped by the survey, but yet appear to be fundamental to his teaching, namely:

1. Students realise their cognitive potential by the end of grade eight.
2. Students do not like (school) mathematics because it is irrelevant.

Nick also showed some awareness of the tension in relation to items 2 and 10 of his beliefs survey. While one could attribute this to the difficulties of managing a class of 25 students, his lack of perseverance in relation to allowing students, particularly in higher grades, to resolve their own difficulties and in helping them to understand the work are logical consequences of the first of these fundamental beliefs. The two beliefs together also account for the contrast in teaching methodologies employed in grade 8 compared with his other classes, and the second may also have resulted in Nick's emphasis on relating well to students as a means of motivating them in what he perceives as the absence of anything intrinsically motivating in the content of mathematics courses beyond grade 8.

While it seems that the recent reform agenda in mathematics education has impacted upon Nick's teaching in lower grades it appears not to have significantly influenced his teaching of higher grades. Clearly unless Nick believes that older students are able to learn mathematics he will be unlikely to make efforts to improve his teaching of them. He is also unlikely to attempt to provide these children with motivating and relevant mathematics problems while he believes that they do not exist. In short, unless these fundamental beliefs are addressed, professional development activities, particularly those relating to the teaching of senior students, are likely to be ineffective.

Concluding Comments

The results reported here support Munby's (1982) assertion that apparent conflicts between teachers' beliefs and practice are evidence of the researcher's incomplete understanding of the relevant beliefs, as they illustrate that the use of data from multiple sources can reveal beliefs with which practice is in fact consistent. They also suggest that the methodology employed in this study may be able to shed light on some of the fundamental beliefs teachers hold that may account for failure of repeated attempts to reform mathematics education (Southwell, 1995). Once identified efforts to modify them can be better targeted.

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Curriculum and the Reality of Primary Teachers

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Over the years, national mathematics curricula have tended to be imposed on New Zealand teachers without regard to their realities. As a means of (potentially) informing future curriculum development the study reported here sought to do two things: (1) identify the actual concerns held, and mathematics teaching difficulties encountered, by classroom teachers, and (2) find ways to support the teachers in addressing these concerns and difficulties. The six month collaborative project with staff in a small urban primary school revealed some insights into difficulties experienced by teachers, the process of teacher professional development in mathematics, and possible directions for mathematics curriculum development.

Just eight years ago Ellerton and Clements (1994) characterised politically-driven mathematics curriculum changes in Australia at that time as a "... national curriculum debacle". New Zealand has had a similar experience. The last mathematics curriculum documents developed democratically with primary teachers were published in 1985 and 1987 (Department of Education, 1985, 1987). These documents had been in the hands of teachers only a short time when the New Zealand Ministry of Education required them to implement a new mathematics curriculum (Ministry of Education, 1992). This new curriculum document was written under private contract and, although the contractor and his team did their best within serious constraints, the resulting 'Mathematics in the New Zealand Curriculum' has been characterised as a document of contradictions (Gillingham & Begley, 2001) which can be seen to exist within a broader curriculum of social control (Codd, 1999).

In our experience this traditional and dominant process of central control, curriculum imposition, and avoidance of rigorous critique (Klein, 1990) is still being used in New Zealand, although there are strong indications that some Ministry of Education staff recognise that teachers are central figures in curriculum development considerations. In this respect, Gillingham and Begley (experienced primary school teachers) have recommended, in curriculum development efforts in mathematics, "... cognisance must be taken of primary school teachers' actual classroom reality, particularly their understanding of mathematics and their teaching of it" (2001, p. 44). This is similar to the conclusion reached by Harnett and Naish (1993, p. 346), "... there can be no educational advance without classroom teachers. They do the work of education and their knowledge, imagination, sensitivities, and skills are central to that advance. And that means that teacher development is too."

We therefore considered it important to try to understand the mathematics teaching world of classroom teachers. This paper describes some first steps in this project emanating from an opportunity to engage with a small sample of teachers in mathematics professional development.

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Theoretical Perspective

Our theoretical perspective encompassed ideas about curriculum, learning and professional development. With respect to curriculum, we believe that it is "... a socially-constructed cultural artefact" (Carr, 1998, p. 330), that in the final analysis it is constructed in the classroom by teachers and students (Chandler, 1992; Remillard, 1990) in a negotiated, co-emergent way (Davis, Sumara & Kieren, 1996), and that its primary purpose is to promote (participatory) democratic values (Biddulph, Taylor & Biddulph, 2000; Carr, 1998; Ministry of Education, 1992). The implication of this broad view of curriculum was that we conceived our role to be one of supporting the teachers to see themselves as classroom curriculum developers in their own right, rather than as mere implementers of someone else's curriculum.

In terms of learning and professional development, we construed teacher professional development as adult learning. Although our recent ideas about learning had their genesis in social constructivism and humanistic learning theory (Biddulph & Carr, 1999) they have increasingly been informed by the principles of enactivist learning. We have therefore used the conceptual framework of enactivism (Davis, 1996; Davis & Sumara, 1997; Reid, 1996,) to both shape our work with the teachers involved in our study, and to interpret our data. We outline some features of enactivism below.

Enactivism presents an ecological view of learning; that is, all learning is considered to be interactive, interrelational and interdependent, and is influenced by biological, social and historical factors. Learning is not about getting knowledge across the big divide that Descartes believed separated the mind from the body (Lakoff & Johnson, 1999) but rather it is about making sense of our actions and interactions – with others, with our environment, and with our personal histories. In essence this means that the world comes into existence for us only in and through our ongoing and ever-changing interactions with, and interpretations of, our physical and social environment. As such, a classroom (and indeed a school staff group) has the property of being an 'autopoetic' system, that is, a complex, dynamic and spontaneously self-organising system (Reid, 1996). It develops and changes as a whole rather than on an individual basis within that setting. In practice, enactivism involves working in a collaborative, collegial-type way and listening constructively (Davis, 1996; Mason, 1998) in order to sensitively support and challenge perceptions.

In adopting an enactivist perspective we considered that we were likely to learn as much from our interactions with the teachers as they from us (Dawson, 1999). In addition, in terms of professional development, we recognised that providing support over a relatively extended period of time, offering alternatives for consideration, and encouraging the teachers to be thoughtfully reflective were all consonant with our theoretical framework of democracy and enactivism.

Methodology

This study involved both professional work with teachers in a school, and concurrent collection of qualitative data related to that work. The data consisted of field notes written by the authors during the meetings with the teachers, together with written records of joint work. The field notes and written materials were regularly compiled into Working Papers, many with the teachers as joint authors, which were shared with all participants as they were produced. A total of 11 Working Papers resulted from the collaborative work. This paper draws extensively from those Working Papers.

The principal of a small urban primary school in a New Zealand provincial city invited us to support the school staff in their mathematics professional development efforts. There were six main classroom teachers in the study and all participated most of the time. The principal was also able to join in most of our deliberations. The teachers represented a range of experience from very little (being first or second year teachers) to very experienced. At least initially, they also appeared to vary in their commitment to the professional development programme.

The children attending the school were from a low socio-economic area of the city, and comprised a cultural mix with more than half being Maori. There was a high proportion of transient children. The school catered for children from Years 1-8 (approximately 5 to 12-year-olds).

Meetings with the teachers occurred at least fortnightly for most of the time, and covered a six month time span from roughly mid-May to mid-November 2001. The meetings were normally held from 3:30pm until 5:00pm on the day set aside for a whole staff meeting. The meetings began with the teachers sharing with us their concerns and difficulties, but gradually moved on to pairs of teachers and university staff trying to identify key mathematics ideas across the school class levels in the areas of number, algebra, measurement, geometry, statistics and probability. During this time, all participants came together for the last part of each session to share, discuss and question what they had managed to do to date.

Prior to our first visit to the school we discussed among ourselves what our part would be during the meetings, given our theoretical perspective and our desire to achieve democratic-type power relationships with the staff. This philosophy pointed to us being teacher colleagues of the school staff rather than outside 'experts', handing over to the teachers as much control and responsibility for the professional development process as possible (Dawson, 1999), and acting as their assistants by keeping records of our deliberations. We therefore implemented these features. We have all had ongoing experience working in classrooms with children so we could identify fairly readily with what the teachers were saying and how they were feeling. In addition, the first author had already built up a close working relationship with several of the teachers having had several groups of preservice teachers working with children in their classes in mathematics over the previous 12 months.

Results and Discussion

Our data revealed a number of major themes. Here we describe and discuss five of them. Frequently our text includes terms used by the teachers, but we do not highlight these. Rather, in keeping with out theoretical perspective, we prefer to let the authenticity of the teachers' voices emerge 'naturally'.

1. Need to understand the mathematics and gain a sense of 'progression'

From our first meeting, the importance of needing to understand the mathematics emerged several times in the teachers' comments. At the outset they acknowledged that teacher knowledge of mathematics is important and they stated explicitly that they felt they could benefit from gaining further understanding. The need to gain more understanding of the mathematics was also reflected in questions asked by the teachers several weeks into the professional development programme. For example, one teacher's question about whether fractional numbers could be 'even' numbers led to an exploration of the various number systems, which in turn led to an interesting discussion about the fact that the sum of 1 + 1 could be numbers other than 2, such as 10 (base 2) or 0 (modular system), something that the teachers had never considered. This exploration, and other similar ones, also represented our quiet challenging of the teachers' belief that there is just one right answer to any mathematical problem.

The importance of understanding the mathematics also came through in the teachers' desire to gain a sense of 'progression' of mathematics ideas across the school. This was raised in the first professional development session and became an ongoing theme - to the extent that the last four months of the programme were devoted to identifying the mathematics ideas that they could reasonably expect most children to have developed in number, geometry, algebra, measurement, statistics and probability by the end of the 2nd, 4th, 6th and 8th year of their schooling. This seeking of a sense of 'progression' stemmed partly from the teachers urgent need to cater mathematically for the numbers of transient children who appeared in their classes.

In the midst of our work identifying the main mathematics ideas (we worked in teams of two, one teacher and one of us on each mathematics strand) several interesting developments occurred. For instance, most of the teachers gradually became involved to the extent where they were genuinely seeking to understand the mathematics more deeply. An experienced teacher working with the first author at one point made the comment that it was all coming back to her now. What she meant was that the mathematics ideas which she had been familiar with in the 1980s were re-emerging in her consciousness as we worked at identifying the major number concepts at the primary level. This was intriguing to say the least! Why had she 'lost' this understanding? Discussion with this teacher and the others revealed that they found the work of identifying the key mathematics ideas particularly rewarding, and they were feeling a sense of relief at beginning to get a sense of the big picture of the mathematics ideas - after having been buried for years in the perceived myriad of learning outcomes and achievement objectives in the mathematics curriculum document (Ministry of Education, 1992). One used the metaphor of previously not seeing the mathematical wood for the achievement objective trees.

2. Complexity of teaching mathematics

The teachers discussed how teaching children can be a real challenge, particularly the children in their school, many of whom apparently lack motivation, are in emotional turmoil, and appear to have big 'holes' or gaps in their mathematical understanding. The teachers were also concerned that they did not have quick ways of identifying such 'holes' and indicated that they were looking to us for guidance. In the end we were able to help them develop their own solution to this problem, namely the 'progression' of mathematics chart mentioned above. At the same time they recognised that if they really listened to children's ideas and determined the strategies they were using then it could have the effect of raising expectations of what the children are capable of understanding. They also began to recognise that teacher questioning improves when teachers know where they are going mathematically, that is, when they know the mathematics and are aware of the 'progressions'. They realised too that encompassing both of the above enables them to have more meaningful interactions with the children.

Early in our meetings the teachers commented on the confusing array of resources in mathematics education, and some added that they had found safety in one commercially produced, structured mathematics programme. However, they did express some reservations about whether it was actually meeting the learning needs of their children.

A growing understanding that children's mathematics learning is not instant, that it takes time for ideas and intellectual strategies to become stable, that sense-making is at the heart of mathematics learning, and that it is therefore possible to begin with problems rather than so-called skills, led one teacher on her own initiative to develop a mathematics unit along these lines. The unit was not planned in its entirety in advance; rather aspects evolved as children's input was sought. This teacher also came to realise that periodically stepping aside from the course of the problems in the unit to give children specific help with aspects of the mathematics was still an important part of teaching.

Overall, the teachers came to express the view that if teaching is constantly informed by children's ongoing learning and learning needs in mathematics then it is a complex process. This differed from their earlier perception that teaching mathematics involved pursuing the behaviourist-type objectives (translated into measurable learning outcomes) found in the current New Zealand mathematics curriculum document.

3. A critical approach to curriculum documents is needed

As early as the third professional development session, the teachers were beginning to express concerns about the achievement objectives in the New Zealand mathematics curriculum document and considered that they could be a hindrance to thinking about and teaching mathematics because they seemed to be either too broad or too specific to provide the guidance they sought. These concerns were confirmed in the teachers' minds as they tried to make use of the curriculum document to help identify the key mathematics ideas that children could be expected to develop at two-year intervals across the school. They found that the document had statements whose meaning eluded them, or important ideas were obscured in the verbiage, or key ideas were omitted entirely (for example, in the algebra strand).

These experiences suggested to the teachers that they should not accept the mathematics curriculum document as some kind of 'bible'. Instead, in the interests of their children's mathematics learning, they considered that they should approach it quite critically and think about the mathematics in their own terms.

4. Growing confidence

The teachers' growing confidence in themselves as professionals emerged in several ways. For example, as mentioned above, they began to feel that it is imperative that they do not accept without question mathematics materials, official or otherwise; they now realised the need to adopt a far more critical stance, even though it felt rather daring at first. This developing understanding, however, also enabled the teachers to observe that, whether it is officially recognised or not, true curriculum development is really a shared effort between the Ministry of Education and themselves as front-line professionals.

Other evidence of growing confidence was the recognition that they had a lot of expertise among themselves. They began to feel comfortable about both sharing their ideas and asking questions to further their understanding. Interestingly, they considered that their expertise was complemented by our expertise. It is noteworthy that the format for the display of key mathematics ideas across the school was constructed by one of the teachers. Almost all of the teachers found it to be a powerful format in that it enabled a vertical view of what mathematical understandings they might expect from children in two-year age bands, and at the same time permitted a horizontal perspective of the 'progression' of mathematics across their primary school in terms of the strands of number, geometry and so forth. As one teacher said, without the sense of 'progression' provided by the chart, it is all too easy to become narrowly focussed on just the mathematics that is supposedly relevant for your single class.

Further evidence of growing confidence was the remark of one teacher (who gradually became an enthusiastic participant) that investigating one strand in depth provided an experience of how it might be possible to focus on the other mathematics areas in the same kind of depth.

5. The professional development process

As a result of their experiences in the six month professional development programme the teachers provided a number of insights into the development process itself. We have already mentioned the growing teacher realisation that they had considerable expertise to contribute, particularly relating to the reality of their children and their classrooms, and how they felt that this complemented the expertise that we ourselves provided. In their eyes, we were able to access relevant articles and mathematics education resources, and provide personal support and encouragement to enable them to make decisions about mathematics education in their school appropriate for their children.

The teachers said that their growing realisations were helped considerably by having their ideas recorded and available to them regularly in Working Paper form. This surprised us because we had never thought about this possibility. From the teachers' comments it seems that the Working Papers gave legitimacy to their ideas, facilitated reflection of the issues involved, and evoked ideas about further development.

We also learnt from the teachers that they now favour longer-term professional development support (of the kind we had engaged in) because they found that it allowed them to explore in some depth issues that matter to them. On reflection this makes good sense because, as we recognised, the teachers carry all sorts of other responsibilities, and the call on their time is such that to ask them to be totally focussed during a short professional development course is somewhat unrealistic.

Furthermore, most of us concluded that professional development is quite complex because as it evolves we become aware of new dimensions that need to be examined; hence the professional development programme can take new and unanticipated directions. In our case it was clear to the teachers that we had still not addressed the issues of mathematical processes (especially problem solving), assessment, and effective ways of helping children learn the major concepts that were identified.

Conclusion

Although somewhat sketchy, we think that our limited data are quite illuminating. For example, there is a strong indication that curriculum change that proceeds without research into teacher and classroom reality runs serious risk of being ineffective. It is somewhat ironic that previous Ministry of Education curriculum initiatives intended to increase children's achievement, seem to have had the effect of stifling teachers' professional understanding and turning the teaching of mathematics into an extremely technical and problematical task.

The complexity of curriculum development also surfaces. We agree with the teachers that effective curriculum development requires a partnership between education authorities and teachers (Fullan, 1993). Our data suggest that, at the very least, teachers need to be consulted about the structure of future mathematics curriculum documents to determine whether the teachers are likely to take ownership of them in ways that do justice to children's learning in mathematics.

The complexity of professional development is also evident. The insights provided by the teachers themselves on this issue are significant, in our view. From an enactivist perspective, the evolving, recursive (Pirie & Martin, 2000) and reflective (Irwin, Britt & Ellis, 1993) nature of professional development apparent in this study, and the need "... to acknowledge the authenticity of the context in which teachers work" (Moon, 1998), stand in stark contrast to the packaged, contract delivery model so favoured in New Zealand over the last 10 years. To add to the complexity, we recognise the importance of the conclusion reached by Tzur, Simon, Heinz and Kinzel (2001, p. 250) that "Teachers' perspectives consist of intricately related webs of understanding of which little is known, particularly as these perspectives have evolved in relation to current mathematics education reform efforts." Perhaps the enactivist foundation of our facilitation provides an alternative approach to professional development in mathematics education that needs serious consideration. In our experience it provides for the significant features of successful professional development identified by Garet, Porter, Desimone, Birman and Yoon (2001), namely (i) a sustained and intensive programme, (ii) collective participation of teachers from the same school, (iii) encouragement of professional communication among teachers, (iv) opportunities for active learning, (v) activities linked to teachers' experiences, and (vi) a focus on content knowledge.

Acknowledgments

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Mental Computation Competence Across Years 3 to 10

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Mental computation is an important aspect of numeracy. There is little information available, however, about appropriate sequences of learning, or what can be expected of students in different years. The responses of 3035 students in years 3 to 10 to linked tests of mental computation were placed on a single scale using Rasch modelling techniques. An eight level scale of mental computation development was identified, and student achievement across the years was mapped onto these levels. The findings and their implications for the classroom are discussed.

There is an increasing emphasis on the place of mental computation in today's classrooms as part of a suite of computational strategies that students should develop over the years of schooling (e.g. Australian Education Council, 1990). This emphasis implies that teachers will monitor students' progress in mental computation. There is little research base, however, to inform the assessment process. This leads to varying expectations as to what students should be expected to be able to do at different stages of education. At the end of year 2, for example, the Victorian Curriculum and Standards Framework (Board of Studies, 1994) expects automatic recall of addition and subtraction facts up to 20, whereas at the same stage of schooling the Tasmanian Key Intended Numeracy Outcomes (Department of Education, Community and Cultural Development, 1997) expect addition and subtraction of numbers to 10 and extensions based on place value, with only some facts committed to memory. The National Numeracy Benchmarks for year 5 require basic multiplication facts to 10 x 10 and some extensions of these facts (Curriculum Corporation, 2000) and these same outcomes are expected for most students by the end of year 4 in the New South Wales Outcomes and Indicators statements (Board of Studies, 1998). There is very little expectation for mental computation outside of the "basic facts" for whole numbers, and none at all for decimals, percentages and fractions.

The research base that exists is generally based on small interview-based studies that focus on strategy use rather than outcomes. In one larger scale study, Bana and Korbosky (1995) used sets of ten "basic facts" in each of the four operations with 390 students across years 3 to 7. Using raw scores as the basis they demonstrated an increased performance of students from years 3 to 4 but a much smaller increase between years 5 and 6. This did not, however, provide generalisable information about generic kinds of mental computational problems that students might be expected to be able to solve at different stages of schooling. McIntosh, Nohda, Reys and Reys (1995) used a test of mental computation in a study across three countries at four different year levels. Although there were some common items, they were unable to compare years directly since the test administered to each year was different. There was some evidence of growth across years from performance on the common items, and also of curriculum effects across the three countries. Callingham and McIntosh (2001) used Rasch modelling techniques (Rasch, 1980) to produce a developmental scale of mental computation based on the responses of nearly 1500 students across years 3 to 10. Although an initial comparison across years was undertaken, no attempt was made in this study to suggest reasonable expectations for

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students' outcomes in each year. The study did, however, establish Rasch modelling as an appropriate approach to the problem associated with using a large item pool across multiple years.

The aim of this current study was to confirm the developmental nature of the mental computation construct, and to establish reasonable expectations for student outcomes in each year from 3 to 10. To this end the research questions were:

1. What are the features of a hierarchy of mental computation development?
2. What happens to the performance of students as they move through school?
3. What are reasonable expectations of mental computation achievement for students in years 3 to 10?

Methodology

Tests of mental computation were developed for adjacent years, years 3/4, years 5/6, years 7/8 and years 9/10, based on the tests used in a previous study (Callingham & McIntosh, 2001). Since increasing the wait time did not make the item easier, it was decided in this study to reduce the pause between questions from five seconds to three seconds for the first part of each test (*Short* items) and to maintain the 15-second pause for the second part of the test (*Long* items) to differentiate between "instant recall" and "working it out" approaches. In addition, nine items that were common across all year levels of the test were presented with a 5-second delay to enable linking to the previous study (*Link* items). Items were recorded and schools provided with audiotapes of the appropriate tests for each year that also contained all instructions for students. In this way the item presentation was consistent across all years. The Short and Link items comprised the first part of the test, with a short break before the Long items were administered. Students in years 3 and 4 were given slightly fewer items than older students to minimise test fatigue. Students were provided with an answer sheet and a pencil only, and instructed that they were not to write anything other than the answer. The test formats are summarised in Table 1.

Table 1
Test Formats for Mental Computation Tests

Year	3/4	5/6	7/8	9/10
Short items	24	24	24	24
Link items	5	5	5	5
Long items	21	36	36	36
Total	50	65	65	65

Each year level test had four forms linked within years and between years by common items. This allowed a large pool of items to be presented. In all 244 items were used, 87 Short, 9 Link and 148 Long items. These addressed whole number operations, fractions, decimals and percentages. Each year level test contained mainly items that were considered to be appropriate to the particular year level and the tests were organised by item topic, addition, subtraction, multiplication and division, rather than using a mixed operation format, following methodology from previous research studies (McIntosh, Bana and Farrell, 1995).

The tests were administered to 3035 students in 8 primary, 2 high and 2 combined primary/high schools in the Tasmanian Government and Catholic sectors and the ACT Government sector. The distribution of students across years is shown in Table 2.

Table 2

Number of Students Across Years

Gr3	Gr4	Gr5	Gr6	Gr7	Gr8	Gr9	Gr10
436	441	420	445	403	344	343	203

The data were scored as correct or incorrect; items presented to a student but omitted were scored as incorrect. Rasch analysis of the data was undertaken using Quest v2.1 software (Adams & Khoo, 1996). Initially, the Link items were calibrated as an anchor file. In the subsequent analyses, the tests were simultaneously calibrated and equated, anchored to the Link items. This placed all items and students on a single scale; the same metric, the logit, is used to describe the difficulty level of the item and the ability of the student (Callingham & McIntosh, 2001). Ability measures for each student were obtained as a basis for further analysis.

A variable map produced by the analysis showed the distribution of the items and students along the mental computation variable. Item clusters along the variable were interpreted in terms of their content and skill demand. Points along the scale where there was a qualitative change in the demands of the item were identified as thresholds between levels of different mental computation competence. In this way eight levels of competence could be described.

Results

Hierarchy of Mental Computation Development

A content and skills analysis of the item clusters suggested eight levels of mental computation competence. Levels 1 to 3 address only whole number computation, and, apart from "one half add one half", Level 4 also addresses whole number. These first four levels generally reflect curriculum emphases in the lower primary years, except for the early emergence of doubling and multiplication by 10. Simple decimals and percentages begin to appear in Level 5, together with addition and subtraction of halves and quarters. Levels 5 and 6 are where all basic multiplication facts and inverses become secure. In Level 6 the whole number emphasis is on more complex 2-digit computations. Addition and subtraction of decimals with one decimal place appears at this level, as does addition of halves and quarters, including simple equivalences. Unlike whole numbers where addition is typically one or two levels earlier than related subtraction computations, addition and subtraction of decimals and fractions begin to appear at the same level. Level 7 includes a range of complex computations, such as multiplication of 2-digit by single digit whole numbers, multiplication and division of unit fractions and whole numbers by one-half and quarters, and calculation of key percentages. Level 8 predominantly comprises decimal, fraction and percentage computations, and particularly division calculations. These levels are summarised in Table 3.

Table 3
Levels of Mental Computation

Level	Content/Skills	Example
1	*Whole Number*: Add single digits; Add/subtract 0; Double single digit; Multiply single digit by 10; Double small multiples of 10	5+3(S); 7+6 (L); 6+7 (L); 8+0; 4x10; 8x10; Double 30
2	*Whole Number*: Add larger single digit numbers; Add/subtract single digit to/from 2-digit (no exchanges); Add small single digit to 2 digit over next 10; Small multiples of 3, 4 & 5	9+8; 8-3; 3+28; 72+6; 36-5; 4x3; 6x5
3	*Whole number*: Basic addition facts to 20/related subtraction facts; Add single digit to 2-digits; Add/subtract multiples of 10 (total <100); 3, 4, 5 tables; Double simple multiples of 10 and 5; Simple halves <20	14-8; 11-4; 4+39; 16+9; 20+70; 80-30; 3x6; 8x4; 15x2; 40x2; Half of 18
4	*Whole number*: Add two 2-digit numbers (total < 100); Subtract single from 2-digits; Subtract simple 2-digit numbers (no exchanges); Related division 3, 4, 5 tables; *Fractions*: 1/2 + 1/2	42+16; 27+25; 36-8; 43-12; 30÷5; 12÷4; 23x3; 12x4; 1/2 + 1/2
5	*Whole number*: Table facts/inverses; Add/subtract two 2-digit numbers; Multiply 2-digit number by 10 and 2; *Decimals , fractions & %*: Add/subtract halves & quarters (Total<1); 50%/100% of even 2-digit number	9x8; 72÷8; 79+26; 65-35; 27x10; 40x5; 1/2 + 1/4; 0.25+0.25; 2/7 + 3/7; 50% of 24
6	*Whole number*: Table facts/inverses; Add 2-digits to a 2- or 3-digits; Subtract 2-digits from 2-digits; Multiply 2-digit by small single digit; Halve even 2-digit number; *Decimals, fractions, & %*: Add/Subtract decimals (1 place); Add halves & quarters beyond 1; Subtract familiar unit fraction from 1; Half/quarter of some 3-digit numbers; 25% of some 2-digit numbers	54÷9; 68+19; 92-34; 105-97; 60x7; 5x24; Half of 76; 0.3+0.7; 1-0.6; 1/2 + 4/8; 1/2 + 3/4; 1-1/3; 1/4 of 120; 25% of 80
7	*Whole number*: Subtract 2-digits from 2 & 3-digits; Multiply 2-digit by single digit; Divide 2-digits by single digit; Multiply some 2-digit numbers by multiple of 10; *Decimals, fractions, & %*: Add/subtract decimals (1 or 2 places); Multiply 2-digit number by 0.1/0.5; Multiply decimals (2 places) by 100; Multiply/divide familiar unit fractions, single digits by half and quarters; Key % of simple 2- and 3-digit numbers	111-67; 49x3; 150÷6; 92÷4; 40x25; 12x20; 6.2+1.9; 0.19+0.1; 1.25-0.5; 0.6x10; 0.37x100; 0.5x48; 0.1÷0.1; 1/2 of 1/3; 3÷1/2; 20% of 15; 75% of 200
8	*Whole number*: Divide 2- or 3-digits by single digit; *Decimals, fractions, & %*: Divide single digit by 0.1; Divide decimal (1 place) by 5; Divide simple whole number by 0.5; Add familiar unit fractions; Straightforward % of simple whole numbers	343÷7; 0.1x0.1; 2÷0.1; 0.2÷5; 90÷0.5; 1/2+1/3; 1/2x3/4; 331/3% of 600; 12.5% of 24

Student Performance Across Years

Box and whisker plots of student ability (in logits) across years are shown in Figure 1. The box shows 50% of the ability range and the heavy line in the box is the median value. The ends of the whiskers are at the 10th and 90th percentiles.

There is a general increase in ability across the years. The growth is more marked in the lower years and plateaus through the middle years of schooling. In year 7, although the median value and 50% box increases slightly over the values for year 6, at both the 90th and 10th percentiles the achievement appears to drop compared with year 6.

There is also a wide spread of abilities across all years, with considerable overlap among years. The best year 3 students appear to have an ability above the median of those in year 10, for example. Conversely, the weakest year 10 students have an ability level below the year 3 median. The implications of these findings are discussed further below.

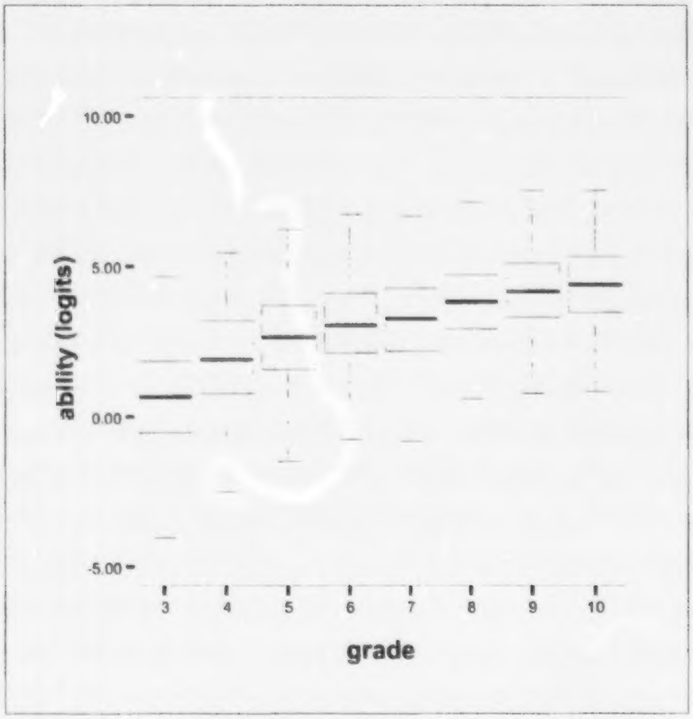

Figure 1. Student ability across years

The growth in competence across years is more clearly seen if the difference in mean ability across each pair of years is calculated. This provides a measure of growth rate across the years of schooling, shown in Figure 2.

The greatest period of growth in mental computation competence is between years 3 and 4. Growth is similar between years 4 and 5 and years 5 and 6 but drops sharply between years 6 and 7. It then increases again between years 7 and 8 but there is a gradual decline in growth as students enter the later years of school.

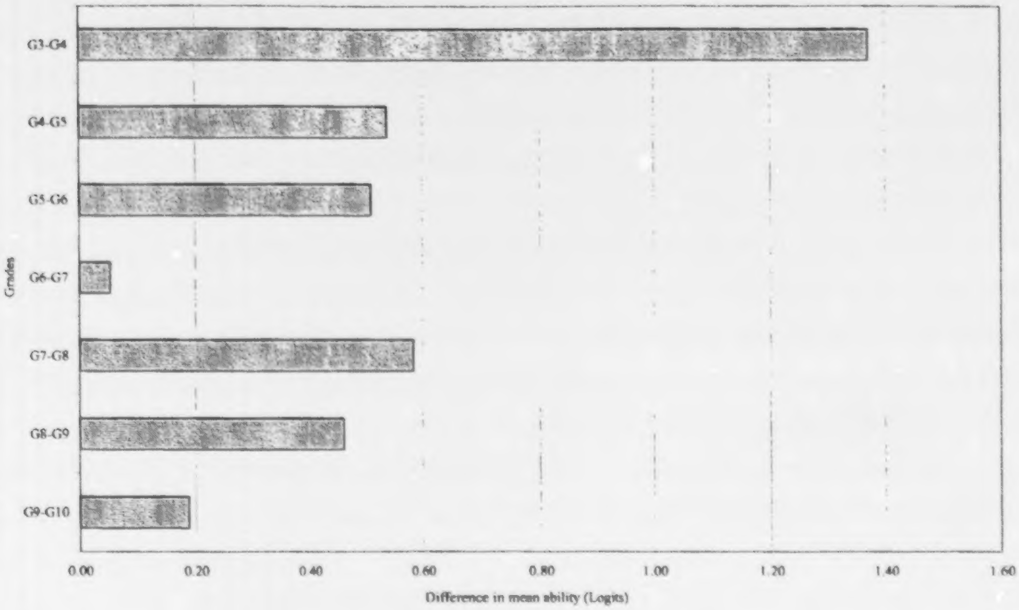

Figure 2. Growth in mental computation competence across years.

Expectations of Students in Years 3 to 10

The percentage of students in each year that were in the different levels of competence identified was calculated to determine what was a reasonable expectation of students in the different years. These results are summarised in Table 4. The shaded cells show levels in which 20% or over of the students in any given year appeared, and show a general increase in competence across the years.

Table 4
Percentage of Students in Each Level by Year

Year	L1	L2	L3	L4	L5	L6	L7	L8
3	8.03	13.99	26.61	20.64	20.18	6.42	2.29	1.83
4	1.36	6.12	16.10	18.37	27.21	15.42	11.56	3.85
5	1.43	2.62	5.48	15.71	28.10	23.10	18.10	5.48
6	0.45	0.67	3.60	10.56	27.64	25.39	21.35	10.34
7	0	1.74	2.73	9.93	25.81	22.83	26.55	10.42
8	0.29	1.16	1.16	5.23	15.70	24.71	33.14	18.60
9	0	0.29	0	1.75	10.79	23.32	38.19	25.66
10	0	0	0.49	1.97	10.84	18.72	38.92	29.06

In all years there is considerable spread across levels. In year 3, for example, although the largest group of students was at Level 3 (116/436 or 27%), over 50% of year 3 students were in Level 4 or above. This suggests that year 3 students have greater facility with mental computation than might be expected. Although nearly 30% of all year 10 students appeared at Level 8 the largest group of students in all high school years was in Level 7. Given the content and skills required at the level, this may be a reasonable level to which

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to aspire by the end of year 10. Similarly, Level 5 would seem to be a reasonable level of expectation for students at the end of primary school. Over 50% of year 6 students are above this level, but the largest single group of years 4, 5 and 6 students appears in Level 5.

Discussion

The scale of mental computation broadly confirms the scale developed by Callingham and McIntosh (2001) in scope and sequence. Some lower difficulty items were omitted from the tests used in this study and more difficult items included. This has improved the description of the upper levels but has perhaps removed some detail from Levels 1 and 2. The general structure of the scale, however, remains. By Level 5, when decimals, fractions and percentages appear, students would seem to have a good understanding of the principles of the four operations, allowing addition and subtraction, and, in Level 6, multiplication and division to appear together, rather than in adjacent levels as is typically the case with whole numbers. Basic multiplication facts and inverses are not completely secure until Level 6, although multiplication and division by 3, 4 and 5 appear by Level 4. The pattern of multiplication of 2-digit numbers mirrors that of single digit numbers: multiplication by 10 and 2 at Level 5, followed in the next level by multiplication by 3, 4 and 5, and, in Level 7 by any single digit number. The recurrence of these patterns suggests an appropriate teaching sequence for multiplication and related division facts.

When the competence of students in different years is considered the range of ability is a notable factor. The high growth rate between Years 3 and 4 is similar to that described by Bana and Korbosky (1995), and the overall pattern of growth is similar to findings in other research studies that have described a dip in the transition years between primary and high school (Hill, Rowe, Holmes-Smith & Russell, 1996). These findings may reflect curriculum change as well as changed contexts. There is a greater emphasis on knowledge of addition and subtraction facts, and basic multiplication facts in the early years of primary school. This is consolidated in upper primary classrooms, while Year 7 in Tasmania marks the transfer to high school and, often, a difference in the approach to mathematics and the introduction of harder content such as decimals and percentage. The smaller growth in the upper years of high school may be related to the test structure not allowing the best students to demonstrate their competency at the highest levels. However, Figure 1, the box-and-whisker graph, also shows a levelling out at the 10th percentile and this would not be expected if the growth pattern were due entirely to a test ceiling effect. An alternative explanation for the slowdown in growth could be that curriculum documents have no expectations for mental computation of fractions, decimals and percentages, and thus this area receives little emphasis in high schools.

In terms of reasonable expectations of students in different years, it seems that some of the current expectations are a little unrealistic. If the National Numeracy Benchmarks are considered as the current standard, the Year 3 benchmark is about Level 3 on this scale, Year 5 lies at about Level 6 and Year 7 at about Level 7. The National Numeracy Benchmarks are expected to be achieved by about 80% of the population. From the results of this study, the standard set would seem to be reasonable for Year 3 benchmarks, with 78% of Year 3 students at Level 3 and above. For the later years, however, this is not the case. Less than 50% of Year 5 students are in Level 6 or above, and this figure drops to less than 40% of Year 7 students meeting the equivalent benchmark standard of Level 7. More reasonable benchmark expectations, using the current definitions, would seem to be Level 5 for Year 5 and Level 6 for Year 7.

The scale as presented here does not show differences between Short and Long items. There was some evidence that contracting wait time to three seconds did make items slightly harder at the lower levels. This is an area that requires further research into students' strategies, and how these are applied.

Finally, the finding that small numbers of students are achieving at much higher levels than might be reasonably expected for their year level does not necessarily imply that these students can deal with all the content associated with the high level. It is probably unreasonable, for example, to expect a Year 3 student apparently in Level 8 to be able to do all the decimal calculations. To some extent the placement of students in these high levels is an artifact of the form of analysis. Year 3 students were not asked any decimal questions, for example, and the fractions questions were limited to those appearing in Level 5. There is, however, a small group of students who can be identified as very competent at mental computation and these students need to be provided for in the classroom. Similar comments could be made about those students at the other end of the scale.

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Exploring Mental Computation in the Middle Years

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Following earlier interest in mental computation and its relationship to number sense in the early and primary years, this exploratory study investigated the mental computation competence of students in the middle school. The study involved clustering the responses of four students in grades 6 and 8 based on individual interviews. In each grade a student was identified as more or less competent in mental computation performance on a pencil-and-paper class test. Several contrasting content and performance features were identified and examined with respect to implications for teaching mental computation in the classroom.

In an educational climate that endorses the development of a flexible and integrated approach to numeracy, mental computation is recognised for its educational and its utilitarian value. Interest in the "ability to calculate exact numerical answers without the aid of calculating or recording devices" (Reys, Reys, & Hope, 1993, p. 306) is by no means new. However, redirecting the practice of mental computation in the classroom away from the instant recall of rehearsed facts to incorporate computational strategies and calculations larger than those covered by basic number facts is new. Mental computation research has mainly focused on the early and primary years of schooling yet in light of the current attention to numeracy in the middle years (Siemon, 2001), there seems reason to examine what mental computation involves at this level.

The middle years of school represent a period where the scope of the curriculum expands and children begin to reason abstractly (Watson & Callingham, 2001). In mathematics, students incorporate their understanding of number and place value to embrace relational concepts, measurement and space (Siemon, 2001). Similarly, as the curriculum demands increase during this period, the span of student achievement also becomes increasingly differentiated (Hill, Rowe, Holmes-Smith, & Russell, 1996). In terms of its educational value, mental computation skill is linked with number sense and is posited to encourage a creative approach to solving problems and facilitate students' understanding of number properties and operation (McIntosh, De Nardi, & Swan, 1994). Yet Weber (1996) contends that middle grade students do not appear to regard numbers as quantities with multiple internal relationships, but rather as symbols to manipulate. What then happens to the mental computation abilities of students as they make the transition into middle school?

Mental computation research has rarely extended beyond year 6 (or year 7 depending on the primary school classification system) to incorporate the early years of secondary school. Large-scale studies that do include these middle grades tend to do so within a range of grades, providing only snapshots of how middle grade students perform (McIntosh, Nohda, Reys, & Reys, 1995; Reys, et al., 1993). One such Australian study explored performance across grades 3, 5, 7, and 9. The authors, McIntosh, Bana, and Farrell (1995), considered three perspectives of mental computation including preference, attitude, and performance assessment. This research however provided an overview of quantitative data that did not extend to cover strategy choice or to detail related cognitive components of students' thinking.

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In terms of qualitative data, individual interview-based research has been invaluable in advocating mental computation as a valid computational method and one which contributes to mathematical thinking as a whole (Sowder, 1990). In-depth interviews have led to descriptions of strategies that students utilise to solve mental computation problems, particularly addition and subtraction (McIntosh, et al. 1994; Heirdsfield, 2000). Interestingly strategy use can be described as idiosyncratic and self-taught (Bana & Korbosky, 1995), with younger children particularly employing spontaneous strategies before direct teaching (Heirdsfield, 2000).

Mental computation interviews have also been a rich source of information in terms of looking at the relationships among factors that mediate mental computation acquisition and use. A complex set of connections involving number sense, cognitive factors (specifically memory), affective factors, and metacognitive processes have been identified in the literature. Predominantly these factors have been combined to look at accuracy in mental computation (Heirdsfield, 2001). Perhaps the most frequently linked factor with mental computation is number sense (McIntosh & Dole, 2000). Two findings in particular with primary students raise some interesting questions for the middle school level. First, good mental computation ability may not necessarily be supported by well developed number sense (McIntosh & Dole, 2000). Second, strategies start to reflect pen-and-paper algorithm procedures as influenced by instruction towards the end of primary school (Heirdsfield, Cooper, & Irons, 1999). These findings, observed in interviews with upper primary students, seem relevant in examining mental computation in the middle school sector and suggest many research questions. The one under consideration in this paper is: What features distinguish middle school students at different competence levels?

The research reported in this paper constitutes an aspect of an on-going mental computation research project currently being conducted in Tasmania and the ACT. The mental computation interviews from which data were extracted for the study form the basis for documenting the variety and type of strategy use to be expected at successive developmental levels.

Method

Subjects

Students in grades 3-10 at a primary and a secondary school in the Tasmanian Catholic sector completed a paper-and-pencil mental computation class test (Callingham & McIntosh, 2002). Three students in each grade (3-10) were selected for interview as representatives of their grades based on three ranges of raw data scores: 50%-60%, 70%-75%, and >85%. From the 12 middle school students (grades 5-8) interviewed, four students were chosen for case study analysis, two from each of grades 6 and 8. These students were two low scoring males (raw score 50%-60%, one in each grade), and a high scoring male and a female (raw score >85%, one in each grade). They were also assigned a Rasch mental computation competence level within the range of 1 to 8 (Callingham & McIntosh, 2002). In grade 6 the low and high scoring students were level 5 and 7/8 respectively and in grade 8 the low and high scoring students were level 6 and 8.

Protocol

Initially a set of base interview questions was selected from the paper-and-pencil mental computation tests (Callingham & McIntosh, 2002). These questions were selected to incorporate all whole number operations with some decimal, percent, and fraction questions mainly for the secondary students. Each base question was then extended into a set of related questions (for example see Table 1). It was this set of questions that was presented to students during the interviews.

Table 1
Example of an Interview Question Set

24 x 3	Base question from test	Double-digit multiplication question
24 x 6	Related question	Related double-digit multiplication problem in the same form
24 x 30	Related question	Related double-digit multiplication problem with a multiple of ten

Interview Procedure

The students were interviewed individually in a separate room in their respective schools. The interview sessions were approximately 30 minutes for both the primary and the secondary students and were videotaped with parental permission. The interview protocol was semi-structured in that the questions were specifically constructed and a number of potential directions were anticipated depending on how the students responded. Nevertheless, the interview was still guided by individual characteristics, responses, and comments from the students. Participation in these interviews was voluntary and the students were told they could conclude the session any time they desired. None stopped early.

Data Analysis

A cyclic clustering procedure (Miles & Huberman, 1994) was employed using the responses and discussion recorded for the four case study students. A case-by-attribute matrix was the result, with each row representing a question asked in the interview and the responses being distributed over the attribute columns. Initial clustering of columns was informed through observation and with reference to a framework for examining number sense (McIntosh, Reys & Reys, 1992) and a framework for proficient mental computation (Heirdsfield, 2001). A second part to this process involved discussion among three members of the research team to identify common characteristics among the rows and columns of the matrix. Overall six features were identified as distinguishing the more competent students from the less competent students in their mental computation interview performance.

For the following discussion the four students will be identified based on their raw scores and Rasch levels as: L6 (less competent grade 6), M6 (more competent grade 6), L8 (less competent grade 8), and M8 (more competent grade 8).

Results

Each of the six features identified as distinguishing students of different mental computation competence will be presented with examples from the interviews where appropriate. These examples will be presented in the following format: the identified student and the question (both in italics), and then the student response. Additional questions by the interviewer will also be in italics. The observations are considered in three content features, two performance features, and a feature concerning making connections.

Use of Elementary Number Work

Strategies that the students used to solve mental problems showed that in some cases the less competent students relied greatly on aspects of their elementary number work. In solving the multiplication questions, for example, the less competent students used repeated addition strategies to solve mental problems.

> *Student L8 (Question 24 x 3).* 72. I went up 24 + 24 + 24.

The more competent students however, used strategies that reflected an understanding of place value.

> *Student M8 (Question 24 x 3).* 3 x one 4 is 12, 3 x one 20 is 60, plus them together is 72.

This facility with place value also enabled these students to make more calculated decisions for whether to use either the units first or the tens first, depending on the properties of the operation. The use of repeated addition is related to the use of doubling in solving multiplication problems. Instances of doubling were noted among the less competent students. Again, it could be argued that this strategy reflects the influence of elementary number work from the early primary years.

Extended Number Facts

The questions based on extended number facts provided another feature where the more competent and less competent students displayed some differences. The term "add a zero" was employed by both categories of students when multiplying by ten (or twenty in one case) and when dividing a number by a multiple of ten. However the more competent students seemed to utilise this to a greater extent and were able to adapt their 'zero' rules to solve the division question. The facility with this rule for these students, in that zeros were purposely added and removed, was reflected by the wide range of expressions used to describe this strategy, for example, 'dissect the ten into zeros', 'put the zero on', and 'cancel down the zeros'.

> *Student M8 (Question 7 x 30 (presented after 7 x 3)).* 210. Just add a zero *(because?)* because it's 10s. *(Question 70 x 30)* 2100. Just add two zeros. *(Question 420 / 70 (presented after 42 / 7)).* 6. Because they're both 10s so you cancel down the zeros.

In contrast the less competent students did not rely on the 'zero' rule. Student L8 identified two ways in which to solve the problem 12 x 10. The first method involved 'adding the zero to the 12 from the 10', whereas the second method involved the 12 being split into 10 x 2 and 10 x 10. On the subsequent problem, 22 x 10, student L8 acknowledged that he would 'add a zero' in this case as the other way was 'too hard'. Student L6 only 'added a zero' once and in this instance he answered incorrectly.

> *Student L6 (Question 70 x 30 (presented after 7 x 3)).* 210. 3 x 7 is 21, add the zero.

Fractions and Decimals

When dealing with fraction and decimal questions there were a number of observable differences between the more competent and less competent students. Primarily the less competent students tended to translate fraction questions into decimals. As illustrated by the examples below, $\frac{3}{4}$ became .75 and $\frac{1}{2}$ became .50. One student in particular indicated a preference for thinking in decimals. With the two less competent students, one answered in decimals, the other in fractions, but both explanations were translated into language associated with decimals.

> *Student L6 (Question $\frac{3}{4}-\frac{1}{2}$).* 25. You just take away 50 which equals 0.25.

> *Student L8 (Question $\frac{3}{4}-\frac{1}{2}$).* $\frac{3}{4}$ is like 75, 50 is a $\frac{1}{2}$, take away the 50 is 25 so $\frac{1}{4}$.

In contrast the more competent students demonstrated facility in moving easily between fraction and decimal representations. There seemed to be a clear boundary separating the two domains, with answers and explanations given in the appropriate terminology, respectively.

> *Student M6 (Question $\frac{3}{4}-\frac{1}{2}$).* $\frac{1}{4}$ 2/4 is a $\frac{1}{2}$ so take that away.

The tendency for the less competent students to apply their decimal knowledge to fraction questions is interesting and it raises several points for consideration. First, the translation from fractions to decimals in one sense makes it a complicated way to solve what is not a particularly difficult question for students at the middle school level. Being able to recognise that $\frac{1}{2}$ is the same as $\frac{2}{4}$ appears a much easier route to solving $\frac{3}{4}-\frac{1}{2}$. Second, the translation is useful only in solving a handful of fraction questions. When given the question $\frac{1}{2}-\frac{1}{3}$ one of the less competent students demonstrated that he had no available means by which to proceed with this question.

> *Student L8 (Question $\frac{1}{2}-\frac{1}{3}$).* 2.5…1/3 is 75. I don't know. *(Do you know what $\frac{1}{3}$ of 100 is?)* 35?…40?

Errors

A feature of these individual interviews in comparison with the timed tests completed earlier is that students were given the opportunity to discover and rectify their own errors during discussion. Overall, the students appeared to be differentiated by the way they managed errors. Less competent students seemed less inclined to check their answers, at times seeking assurance from the interviewer. There were more instances when the less competent students gave initial incorrect answers but rectified their responses during the explanation as to how they worked out the answer.

> *Student L8 (Question 7 x 6).* 51? 41. 7 + 7 = 14, + 14 is 28. Add another 14 to get…42.

In addition the less competent students had to be prompted through their own explanations to assist them in identifying an answer. When given the opportunity to attempt a related question with an incorrect strategy, one of the less competent students completed the question but did not acknowledge the error. The confidence displayed by these students seemed to be at a much lower than for the more competent students.

> *Student L6 (Question 125–99).* 24. Take away 100 first, 25 and then take the 1. *(Question 135–99).* 34 *(Did you use the same strategy?)* Yes.

Overall, the more competent students made very few errors. If in doubt they seemed to recheck again before confirming they were satisfied with their answers. It appeared from the demeanour of these students that those errors that did occur could be classed as a type of 'slip up' error due to speed rather than a conceptual error. The more competent grade 6 student when given two harder questions that she could not complete (1.2 x 10 and $\frac{1}{2}-\frac{1}{3}$), acknowledged the difficulty she was having yet persisted in working through the question with the interviewer.

Timing

During the interviews and when watching the video tapes the speed at which the more competent students processed the questions and produced an answer was striking compared to the less competent students. Whereas the more competent students appeared to process several components of the information concurrently, the less competent students appeared more laboured in decomposing and recomposing the information to arrive at an answer. Subsequently these students answered fewer questions overall and with less coherent explanations in the interview session as to how they arrived at their answers.

Making Connections

The sets of related questions were constructed specifically to explore the information students recognised and exploited from related questions. Providing the first question in the set was answered correctly, the students had the opportunity to either: recognise and use the previous question to assist them in solving a new problem without having to start the computation process from the beginning or, recognise that a previous strategy was successful and apply it again (although it could be argued that a combination of both was also possible). Whereas all students recognised the similarities in subtracting 7 (or 70) from numbers that ended with 7 (or 70), the less competent students could not respond similarly to harder questions. Both less competent students had successful strategies to solve 100–55 but struggled to adapt their strategies to answer a related question, 100–34.

> *Student L8 (Question 100–55)*. 45. Go from 55, add 40 which adds up to 95, then you add the 5 which equals 100. *(Question 100–34)*. 76 (student asks the interviewer if this is right and decides that it's wrong and starts again). 66. Worked it out using my hands. Have your 60, add the 30 so 90, 4 on your 34, add 6 to that and get 100.

The more competent students, on the other hand appeared to move without difficulty from 100–55 to 100–34.

> *Student M8 (Question 100–55)*. 45. Took away 50 from 100 and took away another 5. *(Question 100–34)*. 66. 30 (take) 100 is 70, take another 4.

Examples from the more competent students emphasise that these students brought to their mental computation performance an appreciation of number that facilitated making connections. It is possible to see the students utilise both a successful previous strategy and the number information from a related question.

> *Student M6 (Question 125–99)*. I took the 25 from the 99 which is 74. Then I took the 74 from the 100 which is 26. *(Why did you take 25 from 99?)*. Because it's easier to work in 100s so we have to take the 25 off the 125 so we can work in the 100s. OK, so we're left with the 74 so we take the 74 out of the 100 and that leaves us with 26. *(Question 135–99)* 36. *(Did you do that from the last answer)*. Yes. I remembered that the last answer was 26 so then I was working with a bigger number so I just add the 10. *(Question 125–89)*. 36. Sort of the same thing but it's a different number. I knew that 99 out of 125 was 26 and we were working with a different number so the total is going to be bigger because the number was smaller and I knew it was bigger by 10.

> *Student M8 (Question 125–99)*. 26. 99 to 100 is 1 so I just plussed the 25. *(Question 125–89)*. 36. Because there's 11 up to 100 and plus the 25. *(Question 125–79)*. 46. It's 21 up to 100 and plus the 25.

Discussion and Implications

This was an exploratory investigation impelled by the question, what features distinguish students of different mental computation competence at the middle school level? Although it is not possible to generalise from the results of four individual students, there were some features that differentiated the more competent and less competent students, and some interesting points for discussion about teaching mental computation at the middle school level.

In terms of elementary number work the less competent students used repeated addition and some related doubling strategies, whereas the more competent students preferred place value strategies. The latter may reflect a familiarity with place value as influenced by pen-and-paper procedures during the upper primary years (Heirdsfield, et al. 1999). Although a repeated addition strategy reflects that students implicitly understand the properties of the operation they are using, continuing to rely on these strategies may prevent students from being able to broaden the scope of the problems they can manage, particularly for multiplication and division problems. Similarly, for the less competent students the use of 'adding zeros' with extended number facts appeared to be utilized but not necessarily understood well. Zeros appeared to be regarded as independent entities from the rest of the number by all students but the more competent students were still better equipped to manipulate with understanding, the role zeros play in performing division operations. The role of teaching practices for these two features involves setting up environments where students will be exposed to the range of strategies for a particular sort of problem, and learn to value the process of explaining answers and sharing strategies. Heirdsfield, et al. (1999) showed in a longitudinal case study that 'adding zeros' was used early in year 4 so it is likely this is a strategy that has been explicitly taught in the classroom.

The fact that the less competent students translated fraction questions into language more closely associated with decimals was an interesting finding of this study. When investigating the effects of instructional materials on mental computation procedures, Weber (1996) described 8th grade students' understanding of fractions, decimals, and percents as limited, in that conceptual knowledge bases were inadequate. It would appear in the present study that while all the students had some strategies to solve basic decimal problems, the less competent students did not respond similarly when faced with fractions. Conversely the more competent students were able to access a much greater range of fraction problems. Part-whole number relationships constitute a greater part of the curriculum at the middle school level than in the earlier years. Since the inter-relatedness of part-whole numbers (fractions, decimals, or percents) is essential in establishing conceptual understanding, and in being able to solve these problems mentally, teachers have a role to build clear representations of both using the appropriate terminology.

The cautious way that the less competent students dealt with errors, along with their lack of inclination to check answers in the interviews, seems to be an issue related to confidence. Connections between mental computation and perceptions of ability have been identified in proficient mental computers (Heirdsfield, 2001). The observation in the present study that the more competent students appeared to process several components of information concurrently verifying Heirdsfield's findings regarding the function of memory in supporting and rehearsing interim calculations.

The sets of related questions provided an encouraging avenue in exploring the types of information students recognise and subsequently use to formulate a computational plan. Although it was often difficult to obtain an 'uninterrupted' set of questions, the potential of this type of questioning will be further examined in research particularly with regard to how it might be employed in the classroom. Using related questions may assist in making students more aware of alternatives when choosing a strategy. It might be the case that a well-rehearsed strategy may not necessarily be appropriate or the easiest to use; helping students to learn to discriminate between related questions may prove a useful exercise.

This study identified some features that appear to distinguish the competence of the mental computation performances for four middle school students. The transition into middle school will be the subject of ongoing research particularly in establishing the role of mental computation practices within the middle school curriculum.

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Secondary Teachers' Attitudes toward Probability and their Teaching Strategies

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This exploratory qualitative research study uses observations and semi-structured interviews to examine secondary mathematics teachers' attitudes toward probability and their use of probabilistic reasoning in teaching two activities designed to extend students' models of exponential relationships. Student learning difficulties present problems for the teachers. Teachers' preparation in probability is typically weak and they are uncomfortable teaching probability. Thus, they emphasised the exponential aspects of the activities, concentrating on developing generalised deterministic equations that model growth and decay.

Many decisions faced in everyday life require probabilistic reasoning for effective evaluation of the alternatives. In addition to obviously probabilistic situations such as games and lotteries, many everyday events are the result of many complicated and interrelated causes that cannot be identified or measured satisfactorily. Media reports involving information explicitly or implicitly based on stochastic data inundate us. Using media information or misinformation, people form deterministic opinions about many topics that affect their actions, opinions, and positions on controversial issues. Without the ability to reason probabilistically about data and data analysis, otherwise well-informed citizens and consumers may make costly mistakes affecting their lives and the lives of others. A review of the literature suggests that student difficulties in learning probabilistic reasoning present unusually difficult problems for the teacher. Watson's (2001) recent study confirms that teachers are aware of these difficulties and it provides some insight into teachers' attitudes, expressions of feeling or behavior intention, toward probability. Many authors (including Borovcnik & Peard, 1996; Konold, 1989; Wilensky, 1997) maintain that teachers need to introduce situations and hold conversations that confront students' misconceptions about probability.

This study is set within the framework of a larger research project, led by the second author, on understanding the development of teachers' knowledge as students engage in technology enhanced modeling activities (Doerr & Zangor, 2000). Two of the activities were designed to elicit students' models about phenomena that involved probabilistic elements. In this study, we explore how teachers responded to these elements of uncertainty embedded within a sequence of deterministic modeling tasks. In this paper, we report on the teachers' attitudes toward probability and their general treatment of the probability aspects of the activities.

Background and Theoretical Approach

The well-known difficulties in student learning of probabilistic concepts has led to criticism of traditional approaches to teaching probability because they reduce uncertainty to a determined formula and fail to confront students' misconceptions (Batanero, Green & Serrano, 1998; Konold, 1991; Lajoie, Jacobs, & Lavigne, 1995). Recommended approaches rely on: teacher understanding of student thinking; teacher initiated conversations that confront student misconceptions; and, learner owned investigations that

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allow students to refine their ideas of probability. For instance, Konold (1989) asserts that students must be forced to confront their probabilistic misconceptions by facing situations that demonstrate that their approach is invalid. He concluded that teachers' "understanding of how students are thinking about a topic … puts the teacher in the position of being able to initiate conversations and design curriculum that can facilitate the development of more adequate concepts of probability" (1989, p. 147). He suggests that teacher initiated conversations should encourage students to evaluate their current probabilistic misconceptions against three criteria: how they fit the beliefs of others; how they fit their own related beliefs; and, how they fit their own observations. These three criteria provide an organisational background for teacher-student interaction in the modeling probability classroom.

After reviewing several possible approaches to teaching probability, Borovcnik and Peard conclude, "Teaching has to focus on establishing stable intuitive insights and thus cannot rely on the presentation of the mathematics" (1996, p. 244). In other words, students have to construct and reconstruct probability concepts themselves. The teacher cannot present probability theory to the students and expect them to internalise and use the concepts appropriately. Wilensky (1997) also criticises traditional teaching, which relies on formalism. He focuses on the student's epistemological anxiety, which he defines as a feeling that one does not comprehend the meanings, purposes, sources, or legitimacy of the mathematical objects being manipulated. A popular response to the difficulties in learning probability has been what he calls the accomodationist's view: a deference to the mechanical manipulations as a guard against unreliable intuitions. Wilensky argues that this avoidance not only exacerbates the anxiety but it also fails to confront the student's misconceptions, precluding the necessary construction of appropriate conceptions. His solution focuses on learner-owned investigations followed by reflections. This approach validates the importance of the student's personal views and relationship to personal experience. It seems that a models and modeling approach (described below), with the teacher actively identifying misconceptions and adapting the curriculum and activities to confront those misconceptions, might be able satisfy all of these teaching strategy recommendations

Are teachers initiating conversations and posing situations that challenge their students to refine their concepts of probability? The response patterns to Watson's (2001) survey seem to indicate that a telling teaching style is common among the secondary teachers in her sample. In the only reference specific to secondary teachers in her discussion section, she reports that senior secondary teachers "recognised the difficulty of the topics for students but have made little effort to introduce activity-based aspects, such as simulation or actual sampling, that would reinforce the theory" (p. 325).

However, teachers must understand the concepts themselves and be comfortable teaching them. Watson's (2001) multi part survey instrument provides a model for those attempting to profile quality teaching and is our only current source for data on teachers' understanding and attitudes toward probability. She reported that at least some teachers did not like probability, but it was unclear whether any of these were secondary teachers. She also reported that there were no large differences between primary and secondary teachers' confidence in handling statistics in social settings, the need for statistics, and sampling. However, high school teachers did express more confidence in understanding statistical terms in the media and felt more confident teaching statistical topics. Teachers were less familiar with sampling than with average and provided less sophisticated responses. "This may have been due to either a lack of familiarity with the topic or a belief that it was peripheral to the curriculum " p. 318). She reported that the teachers' success rate on a particular item was 56%, higher than the 23% success rate reported for grade 11 students in a previous survey. These findings indicate that teachers have a more sophisticated understanding of probability and statistics topics than their students do and that secondary teachers know more probability and statistics than primary teachers do. However, the weakness in sampling indicates that these teachers may have trouble helping students develop sophisticated models of probability and statistics.

This paper provides additional insight into secondary teachers' attitudes and knowledge of probability and their use of the recommended teaching strategies by addressing these research questions.

i. What are the participating teachers' attitudes toward probability and toward their knowledge of probability?
ii. Do they think probability is important in the curriculum?
iii. Do they initiate conversations to confront their students' misconceptions about the probabilistic aspects of two exponential modeling activities?

The models and modeling perspective of Lesh and Doerr (2001) provides a framework not only for understanding the learning process but also for developing curricula and teaching strategies. Student concepts are developed in a multiple, non-linear, cyclical, interactive modeling process that allows modelers to express, test, revise, refine, extend, and sort out their ways of thinking" (Lesh & Doerr, 2001). These models "consist of conceptual systems that are expressed using a variety of interacting media ... for constructing, describing, explaining, manipulating, predicting or controlling systems that occur in the world" (p. 5).

The models and modeling perspective seems uniquely suited to the difficulties of teaching probability because:

i. The students' activities must uncover, test, sort, refine, and extend the students' pre-existing models developed in and out of school. This requires exposure to a variety of alternative models presented by other students.
ii. The persistence of the misconceptions requires gradual and iterative re-construction using tasks and teaching strategies that requires conceptual not procedural understanding.
iii. The student's misconceptions will not respond to traditional linear teacher oriented presentation of the material.
iv. The complementarity of chance and probability requires iterative modeling cycles, alternation of activities and mathematisation with much reflection.

This study uses this framework to interpret the teachers' perspectives and experiences with the probability aspects of two modeling tasks.

Methodology

Participants

The six teachers, observed in their classrooms in large suburban districts are white females with at least 13 years experience teaching high school mathematics courses. Five teachers in one district were using the activities for the first time in their classrooms. The sixth teacher, who had participated in a pilot study the previous year, was using the activities for the second time. Before using the activities in their own classrooms, all of the teachers attended a two week long summer workshop led by Doerr exploring and discussing the activities, the technology, and appropriate teaching strategies. Before joining this project, the teachers had limited experience in leading extended activities that engaged their students in creating their own solutions to mathematical problems. The students had a range of abilities, motivation, and confidence as mathematical learners.

Probabilistic Activities

Two activities were taught as part of a course designed as a fourth year in the secondary mathematics curriculum. The M&M task, was designed primarily to extend the students' understanding of exponential functions to include exponential decay and only secondarily to introduce an understanding of the role of uncertainty in their models. Students were given a small cup full of M&M's. Starting with a few M&M's in a second cup, students were asked to roll their M&M's out of their cup, add one M&M to their stock for each M&M with an M showing, and then roll again. They repeated this process until they ran out of M&M's. Then the students were told to put all of their M&M's in a cup and roll them out, removing all of the M&M's with an M showing before they rolled again. They continued rolling until they ran out of M&M's. As a third task, students were asked to repeat the first two experiments with hypothetical 4-sided M&M's. For each situation, they developed an equation modeling their results and were asked to predict results outside of their domain.

The Rumor Mill activity was primarily designed to extend the students' exponential model by introducing the logistics function and secondarily to reinforce the role of uncertainty in their model, first seen in the M&M activity. All of the students in a class were given small pieces of paper with the numbers one through ten written in a column. Next to the number one, one student had an R, meaning that that student knew the rumor. Each student was told to show his/her paper to one other student during each round. If the student saw an R on any of the papers that s/he saw, s/he then knew the rumor and wrote an R next to the number of that round. The rounds continued until all of the students knew the rumor. Then the students were asked to plan and execute a simulation of the spread of a rumor. To support this activity, they were given a chart with a grid of the numbers from one to a hundred and an instruction sheet for generating random numbers on their calculators. They were not told that they had to use these items or what to do with them.

Data Sources and Analysis

The qualitative data collected for this study included audiotape transcriptions and field notes of each class session, teacher interviews before and after each activity, teacher workshops before and after the school year, and monthly teacher meetings. The workshops and meetings were used to review and discuss all of a larger set of tasks, not just the two activities of interest in this project, so only parts of these sessions were of interest to this study. We coded the transcripts and field notes using grounded theory (Strauss & Corbin, 1998), which allowed the codes to emerge from the data in the first pass. Subsequent passes refined and modified the coding. Copies of student work from each activity were coded for validity and misconceptions and this information was used in the post interviews. The issues from the M&M observations and interviews were compared with the issues that emerged for the Rumor Mill observations and interviews. The results were then related to the research questions and the literature.

Results

The results of this study include three major findings related to teacher knowledge and teaching strategies. First, this study confirmed the finding of Watson (2001) indicating that some teachers dislike probability and feel unprepared to teach it. Second, these teachers recognise the importance of teaching probability. Third, the teachers emphasised only the exponential aspects of the activity.

In this study, we examine the teachers' responses to two of these activities primarily designed to extend the students' models of exponential functions. Since the secondary purpose of these tasks was to refine students' models of probability, we thought that we would be able to observe the teachers' general probabilistic reasoning and attitudes toward probability without the specific impetus of a probability agenda. We were keenly interested in how the teachers responded to the probability misconceptions of their students and how they handled those misconceptions because of the importance of confronting those misconceptions with situations that would lead their students to re-construct their concepts.

Some Teachers Dislike Probability

Three of the teachers do not like probability and four feel that they are not prepared to teach the subject. Penny addressed this issue:

> I know most math teachers are confused. If you ask the entire floor (of math teachers) what they like to teach the least, what they feel the least comfortable with, it's gonna be probability. Anytime you walk into the faculty room, people talk about probability and everybody saying they hate it. The first time I had to teach it, I always had the answer key out in front of me. And I work ahead to make sure that I have it all down. Teachers would routinely say they did this and they did it wrong and they had to go back and figure out why, and the kids lose confidence.

Only one teacher had taken more than one probability or statistics course. Three had never taken any. None of the teachers remembers any discussion of probability and statistics in any mathematics education courses. Joyce thinks that her "students know as much as she does". She believes that any group of three of her honors students will solve any problem at least as well as she can and their reports seem to confirm this. The other teachers are sure that their students have limited procedural knowledge and that few have a conceptual understanding of what they learned to pass the standardised state exams. The teachers do not voluntarily make connections to probability units taught to the students in prior courses, encourage discussions of questions raised by the students, or raise issues that the students have missed. Even the two teachers who feel comfortable with high school level probability do not often encourage their students to connect the concepts in these units to previously learned material or provoke students to re-conceptualise their probabilistic misconceptions.

Teachers Recognise the Importance of Probability

The teachers think that the basic conceptual knowledge of probability required to explore the issues raised by these activities should have been taught in earlier courses, as early as the fifth or sixth grade. In the interviews, all of the teachers agreed that probability education was important and that there was not enough probability in the curriculum. They agreed with the NCTM (2000) position that probability should be a part of the mathematics curriculum at all levels. They resent the pressure of the standardised state examinations in the earlier high school courses that forces teachers to expose their students only to the procedural aspects of probability that are tested. There is not enough time to cover everything on the test. As Joyce said, "Probability is almost always left out except for the test questions", and so teachers tend to present solution strategies only for likely test questions. They suggested that the activities should be rewritten to specifically address the probability issues involved. However, they were worried about the time this would take away from their current mathematical agenda.

Teachers Emphasised only the Exponential Aspects of the Activities

At several points in the activity, there was an opportunity to help the students refine their model of probability by revisiting the relationship between experimental and classical probability. In the M&M activity, for example, students typically averaged the successive ratios of the total number of M&M's at each roll to find a base for an exponential equation that fit their data. Only one teacher encouraged students to think about the relationship between the average of the successive ratios and the theoretical probability of rolling out an M&M with the M showing. In the interviews, the five teachers who did not explore this opportunity said that they "did not think about it" at the time. Their focus on the primary purpose of the activity and their concern over the amount of time that the activity was taking precluded the exploration of issues arising from the situation. One of the first time teachers said, "We don't have time to discuss … (all the) details and have them sit and think." Although the teachers value student think-time, they feel pressured to save time. Dealing with student frustration was a recurrent theme at the workshop given the summer after they taught the units. They recognised that students needed to struggle with the issues raised by the activities; however, exactly how and when to rescue a student in distress was problematic. According to the teachers, their students had always been taught mathematics traditionally. Students were used to group work in other content areas, but considerably less so in mathematics. The students were generally proficient at applying procedures that they had been shown, but they had little, if any, experience in solving problems that they had not seen before. Using modeling activities to revise and extend mathematical knowledge was a new experience for the students and the five of the teachers.

After a lengthy discussion with Joyce about why she and the other teachers had not discussed distributions with the students, the first author summarised what he had heard:

> First author: Number one, you didn't think about it. Number two, it is not the purpose of the lesson, so why would you think of it? And number three, some teachers have said that they don't like probability and don't want to mess with it because they might make mistakes because they are uncomfortable with it. The kids will ask questions that they can't answer and they are not comfortable with that.
>
> Joyce: That is scary, but I'm sure that is very truthful … And that is why the kids are where they are. Nobody feels comfortable with it…I would just add that the other critical thing here is time. We are crunching and if you are in the middle of an exponential unit, how much extra stuff can you fit in…As soon as you start with this stuff, we are in never never land now because you don't know what in the devil is going to come up.

Most of the teachers felt unprepared to teach probability and all of the teachers felt pressed for time. They believe that there is too much in the curriculum to treat all of the possible topics and issues. Given the challenges that the teachers faced with the basic process of teaching using a modeling approach, it is not surprising that they did not emphasise the secondary, probabilistic aspects of the activities that we observed. After all of these teachers had worked through the activity themselves, discussed the activity several times, and taught the activity at least once, they wrote a teacher's guide to the M&M

activity at the second summer workshop. All of the teachers agreed that probability is one of the key ideas of the activity. However, they did not emphasise probability in the sections of the guide dedicated to questions to provoke student thinking and issues to bring up in whole group discussion. As Bea put it, "Why do probability unless you are asked?"

Discussion

Teachers have a clear agenda; they know what they want to accomplish during a course. They are not going to jeopardise reaching their goals by spending time on all of the secondary issues inherent in a given activity. They have to select what they think is most important for their students to learn. The idea of probability was an important concept in the activity, but exploration and refinement of probability conceptions was not. This is understandable because probability was not the main point of the activity and this study was constructed to explore how the teachers thought about and used probability in this context.

In respect to my first research question, we found that most of the teachers in this study do not like probability and have a weak understanding of the topic. This extends the findings of Watson's (2001) study, indicating that many American teachers may not like probability or understand the topic well enough to be comfortable teaching beyond basic procedural elements. Many do not like to teach it. It typically does not occur to them to think probabilistically when faced with a probabilistic element in an activity, at least when the probability aspect is not central to the teacher's agenda. Sometimes, they did not recognise the problems or misconceptions implicit in their students' work. They tend to think about probability only when teaching a probability unit. They seemed to view the stochastic element as an impediment to getting to the deterministic problem, which was the focus of the activity.

The answer to the second research question is that the teachers did want their students to realise that probability was an essential element in the activities. . However, the teachers focused their attention on the exponential aspects of the activities and did not address or try to resolve many of the probabilistic issues that they noticed. They felt they did not have the time to deal with all of the issues that the activities generated and placed low priority on the probabilistic issues.

Teachers did not often initiate conversations to confront student misconceptions, the third research question. Understanding and confronting student misconceptions is essential for teachers, if students are to refine and extend their models explaining probabilistic situations. In order to do this, teachers must be comfortable with the probability issues and believe that they are important enough to spend the necessary time on them. Revising and extending the activities to include more explicit explorations of the underlying distributions could help teachers manage this difficult process. However, this would extend the time spent on the activity and may not be acceptable to some of the teachers. The teachers continue to struggle with changing their traditional, linear model of teaching. More experience with activities based on the models and modeling perspective may give teachers and students the tools required to address misconceptions and non-linear problems. Extended time on meaningful investigations allows and encourages the testing, revision, and extension of their mathematical and teaching/learning models.

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Evaluating Pre-service Teachers' Understanding of Middle School Mathematics

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This paper reports on an attempt to measure aspects of pre-service teachers' understanding of middle school mathematics together with their perceptions of its difficulties by using a simple questionnaire. Pre-service primary teachers did not, in general, perform as well as their secondary counterparts, and they rated the items as more difficult. While there were differences between perceptions of conceptual difficulty and computational difficulty, it is not certain that these distinctions are clear in the minds of pre-service teachers. Suggestions for improving the questionnaire are discussed.

In recent years there has been a growing interest in the multifaceted nature of teachers' knowledge, and how this affects students' learning. Shulman (1986) and others have identified several key components of teacher knowledge. These include (i) *content knowledge*, which involves both mastery and deep understanding of the content and structures of the subject matter; (ii) subject-specific *pedagogical content knowledge*, which incorporates the way subject matter knowledge is used in teaching, and includes knowing ways of formulating explanations and useful representations of concepts, etc.; (iii) *knowledge of students' thinking*, which incorporates knowing what makes topics easy or hard for students to learn, misconceptions and common difficulties, and the capabilities of students at different ages; and (iv) *curricular knowledge* of curriculum programs and instructional materials.

Ball (2000) notes that one distinguishing feature of knowledge for teaching is to be able to deconstruct it so as to be able to see it from the learner's perspective. Ball writes that "knowing for teaching requires a transcendence of tacit understanding" (2000, p. 245), so that the critical components central to but often invisible in one's own compressed mature knowledge are revealed. There is more to content understanding for teaching than the type of mastery that would be exhibited by an excellent student. In an influential book, Ma (1999) investigates primary school teachers' "Profound Understanding of Fundamental Mathematics" and contrasts the deep conceptual knowledge of Chinese teachers with the shallow procedural understandings of their U.S. counterparts, even in basic mathematics. In her research she used interviews to probe teachers' understanding of subtraction, multiplication, division by fractions, and area and perimeter. Results from other research (e.g., Stacey, Helme, Steinle, Baturo, Irwin, & Bana, 2001; Behr, Harel, Post, & Lesh, 1992) suggest that areas such as decimal and place value understanding, and ratio are also particularly problematic and worthy of further study.

Given the importance of teachers' knowledge of fundamental mathematics content, the focus of the current study is to investigate Australian pre-service teachers' understanding of aspects of elementary mathematics, both Shulman's content knowledge and some aspects of his pedagogical content knowledge. Since there has been a growing awareness of critical needs in the middle school it seems appropriate to consider both primary and secondary pre-service teachers, and investigate understanding of aspects of mathematics important in

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Grades 5 to 9. A comparison of the content knowledge of pre-service primary and secondary teachers is also of interest, since one group has more mathematical training and the other more training in pedagogical content knowledge. In order to investigate this for large numbers of teachers it would be useful to develop a questionnaire that could replace interview methodology, at least for some aspects of a future study.

There is also concern about the effects of teachers' judgements about the difficulty of items, because such judgements may influence what they decide to teach and how. It is possible that items judged very difficult may not be taught, or that teachers who do not appreciate the conceptual demand of an idea may treat it perfunctorily. This concern led to the inclusion of an investigation of teachers' beliefs about the difficulty of mathematical questions. Since it is common in our teacher education courses to distinguish informally between the conceptual and computational difficulty of items, students were asked to rate both these aspects. The outcomes reported here, then, investigate mathematical understanding and beliefs about difficulty, and are preliminary results based on a trial questionnaire that we hope to improve.

Method

Participants

Two groups of pre-service teachers from the author's Australian university were involved in the study. Participants from the BEd cohort (N = 39) were in the final year of a four-year Bachelor of Education program, preparing to become primary teachers. These students had done a mathematics education subject in each year of study, covering both elementary mathematical content and pedagogical issues. For admission to the BEd course they needed to have passed at least Year 11 mathematics. About half of the whole cohort (although not necessarily this sample) also passed a Year 12 mathematics subject. The second group, the DipEd cohort (N = 29), comprised students in the Diploma of Education program who were training to become secondary mathematics teachers. These students had at least a minor in mathematics from a previous degree. Participation in the study was voluntary, which may imply that the study is biased in favour of participants with greater mathematical self-confidence.

The Fundamental Mathematical Understanding Questionnaire

Data for the study came from participants' responses to a written questionnaire. There were two versions of the questionnaire, although most questions were common to both. The items were designed to cover a variety of topics in middle school mathematics, including decimals, ratios, simple geometry, fractions, and probability. These are topics treated in Grades 5 to 9; none involved algebra. Some items required only numerical answers, while others (items 3, 6B, 9, and part of 10) required some explanation. The items are described in Table 1. Many items had been used in or were based on the research literature (e.g., Greer, 1992; Hart, 1981; Henningsen & Stein, 1997; Stacey et al., 2001).

Participants were asked to rate the difficulty of many of the items, separately identifying computational difficulty and conceptual difficulty. This was done on a three-point scale of Easy, Medium, and Hard. In some cases, one response applied to two or more parts of an item. The questionnaire was administered during class time, allowing about 45 minutes, and the two versions of the questionnaire were distributed randomly. Calculators were not permitted.

Data Analysis

For the purposes of this analysis, items requiring elaborated responses were not analysed (item 6B, item 9, and two parts of item 10). Responses to the remaining item parts were scored as either correct (1) or incorrect (0); a partial score was given to some responses to item 1d. Omitted questions were marked as incorrect. The two difficulty ratings (computational and conceptual) were each assigned numerical values of 1 (Easy), 2 (Medium), or 3 (Hard). The data were analysed to compare the performances of the two groups, identify common difficulties in the understanding of fundamental mathematics, examine differences in perceptions of the conceptual and computational difficulty of questions, and investigate the relationship between performance and perceived difficulty.

Results

Comparisons Between BEd and DipEd Cohorts

As might be expected the DipEd cohort performed significantly better overall than the BEd cohort, averaging 88% correct compared with 71%. This overall result was reflected in the individual questions, with the DipEd students performing better on nearly all items (see Table 1). The differences between the groups were greatest on the part of question 10 requiring students to identify which of the two spaceships travelled further (BEd:DipEd = 28%:90%), although the wording may have caused some students (particularly those less confident) to be deflected from the correct answer that the spaceships travel the same distance. Other items proving particularly difficult for the BEd students were question 1d, which asked how many hundredths of a litre are there altogether in 0.485 litres (28%:83%); questions 7 and 9, requiring estimates of the product and quotient of two decimals (see additional discussion later in this report); and question 11, where clearly showing 0.725 of the 8×10 rectangle was essential for obtaining full marks.

The two groups performed comparably on the decimal comparison item (96%:100%; this was part of question 1), which was pleasing given the emphasis that had been placed on this in one of the BEd mathematics subjects, since earlier research (Stacey et al., 2001) had shown this topic had been problematic. They also performed comparably on the easiest of the ratio comparison items in question 3 (87%:90%). Both versions of question 4 required students to compare side lengths and areas of triangles determined by the diagonals of either a parallelogram or a rhombus. The parallelogram version proved more difficult, and although the BEd students did worse than the DipEd students for both versions of this geometry item, the difference in performance was not as great as for the remaining items. This may well be because the BEd students had done some work on geometry as part of their course (18 months prior to the questionnaire), whereas the DipEd students may not have encountered geometry since high school.

Table 1
Description of Questionnaire Items and Average Scores for the Two Cohorts.

Item	Topic	No. of parts	BEd score	DipEd score
1	Comparison of size of pairs of decimals, placement of decimals on a number line, place value understanding (e.g., how many hundredths in 0.485)	5	75%	92%
2	Simple ratio. Mr Short is 6 paperclips or 4 matchsticks tall. Ms Tall is 6 matchsticks tall. How tall is Ms Tall?	1	59%	86%
3	Comparison of pairs of ratios (e.g., which is sweeter: 5 spoons of sugar mixed with 15 spoons of lemon juice or 3 spoons of sugar mixed with 12 spoons of lemon juice?). Item had 3 parts, increasing in difficulty.	3	76%	93%
4A	Comparison of sizes and shapes of triangles in a parallelogram	5	70%	78%
4B	Comparison of sizes and shapes of triangles in a rhombus	5	81%	90%
5	Probabilities using a spinner	1	72%	84%
6A	Design a spinner with regions of probability 1/2, 1/3 and 1/6.	1	69%	95%
6B	Give three explanations for why 3/8 is the same as 37.5%	3	NA	NA
7	Multiplication: Choose value closest to 19.7×3.52	1	54%	90%
8	Division: Choose value closest to 5.67÷0.032	1	44%	83%
9	Explain why you "add a zero" when you multiply by 10	1	NA	NA
10	Describe and graph the movement of two spaceships, one moving at 1.88km/hr for 0.32 hours, and the other moving at 0.32 km/hr for 1.88 hours (values for 1.88÷0.32, 0.32÷1.88 and 0.32×1.88 were supplied)	5	71%	91%
11	Shade 0.725 of an 8×10 rectangle and give the fraction and decimal equivalents.	3	32%	71%

Note: Scores are based on all parts of the whole item, except for item 10 where two parts were not analysed.

A large number of students (12 of the 68, mostly BEd students), did not respond to question 2, which was surprising given that a distinctive large diagram accompanied the question. Questions 11b and 11c were also omitted by some of the BEd students, but since these were the last questions time constraints may have led to their omission.

Common Errors

Although the wording of question 1d—which asked how many hundredths of a litre there are in 0.485 litres—probably contributed to some justifiable variants of the intended answer of 48.5 (such as 48 or 8), there were a large number of answers showing a poor understanding of place value. For example, 36% of the BEd students responded with 485, while another 13% gave other values reflecting inadequate place value understanding, such as 5, 80 or 4850. In contrast, only 4 of the DipEd students (14%) gave such answers. The difficulty of the primary pre-service teachers with such a fundamental aspect of place value understanding in the final year of their course is of great concern.

Question 7 asked students to select which of five possibilities was the closest to the correct answer for 19.7×3.52. Most DipEd students selected the correct value of 69.34 whereas many BEd students, 15 of the 39, and two of the DipEd students selected 75.264. These students may have been distracted by there being 3 digits after the decimal point, as expected if calculating the answer using the conventional multiplication algorithm. Only three students selected answers that involved incorrect powers of ten. In future, an improved question will have the three decimal point answer further from the simplest estimated answer (which here is 70 = 20 x 3.5).

In contrast, the ability to estimate according to place value proved more difficult for the division problem, 5.67÷0.032, perhaps exacerbated by the fact that it involved calculation with a decimal less than 1 (e.g., Greer, 1992). Again, the distractor of a precise answer (17.71875) attracted some (6 BEd and 2 DipEd students), while other students selected 2000 (4 students), 20 (3), 0.2 (3), and 0.02 (2). Whereas experts may have rated this primarily a conceptual question, for some students it was heavily computational, as their scripts showed that they calculated the answer, rather than estimating it.

The first part of question 11 was the worst done of all the questions (18%:55%). It required students to shade an area of 0.725 in an 8×10 rectangle that was marked to show 80 squares. Of those who got it incorrect, most successfully shaded 0.7 by using 7 of the 10 columns of 8 squares, but then had difficulty representing the remaining 0.025. The most common mistake, by 8 of the 38 students who were unsuccessful overall, was to shade 2.5 of the remaining squares, as if each square was a hundredth.

Perceived Computational and Conceptual Difficulty

The students distinguished between questions when rating them for difficulty. The spinner question and the decimal comparison questions were rated as relatively easy both computationally and conceptually (average rating over both cohorts less than 1.5 for computational difficulty, and less than 2 for conceptual difficulty). Question 6B, which required three explanations for why 3/8 is the same as 37.5% and which has been excluded from most of the analysis here, was perceived as the most difficult item. Both groups rated both categories greater than 2.5 on average, and the BEd students were almost unanimous in rating it as Hard conceptually. Other "hard" questions included the parallelogram item (conceptually), the division estimation question, the spaceship distances question (especially for the BEd group), and the item requiring an area of 0.725 to be shaded.

The DipEd students generally rated the questions as easier than BEd students. This is partly apparent in Figure 1. There were some exceptions, with similar ratings being given by both cohorts for the computational difficulty of the first ratio comparison question (question 3a), the multiplication estimation question (question 7), and representing 0.725 as an area, a fraction and a decimal (question 11); and for the conceptual difficulty of the parallelogram item (question 4A), the spinner (question 5), and question 11.

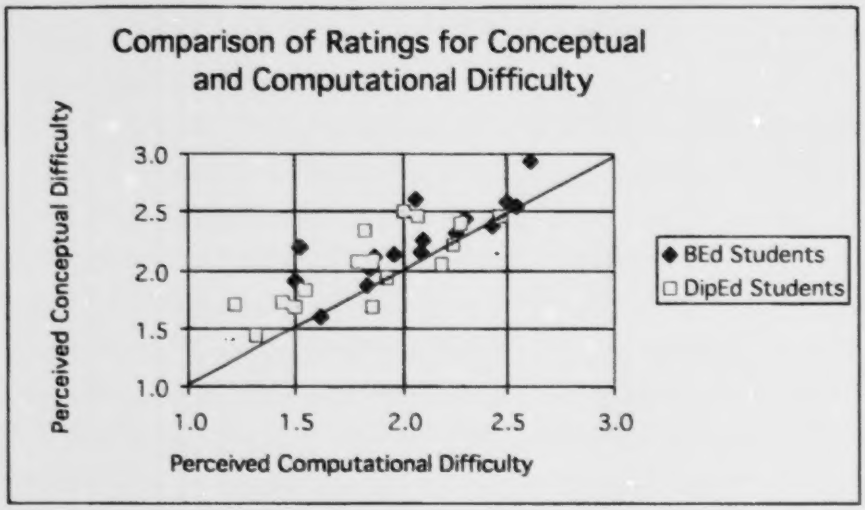

Figure 1. Relationship between the averages of the perceived conceptual and computational difficulties for different items for both cohorts of students.

Figure 1, showing average ratings for the two groups, reveals that students' perceptions of computational and conceptual difficulty were strongly related. Students gave most questions a higher rating for conceptual difficulty than for computational difficulty, with most points in Figure 1 above the line $y=x$. This was particularly true of the decimal comparison and number line items (Questions 1a and 1b), the geometry items (Question 4), and the probability spinner item (Question 5), all of which involved little or no calculation. For Question 10c, which asked students to say which spaceship travels further, the BEd students rated it equally on computational and conceptual difficulty, when in fact no actual computation is required (answers to likely computations were given). Nevertheless, recall that this question proved difficult for the BEd students, suggesting that perhaps they were themselves concentrating on calculation rather than concepts, or that they carried out computations that were not relevant to the question. The DipEd students, in contrast, had a significant difference between their difficulty ratings on this item. The multiplication and division estimation questions (Questions 7 and 8) were the only two questions where computational difficulty rated marginally higher than conceptual difficulty, but only for the DipEd students. Again, this is surprising given that only estimation was expected to have been involved. The largest difference between the average perceived conceptual and computational difficulties was 0.66 for the BEd students on questions 1a and 1b (decimal comparisons), although most differences on items were less than 0.3 for both cohorts.

Perceived Difficulty and Performance

This section examines the relationship between perceived difficulty and actual performance. Not all items were included because it was inappropriate to combine some scores, or because difficulty ratings were not requested. Where a single pair of difficulty ratings was collected for multiple parts, scores were combined. Because of the correlation between the ratings, the computational and conceptual difficulty ratings were added for this analysis. A difficulty value of 2 implies that the question was regarded as easy both computationally and conceptually, and 6 that it was perceived as hard in both aspects.

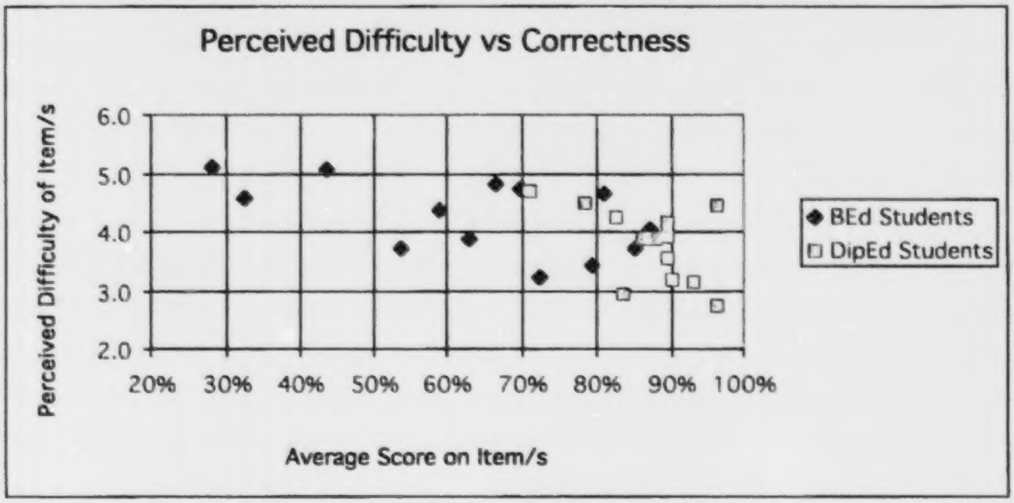

Figure 2. Relationship between the perceived difficulty of the items and average performance for both cohorts of students.

As can be seen in Figure 2, there was a general association between the students' average performance on questions and the average perceived difficulty of questions, with students perceiving as easier those questions on which they scored better (hence the negative slope of a trend line). There are a few interesting outlying values. Apart from one student who omitted it, the DipEd students scored perfectly on question 3c. They recognised the item's conceptual and computational difficulty, however, by giving it a relatively high difficulty rating of 4.45. It was similarly regarded by the BEd students, but they did not perform as well (Score=67%, Difficulty=4.80). Interestingly, my own opinion was that items 3a to 3c increased in difficulty, which was reflected in both cohorts' perceptions and the BEd cohort's performance, but the DipEd students actually improved in their performance over the three parts. Question 10c, requiring students to identify which of the two spaceships travels further, was similarly done well but regarded as moderately difficult by the DipEd students (Score=90%, Difficulty=4.17). In contrast, the BEd students also recognised the difficulty of 10c, but performed very badly (Score=28%, Difficulty=5.09). Question 5, ranked easiest by the BEd students, was not their best-performed question (Score=72%, Difficulty=3.21). The DipEd students also found question 5 easy, but did not perform particularly well on it (Score=84%, Difficulty=2.93).

Question 7 required students to select an estimated value for 19.7×3.52. This was regarded as relatively easy by the BEd students, although their performance was not good (Score=54%, Difficulty=3.70). The DipEd students performed better, but had a similar difficulty rating (Score=90%, Difficulty=3.54). It is interesting to compare these results with those of Question 8, which was similar but involved division instead of multiplication. Both cohorts regarded the division question as much harder than the multiplication item (BEd Difficulty=5.08; DipEd Difficulty=4.24), and yet each cohort's performance does not reflect this as much as might be expected (BEd Score=44%; DipEd Score=83%). The DipEd students, however, performed better than the BEd students.

Discussion and Conclusion

First and foremost it is acknowledged that, as it stands, the questionnaire has some shortcomings. Nevertheless there is scope for refining it so that it better measures the depth of understanding of fundamental mathematics and is also able to address pedagogical content issues, such as how to explain particular mathematical results to students. Future components of the study will involve others who might be called upon to teach middle school mathematics (e.g., science teachers), and pre-service teachers in other countries.

As can be seen from the results, the DipEd students perform significantly better on the aspects of fundamental mathematics used in the study. The area with the smallest gap was geometry, which may indicate that teacher education needs to address this for DipEd students in our state. The better content knowledge of the DipEd students was also reflected in the generally lower ratings of difficulty that they gave to the items.

While there were differences between the average rating levels of perceived conceptual difficulty and computational difficulty, it is not clear that these distinctions are clear in the minds of pre-service teachers, since some of the responses are difficult to explain. Future versions of the questionnaire need to consider this more carefully. This version, for example, did not provide participants with a good opportunity to distinguish between the two (e.g., there was no question that was obviously hard computationally while being easy conceptually, although there were questions that were the reverse). We think most mathematics educators believe that there is a difference between the computational and conceptual difficulty and that identifying it is important, but perhaps we need to think more about what we mean by it and how best to convey it to our pre-service teachers.

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A Problem-Based Schema Analysis in Algebra

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The development of students' algebraic understanding is generally accepted to be one of the major goals of K-12 mathematics teaching. In this paper I attempt to examine this understanding by characterising a group of high school students' algebraic knowledge and patterns of use of that knowledge during the solution of selected problems. Results show that these students tended to show acceptable levels of proficiency with problems that involve substitution of values for variables, and simplification of equations. However, students experienced difficulties with the solution of equations and the interpretation of variables both in symbolic and graphical modes. These results are interpreted as suggesting that the participating students' understanding was buttressed mainly by schemas that were dominated by procedural knowledge of algebra.

Introduction

Algebra provides conceptual foundation for the understanding of other concepts that students encounter in the school mathematics curriculum. The importance of this area of mathematics has been underlined by the increasing attention the teaching and learning of algebra has received over the past decade from teachers and researchers alike. Children's understanding of algebraic concepts begin in the early years of their school life and continues throughout their mathematics learning experiences in high school and beyond.

The ubiquity of the subject matter of algebra in K-12 mathematics curriculum further attests to its critical role in helping students develop an appreciation of links that exists among other topics in mathematics. Indeed, this issue has been given considerable attention in the agenda of major curricular documents (National Council of Teachers of Mathematics, 1989, 2000). The recently concluded 12th ICMI Study on the theme, 'The Future of the Teaching and Learning of Algebra' further highlights the importance of algebra and brought into focus the many difficulties faced by students in learning algebra.

Despite significant strides that we have made in improving students' confidence and competence in using algebraic skills and concepts, it has been suggested that more work needs to be done in this area as students continue to experience difficulty in going beyond the meaningless manipulation of equations and symbols (Chazan, 1996). In the study reported here I address this issue by exploring the nature of algebraic knowledge that drives students' cognition during problem solving.

Theoretical Considerations

Connections and Mathematical Understanding

The development of mathematical understanding has been analysed from a number of vantage points. Of these, the investigation of connections constructed by students has been the theme of recent debate. It has been suggested that children learn mathematics best when they are encouraged to 'organise their information through making many connections and forming relationships' (Sowder, 2001, p. 4). Hence the analysis of connections seem to

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provide an effective research strategy in the examination of mathematical understanding. The focus on connections has had a long history in psychological literature on concept development and problem solving not only in mathematics but also other domains such as geometry, chess and physics. In their analysis, Carpenter and Lehrer (1999) characterised mathematical understanding as involving problem solving, constructing relationships and reflecting on one's own previous experiences with a particular topic of mathematics. As mathematical understanding is a developmental process connectionist models are appropriate for describing relations between the above activities. The quality of the connections can also be expected to have a major impact on how that knowledge is used in a variety of learning situations (Schoenfeld, 1992).

Research from cognitive psychologists and mathematics educators has advanced several theoretical frameworks about concepts and their growth. In this paper, I adopt the network perspective in making judgments about mathematics knowledge development. According to this view conceptual growth and mathematical understanding can be interpreted in terms of the building of organised knowledge clusters called *schemas*. Schemas can be visualised as knowledge structures or chunks having one or more core concepts which are connected to other concepts and/or schemas by relational statements.

According to this framework of knowledge development, the quality of a schema is a function of two variables: the spread of the network and the strength of the links between the various components of information located within the network (Anderson, 2000). A complex schema can be characterised as having a large number of network of nodes that are built around one or more core concepts. Further, in a mature schema, the links between the various nodes in the network are robust, a feature which contributes to the ready accessing and use of that schema during problem-solving and other learning situations. A well-structured schema can also benefit students by helping them assimilate incoming new mathematical information with less cognitive effort.

The acquisition of mathematical concepts in K-12 can be seen as the construction of schemas each with differing levels of organization and complexity. The difference between a good student and a poor student is that the good student has built up schemas that are more complex, dense and better organised than his low-achieving peer. Chinnappan (1998) used the schema framework to compare the quality of geometry knowledge between high- and low-achieving students. Thus, a useful strategy would be to analyse the schemas of students for gaps in their knowledge, and organisational quality of conceptual nodes and links. According to the schema framework of knowledge and performance, an impoverished schema is not conducive to solving novel problems and describing relations among concepts in mathematics because it does not help students extend their prior knowledge to new boundaries of understanding. Such schemas can be characterised as having a limited number of conceptual points to connect with.

Structure of Algebraic Schema

While schema provides a broad theoretical framework for analysing organisational features of students' algebraic knowledge there is a need to disentangle components of the schema that underlies algebraic understanding. Literature on the development of algebraic understanding has advanced two constructs: procedural and structural conceptions. Broadly speaking, students who have attained procedural understanding can be expected to perform operations involving algebraic expressions such as simplifying equations. Conceptual understanding, on the other hand refers to the elucidation of relations between algebraic expressions and components that make up a particular algebraic statement. Sfard (1991) used a *process-object* model to articulate the relationship between procedural and conceptual elements of algebra.

Students' experiences with algebra begin with the acquisition of knowledge about procedures or operations that are used in dealing with algebraic situations. These procedures include strategies and rules for simplifying, factoring and solving equations. Also included in this set of skills is an understanding of conventions and symbols that are used to represent algebraic expressions. One such convention could be the use of letters to represent variables or $f(x)$ to represent function of x. As students' experiences with algebra matures they are able to transfer knowledge of procedures to conceptual characteristics of relations. Kieren (1992) referred to this advancement as the evolution from the 'procedural to structural' (p. 413). She argued that most students learn procedural skills but do not make the transition to structural understanding. Tall and Thomas (1991) also alluded to this link in their analysis of the nature of difficulties faced by students in learning algebra. Students who have developed multiple representations of an algebraic relationship can be expected to show high levels of structural understanding of variables that are embedded in that relationship.

The above analysis suggests that, among other things, algebraic schema consists of networks of nodes that are procedural and conceptual in nature. Accordingly, students who have built up a better connected and organised algebraic schema can be expected to make a smooth transition from the procedural to the conceptual aspects of the knowledge structure. For instance, in order for students to develop a sophisticated schema, say, about solution of quadratic equations, they need to make multiple connections among variables, families of equations and unknowns. As their schema becomes elaborated further one might expect to see information about how to use strategies in order to construct equations to model a problem situation. In this sense the maturation of schemas can be seen as progressing along the procedural-conceptual continuum. Thus, the construction of algebra schemas that are loaded with procedural information can be argued to be less complex than one that has more conceptual information. In a problem situation both components of the knowledge base are necessary but conceptual knowledge is more useful in generating powerful representations of a given problem. In this sense, one could argue that the conceptual part of the schema in indicative of deeper learning of algebra.

The purpose of the present study was to describe the quality of algebraic schema developed by a group of Year 10 students. In particular, I was interested to examine the level of procedural and conceptual knowledge that students could access in a number of problem contexts, and the integration of these knowledge components during problem representation. The research questions for this study are:

- What is the nature of procedural knowledge that Year 10 students activate during the solution of algebra problems?
- What is the nature of conceptual knowledge that Year 10 students activate during the solution of algebra problems?
- Is there evidence of transition from procedural to conceptual understanding of algebra problems among Year 10 students?

Method

Participants

The participants in this study consisted of 58 Year 10 (Form 4) students (28 males and 30 females) in a New Zealand secondary college. The students came from two classes representing 'average to above average' ability levels of the college's Year 10 cohort. All students had completed algebra topic. Students in the study also reflected the socio-cultural composition of the local community.

Instrument

The *Algebra Schema and Access Instrument* (ASAI) was developed for the study. The instrument contained 16 problems all of which required the accessing and use of algebraic knowledge. Problems 9, 14 and 15 consisted of two (9a, 9b), three (14a, 14b, 14c) and two parts (15a, 15b) respectively. An important consideration in the development of the instrument was the identification of algebraic schemas that teachers would expect from their Year 10 students. It is important to point out that I have used a problem-based schema identification approach, and that schemas activated by the students were necessarily limited by the problem contexts. It is possible that a non-problem based strategy could be expected to generate a different set of schemas. However, a problem-driven schema activation and use by the students could be argued to provide a more complete picture about the quality of schemas that students have built up because it has the potential to reveal more complex connections that exist not only within schemas but among schemas. The latter complex of connections among schemas has been argued to exert a major influence in the construction of problem representations (Sweller and Cooper, 1985). This line of reasoning was used in classifying the 16 problems into six representations (Table 1). A selected set of the problems from each category is provided in the Appendix.

Table1
Problem Categorisation

Problems	Category	Representations
1 and 2	A	Factorisation
3 and 4	B	Evaluation
5, 6, 7, 8, 9 and 10	C	Solution of equations
11, 12 and 13	D	Word problems
14a, 14b and 14c	E	Pattern generation
15a, 15b and 16	F	Graphical interpretation

Procedure

The class teachers administered the instrument to students during normal class periods. The study was conducted in the fourth term of the college's academic year. Students were given 60-90 minutes to complete the problems. Students were encouraged to attempt all the 16 problems. They were also asked to write down every step in their solution attempts even if they did not arrive at the 'correct' answer. Students were permitted to use calculators if required.

The following scoring scheme was developed to code students' solution attempts. There were two major considerations in this scheme: solution approach and generation of relevant values. The former was concerned with problem representation and the latter factor provided information on the use of schemas to generate values that were relevant to the problem representation. While solution outcome was important it was not the sole factor in the scoring scheme. The scheme was trialled with two independent coders who were mathematics teachers and researchers in order to resolve potential differences in the interpretation of the codes. The final scoring system was used to code students' solution transcripts.

0 - No attempt was made to solve the problem
1 - Solution was attempted but both the approach and values generated were incorrect
2-- Solution was attempted with a correct approach but none of the values generated were correct or relevant; incorrect solution outcome.
3 - Solution was attempted with a correct approach; one correct value was generated; incorrect solution outcome
4 - Solution was attempted with a correct approach; two correct values were generated; incorrect solution outcome
5 - Solution was attempted with a correct approach; three or more correct values were generated; correct solution outcome.

Results and Discussion

Results of analysis of students' solution attempts is presented in Table 2. The results show that students attempted all the 16 problems with varying degrees of success. The means and standard deviations indicate that some problems were more difficult (Problems 6-10, 13-16) than others (Problems 1, 2, 3, 11 and 12). From the representational angle the solution attempts reveal a number of patterns. Firstly, students constructed correct representations for most of the problems in category A and B, and a few problems in category C. In category D, students experienced more success with problems 11 and 12 in comparison with 13. In general, students experienced difficulty with the solution of all the problems in categories E (pattern generation) and F (graphical representation). In more than 50% of the problems presented students scored mean values of 1 or less indicating failure to attempt or construction of incorrect problem representations.

Table 2
Descriptive Statistics of Solution Scores

Problems	Mean	SD
1	2.55	1.59
2	3.59	1.60
3	4.64	1.18
4	2.19	2.08
5	2.12	1.76
6	1.69	1.85
7	1.71	1.36
8	1.48	1.53
9a	2.17	2.15
9b	0.66	1.19
10	1.09	1.42
11	3.48	1.75
12	4.48	1.41
13	0.93	0.72
14a	2.40	2.25
14b	1.64	2.13
14c	1.21	1.87
15a	1.48	1.75
15b	1.21	1.63
16	1.02	1.10

The present study attempted to answer three questions that are related to high school students' knowledge and understanding of algebraic concepts by examining schema activation and problem representation. The first question aimed at describing the quality of procedural knowledge of their schemas. Analysis of solution attempts to problems that mainly involved application of procedural skills (Category A and B) indicate that students had acquired a reasonable level of procedural knowledge of algebra. This was evidenced by the high proportion of success with problems that required substitution of numerical values into a given expression. Students also showed that they could expand and simplify algebraic expressions. However, students tended to experience difficulty in finding solutions to a number of equations that had a complex structure. Taken together, these results suggest that while students have built up a repertoire of process skills there were also knowledge gaps in their procedural knowledge. It is also possible that the solution of more complex equations in Category C required an understanding which needs to be supported by schemas that are more conceptually loaded. This suggests that students' algebraic schemas had more procedural than conceptual information.

Analysis of solution attempts relevant to research questions two and three focused on a) students' conceptual understanding of algebraic expressions, and b) evidence of establishing links between conceptual and procedural elements in the given problems. The mean scores indicate a high proportion of the students could not generate equations to model a given situation, and solve that equation. In addition, students experienced difficulty in representing and interpreting graphical forms of given equations (Category F). An equation is a symbolic form of a relationship that can be expressed in a graphical mode. A large number of students who participated in this study failed to translate the symbolic to the graphical form of a given equation and vice versa. The present results are consistent with findings of a number of other recent investigations of problem solving that involves modelling of problems in terms of algebraic expressions (Nathan, Kintch, & Young, 1992; Schoenfeld & Arcavi, 1988).

In the present study students performed poorly in a task that involved the establishment of a relation between two variables (M=1.21;SD=1.87). This problem can also be seen as exploring a pattern that exists between two sets of numbers. While a number of students could determine the value of one variable given the other, these students did not describe the overall relationship in any meaningful manner. These results reflect those obtained by Stacey (1989) who found that students had difficulty in reasoning that led to generalising a pattern among variables.

Students also did not seem to understand the notion of ordered pairs (x, y), and that the is a rule that connects values x with values of y. This misunderstanding was evident in the solution attempt of Problem 10 where students were required to decide if a point was a solution to the given equation. It would seem that these students had yet to establish a link between the coordinates of the point and the rule that was expressed in the equation. The mean score for this problem was 1.09 suggesting most students did not attempt or used an incorrect representation. Solution attempts to Problem 10 again provides further support to the claim that students' algebra schema lacked appropriate conceptual information.

On a more general level, the results of the present study are relevant to the debate over the causal relations between procedural and conceptual knowledge not only in the solution of algebra problems but mathematical problems. The present result is consistent with earlier research by Rittle-Johnson and Alibali (1999) who found that conceptual knowledge has a greater influence not only on the understanding of problems but also on the further development of procedural knowledge.

While it is too early to speculate, the present findings do suggest that teaching needs to focus on the development of both procedural and structural or conceptual aspects of algebra. It seems that higher levels of procedural skill development is a necessary but not a sufficient condition for students to solve problems that involve generation and manipulation of variables in an equation. Thus, classroom learning experiences need to make explicit the connections between these two aspects of algebraic knowledge.

This study represents a modest attempt at exploring the construction of representations for algebraic problems and the nature of schemas that support that construction. While there is some support here for the claim that teaching and learning algebra needs to focus on facilitating the building of more conceptually based schemas there is a need for a fine-grained analysis of algebraic schemas that drive problem representation.

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Appendix

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1. Factorise the expression $3x^2 + 6x - 9$

4. If $f(x) = 3x^2 - 7x$, what is the value of $f(2.5)$?

7. Solve the equation $x + \frac{6}{x} = 5$

11. A photograph is 3cm longer that it is wide. Its area is 40 cm^2.
Find its length and width.

14.

Q	*P*
1	1
3	?
4	10
6	16
n	?

a) What is *P* when *Q* = 3?
b) What is *P* when *Q* = *n*?
c) Describe the relationship between *P* and *Q*.

15(a). Graph the equation $5y = -15 + 3x$ in Figure A. (Figure A is a grid with *x* and *y* coordinates)

16 If you start with the light line, what would you do to get to the other line (dark)? (The light and dark lines were provided in a grid with *x* and *y* coordinates)

Year 3 Children's Understanding of Fractions: Are we Making Progress?

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The aim of the study reported here was to examine the quality of understandings developed by young children in the area of fractions and decimals. Analysis of data showed that the existence of great disparity in Year 3 children's knowledge base of fractions. We discuss these results in light of levels of competence that are expected in K-6 curriculum documents and with reference to past research on students' knowledge of fractions. The results of this small study raise doubts about the progress being made in the teaching of fractions.

Background

A principal aim of the primary mathematics curriculum is that students develop a sound understanding of the number system and become confident in using such understandings in describing real-world phenomena and solving problems (National Council of Teachers of Mathematics, 2000; Australian Education Council, 1990). Within the general topic of number, teachers and researchers have paid particular attention to fractions and decimals, for these numbers are used in making sense of problems in daily life of children and adults alike.

A major outcome expected in the areas of fraction and decimals is that primary school students should use these numbers and their relations flexibly to solve problems and reason mathematically (see *Mathematics K-6 Outcomes and Indicators*, Board of Studies - New South Wales, 1998). Despite the critical conceptual link provided by fractions between mathematics strands such as space and measurement, this area continues to present difficulties for some young children in primary schools. Indeed, analysis of state-wide performance of recent Year 5 students in New South Wales (NSW Department of Education, 1999) shows that students' understanding of and reasoning with fractions and decimals is less than satisfactory.

The situation in NSW is similar to what has been observed earlier in Victoria (Clements and Del Campo, 1987) and in other countries. Ellerton and Clements (1994) referred to the general area associated with fractions in Australia as a "weeping sore in mathematics education." In commenting on the poor performance in fractions among American students Brown et al (1988, p. 245) commented that the computational activity of these students seemed to be carried out "without developing the underlying conceptual knowledge about fractions."

Taken together, these findings suggest that the expectations held by mathematics curriculum developers for a sizeable group of students in the upper primary school are unrealistic. It is clear that a sizeable number of students will not achieve the following objective: 'Students develop an understanding of the parts of a whole, and the relationships between the different representations of fractions' (Board of Studies - New South Wales, 1998:61).

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The findings of the state-wide testing in NSW are frustrating. Many Year 5 students, despite having experienced the mathematics activities presented by their teachers, still have difficulty in using their knowledge of fractional and decimal numbers to solve problems that are regarded as being appropriate ones for that level of schooling. Yet, in their work with Australian preschool children, Hunting and his colleagues showed that while these young children did not usually understand fractions language, many of them had developed knowledge about sharing that would provide a sound basis for construction of future knowledge of rational number (e.g., Hunting & Sharpley, 1988). It seems clear that between the time of development of this rudimentary understanding and the later primary years many of these students do not develop knowledge structures that will support the problem solving with fractions and decimals that is demanded in Stage 3 of the K-6 curriculum and beyond.

This area of mathematical learning among young children has received considerable attention from researchers in recent years (e. g., Behr, Harel, Post & Lesh, 1992; Bezuk & Bieck, 1993; Baturo & Cooper, 1995; Owen & Super, 1993). More specifically, a significant body of research has focused attention on the development of rational number knowledge from children's understanding of whole numbers. Rational numbers are complex in character and provide important prerequisite conceptual tools for the growth and understanding of other number types and algebraic operations in secondary school. Young children are exposed to these number concepts at an early age in a variety of real-life situations, such as measuring and dividing continuous quantities, and quantitative comparison of two quantities. These experiences should enable young children to develop an understanding of rational numbers which matures through learning situations they encounter in the classroom. As the students move through the primary years we might anticipate that there will be a gradual integration of their understandings and the formal representations of mathematics.

Almost two decades ago Hiebert and others suggested that we were being too optimistic in anticipating such an integration of form and understanding (e.g., Hiebert, 1984, 1989). Hiebert argued that one of the factors contributing to the poor levels of performance of many students on fraction and decimal problems could be represented as a problem in establishing connections between form and understanding. One interest in this study was to see if this construction of the problem was still valid.

The multifaceted nature of fraction number has made the task of describing its growth difficult. Several attempts have been made to capture the complexity of fraction numbers and children' construction of these numbers. The most detailed analysis of fraction numbers has been undertaken by Kieren (1988). His analysis showed the fraction number knowledge consists of many interwoven strands. He identified eight, hierarchically related, levels in his description of fraction number thinking. An important outcome of this model is the specification of cognitive structures that provide the basis for the maturing of fraction numbers among young children. These structures or schemas which appear at levels three and four consists of what he referred to as *subconstructs*: partitioning, unit forming, quotients, measures, ratio and operations. These subconstructs among others play a key role in young children's understanding and interpretation of fractions. The model shown in Figure 1 builds on Kieren's representation of fraction subconstructs and also draws out potential links among the constructs (Chinnappan, 2000). These links are important for the analysis of the organisational quality of the knowledge of fractions.

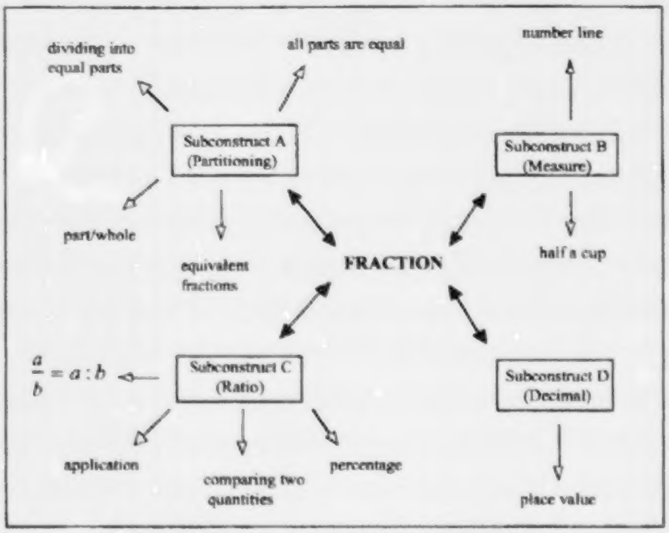

dividing into equal parts
all parts are equal
number line
Subconstruct A (Partitioning)
Subconstruct B (Measure)
part/whole
equivalent fractions
FRACTION
half a cup
$\frac{a}{b} = a:b$
Subconstruct C (Ratio)
Subconstruct D (Decimal)
application
comparing two quantities
percentage
place value

Figure 1. Knowledge components in a Fraction schema.

Although we do ultimately want to investigate the growth in complexity of students' fraction knowledge our concern here was with the gathering of necessary preliminary information about this knowledge. Our objective here was to gain information about how well-prepared students in early primary school might be for a discussion about fractions. Put more crudely, we wanted to find out how well prepared these students might be to get into the 'game' of fractions. We wanted to know if they had the common language, diagrammatic and symbolic vocabularies to participate in the discussions that would occur in their classrooms and whether this vocabulary was linked to well-developed schemas. Thus our initial focus was not on the complexity of the students' structure of fraction knowledge per se. It was focussed on the elements of the interaction that might typically occur between teacher (or textbook) and student.

Method

Participants

Twenty four children in Year 3 (aged between 8 and 9) from a regular suburban school participated in the study. The school was situated in a middle-class suburb of a metropolitan city and was not selective in its intake. The sample contained 13 boys and 11 girls. The children had studied whole numbers and fractions within the Number Strand of the K-6 mathematics curriculum in the previous two years of primary school and at the time of the study had completed the topic on fractions in Year 3.

Tasks and Procedure

A range of tasks were developed for the purposes of assessing children's knowledge about proper fractions. One set of tasks required children to respond to a series of questions that focussed on comprehension of simple sentences containing fraction words. The child's comprehension of the sentences was checked first and then they were asked to identify number words and those words that referred to fractions. Their knowledge of the labels used for fraction words was also assessed. 'Half' and 'quarter' were the fraction words used in this first set of tasks.

A second set of tasks involved children having to exhibit their understanding of fractions via diagrams. In this case the child was asked to represent the fraction words in diagrams and also to illustrate fractional quantities in diagrams. The students were also asked express fraction words in mathematical symbols, and to display their knowledge of both the size of different fractions and of equivalent fractions. In the final set of tasks, we attempted to assess children's understanding of the concept 'fraction' and of the symbols that were used in representing fractions. Figure 2 shows a selection of problems from the seven sets of fraction tasks.

Each child was interviewed individually and was asked to respond to all the focus questions and problems. Children's responses were also probed to generate more complete data about their knowledge of fractions. All sessions were audio-taped and transcribed for subsequent analysis.

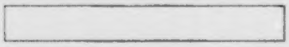
Set 1

'For your breakfast your mother filled half a glass with milk'

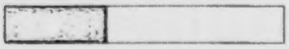
What did your mother do?
Draw a picture to show me what your glass of milk looked like when your mother had finished putting the milk in it.
Pick out any words in the sentence that refer to numbers,or parts of numbers.
Does that number have a special name?
What does 'a half' mean?
Can you write 'half' any other way?

Set 2

John cut a strip of paper into five parts. His teacher said that each part is 'one- fifth'.
Look at this diagram of the strip of paper. Show me how much *is one fifth.*



Look at this second diagram.
Is the shaded part equal to one fifth?
If it isn't could you change the diagram so that one fifth was shaded?



Set 3

In the number $\frac{2}{5}$, what does 2, - and 5 mean?

Figure 2. Fraction tasks.

Results and Discussion

To facilitate the description of students' task performance we formed the eight composite measures shown in Table 1. These measures summarise performance across all the interview questions.

Table 1

Definitions of Composite Measures

Identification of fraction words:	Identification by the student of words that refer to numbers, including fraction words.
Knowledge of labels:	Knowledge that the term 'fraction' referred to words such as 'half' and 'quarter'.
Representing fractions in mathematical symbols:	Ability to represent a fraction using a/b format.
Meaning of fraction words:	Knowledge of partitioning of a whole into equal parts
Representing fractions in diagram form:	Ability to represent fractions in simple diagrams or to modify diagrams to represent fractions
Meaning of a/b symbols:	Knowledge of the meaning attached to the a/b representation of a fraction
Relating and ordering fractions:	Ability to write an equivalent fraction and to order fractions
Explanation of fraction ordering:	Knowledge of how to establish the relative sizes of different fractions

The mean percentage scores for the total group of students on each of the composite variables are shown in Table 2. The figures in the table show that there was a high degree of variability in the scores on each set of tasks. The students performed best on the tasks requiring them to recognize and label the simple fraction words in the sentences provided as stimulus materials. Their knowledge of the meaning of fraction words and of how to represent fraction words in both diagrams and mathematical symbols was only moderately good. They were least successful on the tasks that required them to order and relate fractions written in a/b form.

Table 2

Group Scores (%) for Composite Measures (n=24)

Measures	Possible Score for each student	Percentage scores Mean	Percentage scores SD
Identification of fraction words	6	76.4	34.0
Knowledge of labels	6	68.8	33.4
Representing fractions in mathematical symbols	4	58.3	45.8
Meaning of fraction words	8	56.3	40.4
Representing fractions in diagram form	12	54.9	28.1
Meaning of a/b symbols	2	43.8	45.0
Relating and ordering fractions	6	16.7	27.8
Explanation of fraction ordering	2	8.3	24.1

Clearly, the large variation in performance indicates that the pattern of performance shown in Table 2 hides much interesting detail. Inspection of the performance of individual students shows a very substantial disparity in performance of students within this class. This disparity is illustrated more fully in Table 3 where the patterns of performance of four students is shown.

Table 3

Individual Scores of Four Students for Task Sets

Measure (possible score)	S3	S23	S11	S17
Recognition of fraction words (6)	6	6	0	0
Knowledge of labels (6)	6	6	0	2
Representing fractions in mathematical symbols (4)	4	4	0	0
Meaning of fraction words (8)	8	8	0	0
Representing fractions in diagram form (12)	11	12	2	2
Meaning of a/b symbols (2)	2	2	0	0
Relating and ordering of fractions (6)	6	4	0	0
Explanation of fraction ordering (2)	2	0	0	0

Here we have two pairs of Year 3 students with widely differing knowledge bases and very different capacities to effectively engage in discussions about fractions. Students 3 and 23 performed at a very high level and could cope with much of the fractions work being done in Year 5 or 6 classes. They have the vocabulary necessary to work on fractions problems and have quite complex knowledge related to the partitioning. Student 23, for example, not only responded correctly to all questions about language and symbols that were related to fractions but was also able to generate a correct solution to a problem which required him to work out the number of extra squares that had to be shaded so that $\frac{4}{5}$th of a given rectangle was shaded. Student 3 was able to give an acceptable interpretation of the a/b format and correctly ordered a set of fractions that included one improper fraction.

Students 11 and 17 showed very little evidence of even a rudimentary knowledge of fractions. Indeed their knowledge of even the language of fractions was limited, though they could both show half a glass of milk in diagram form. S17, for instance, could not recognise anything that referred to numbers in the sentence about breakfast and milk (Figure 2: Set 1, Question 4).

It seems obvious that if these pairs of students are to develop more powerful knowledge structures in the short term they need different and differently paced curricula. The first pair, along with several other students in this class, need to be engaged in activities that move them toward development of knowledge of the fraction subconstructs in the bottom half of Figure 1. For this group the expectations in curriculum documents are far too low. For example, according to the Outcomes and Indicators (Board of Studies- New South Wales, 1998, p. 61) students in Stage 2 of development (Years 3-4) are expected to 'represent numbers in tenths and hundredths using concrete material and grids, and record these numbers in words'. While these are necessary skills in the development of fraction schema, students S3 and S23 have exceeded this requirement, and are in need of, and ready for, more complex representations and operations involving fractions and decimals. We suggest that such students need exposure to subconstructs C and D of Figure 1.

The other pair of students is representative of a group that constituted about a quarter of this class, all of whom had a poorly developed understanding of the fraction concept. Although some of these students do have some of the common language and diagrammatic vocabulary, the inconsistency of their performance indicated that even this vocabulary was not strongly established. Students 11 and 17 are poorly placed to get into the fractions game. Once the teaching in class involved more than discussion of 'half', the performance we observed suggests that these two students would struggle to make meaning. Neither student would be likely to have the vocabulary to create links between what they already knew and what might be involved in discussion of 'one fifth' or '$\frac{1}{2}$'. There would be very little of Figure 1 that we could confidently judge to have been established by these students.

Conclusion

The patterns of performance here provide a somewhat more detailed account that shows what might lie behind the patterns of performance in state-wide testing. The results of this study shows that although the participants came from one Year 3 class there existed a wide disparity in their understanding and representation of fractions. The performance of many, if not most, of the students in this class was only weakly related to the specifications established for their year level in curriculum documents.

The results also suggest that the argument made by Hiebert (1984) about students in the US still holds for many Australian students. In his analysis Hiebert focussed on the problems of establishing links among a student's informal understanding and the formal symbols and procedures introduced in the course of mathematics classes. In this study we see strong evidence of problems in the first of Hiebert's sites of connection, the connection of understanding and the meaning of the symbols used in mathematics lessons. Hiebert's emphasis on the importance of connections should make more sense to us than it might have in 1984, or when Skemp (1971) made similar arguments at an earlier time. With our current desire to place all student learning with a constructivist framework (e.g, the South Australian Curriculum Standards and Accountability framework) the view that we should focus on the building of links between form and understanding seems all too obvious. Yet it seems that these links are not being established by many students.

In a recent paper Cobb and Bowers (1999) also use the crude language of the classroom 'game'. They remind us about the consequences if students lack the necessary vocabulary to take part in the game.

> 'All students must have a way to participate in the mathematical practices of the classroom community. In a very real sense, students who cannot participate in these practices are no longer members of the classroom community, from a mathematical point of view' (p. 9).

We must admit that students like Students 11 and 17 in our group might well have been excluded from the mathematical community in their classroom. And there is also the possibility that Students 3 and 23 might 'leave' that mathematical community unless they were provided with opportunities to develop their quite powerful understandings about fractions.

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Drill, Examinations and the Learning of Mathematics

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This paper presents an overview of the findings of three masters-level research investigations that the writer supervised during the period 1998–2001. Each investigation was based on a "multiple perspectives" design, and analyses revealed that most of the Bruneian secondary school children who were taught mathematics in the traditional "skill-drill" way did not learn much mathematics. Data from the studies suggest that emphasis on preparation for forthcoming tests and examinations does not result in students achieving a genuine understanding of mathematical topics. Lim's (2000) summary of a didactical contract between Bruneian secondary mathematics teachers and students is presented. It is argued that the teachers and the students, and indeed whole school communities, are continuing to exercise uninformed faith in a method of teaching and learning which is not "producing the goods."

Schools in Brunei Darussalam place great emphasis on the preparation of students for forthcoming tests and examinations (Khoo, 2001; Lim, 2000; Noridah, 1999). Primary school students are prepared for the high-stakes Primary Certificate of Education (PCE) examinations, and mean PCE performances of schools are ranked and monitored. Similarly, lower secondary students are prepared for the end-of-Form 3 *Penilaian Menengah Bawah* PMB national examinations, and upper-secondary students are prepared for the University of Cambridge Local Examination Syndicate's (UCLES's) O- and A-level examinations. There is also an N-level examination, which is organised within Brunei Darussalam for those not considered to be ready for O-level studies. All mathematics teachers are expected to test their classes regularly (at least once a month), and special lessons are dedicated to revision, tests, and to giving overviews of test results.

The Multiple Perspectives Approach to Researching the Teaching and Learning of Mathematics in Schools in Brunei Darussalam

Mathematics Classroom Patterns in Brunei Darussalam: Lim's (2000) Case Study

Three studies carried out in Brunei Darussalam, by Khoo (2001), Lim (2000) and Noridah (1999) and supervised by the present writer have provided strong evidence that the traditional "teacher introduction-worked examples-seatwork" pattern in mathematics lessons is well entrenched within schools in that nation. To illustrate the extent to which the pattern manifests itself it will be useful to summarise Lim's (2000) study into the teaching and learning of algebra in four O-level classes at a secondary school. Altogether 94 students in four classes at a government secondary school ("School X") participated in the study. These classes were taught by three experienced, professionally trained, and mathematically well-qualified teachers. One teacher taught two of the classes.

Lim's research questions. Lim (2000) investigated four research questions:

- What knowledge and skills did the O-level students possess and how confident were the students about algebraic equations and factorisation in January 2000?

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 203-210). Sydney: MERGA. ©2002 MERGA Inc.

- What teaching and learning strategies did the teachers and students adopt in lessons on equations and factorisation between January 2000 and March 2000?
- What knowledge and skills concerned with algebraic equations and factorisation did the students learn between January 2000 and March 2000, and how did this affect their confidence to carry out related algebra tasks?
- What attitudes, societal forces and assumptions, had the greatest influence on the teaching and learning of algebraic equations and factorisation?

One of the classes (Form 4A) comprised 23 students who had gained high marks at the 1999 *Penilaian Menengah Bawah* PMB examination and were being prepared for the "Additional Mathematics" and the "Mathematics (Syllabus D)" examinations of the University of Cambridge Local Examination Syndicate (UCLES). The other 71 students, in Form 4B, Form 4C, and Form 4D, were preparing for UCLES's Mathematics (Syllabus D) examination, but *not* for the Additional Mathematics examination. All 94 students had passed both Mathematics and English at the 1999 PMB examinations.

The multiple perspectives research methodology. Lim (2000), like Noridah (1999) and Khoo (2001), employed a multiple perspectives methodology, with data being gathered from nine "vantage points". These vantage points were: (a) relevant documents; (b) pre- and post-teaching test performance; (c) responses by students and teachers to questionnaires; (d) classroom observations; (e) student confidence and other affective domain data; (f) students' study habits; (g) textbooks used by the students and teachers; (h) student artefacts; and (i) pre- and post-teaching interviews.

The Didactical Contract Identified by Lim (2000)

The climax of Lim's (2000) dissertation came in his identification of an unspoken, informal, but powerful "contract" which constrained the teaching and learning of mathematics in the fourclasses which he studied. Lim borrowed the idea of a "didactical contract" from Brousseau (see Balacheff, Cooper, & Sutherland, 1997), whose concept of a "didactical contract" embraces the obligations and sanctions that teachers and students impose on each other. It takes into account not only the knowledge that is to be taught and learned, but also the manner in which the teacher will teach and the students will behave.

When classroom observation data associated with the four O-level classes involved in Lim's (2000) study were triangulated with data from questionnaires and interviews, seven common elements and relationships within the didactical contracts between the three teachers and the students in the four classes were identified. Lim's (2000) analyses left no doubt that these didactical contracts were a vital component of the micro-cultures associated with the four mathematics classes he studied. The contracts greatly influenced not only what content was taught, but also how the teachers went about teaching that content and what the students did in their efforts to learn it.

The seven elements of the didactical contract identified by Lim (2000) were:

1. Class lessons should proceed in a formal manner, in which the teacher should teach students how to answer questions similar to those likely to appear on examinations. The teacher should not assume that the students know much about related work done in the past.
2. The teacher should briefly introduce a lesson by explaining the types of questions, which the class was about to consider. The teacher should then write (on the whiteboard) the solutions to a number of worked examples which

illustrate the main types of questions likely to be asked by examiners. After allowing the students sufficient time to copy the worked examples into their exercise books, the teacher should then set students to work on questions from past examination papers. After about 30 minutes of seatwork, during which time the teacher should move around the room providing assistance to students who might need it, the teacher should close the lesson by setting relevant examples for homework.

3. The teacher should not do anything which might embarrass a student publicly, and the students' behaviour should not embarrass the teacher. The teacher should not expect students to ask questions publicly. and the teacher should not ask individual students to answer difficult questions publicly. The only questions which teachers were "permitted" to ask were those at a sufficiently low cognitive level that chorused responses could be given.
4. The teacher should not give detailed mathematical explanations of difficult points, which would not be understood by most students. (Thus, for example, in lessons observed, the quadratic formula was merely stated, and not derived; its use was illustrated through worked examples.)
5. The teacher should not depart from the standard lesson sequence ("introduction to topic, followed by worked examples, followed by seatwork") except immediately before and after tests and examinations, when the teacher might spend the whole lesson going over examples with the class.
6. The students should copy, accurately, into their workbooks any points made on the whiteboard during the introduction, and also the worked examples written by the teacher on the whiteboard. Some homework (but not too much) should be expected of the students.
7. Provided the teacher behaved in the above-mentioned way, the students would co-operate. The teacher should permit the occasional lapse of concentration with students, and the students would tolerate the occasional "tantrum" from the teacher provided there was some justification for it.

The didactical contract outlined by Lim (2000) pointed towards a form of instruction which is similar to a pattern of instruction common in mathematics classes in the United States of America throughout the twentieth century (see, e.g., Cuban, 1993; Goodlad, 1984; Grossman & Stodolsky, 1994; Santagata & Stigler, 2000; Stigler & Hiebert, 1999). The same pattern has been found to be common in mathematics classrooms in Australian schools (see, e.g., Clarke, 1984; Mousley & Clements, 1990). Research indicates that this method of teaching rarely generates deep student understanding of concepts (Farrell & Farmer, 1988; Skemp, 1976; Suydam & Dessart, 1980). Indeed, in a landmark review of research, Resnick and Ford (1981) concluded that the practice of presenting mathematical concepts through stand-alone examples and repetitious practice sets does little to foster understanding or to affect transfer of learning to other areas.

Some Bruneian Performance Data

The Language Issue

Research by Clements (1999), Fatimah (1998), Sainah (1999) and Saman (2000) has shown that many upper-primary and lower secondary students in Brunei Darussalam understand very little of what they are taught in their mathematics lessons. Although that might be associated with the fact that mathematics is taught in *Bahasa Melayu* in Primary 1, 2, and 3, and in *English* in Primary 4 and above, it is salutary to note that upper primary and secondary students in Brunei Darussalam do no better on mathematics questions written in *Bahasa Melayu* than in English (Radiah, 1998; Saman, 2000).

Often the language transition issue is used as an excuse by teachers to concentrate solely on skill manipulation. Noridah (1999) reported that teachers tend to say that it is difficult to teach Bruneian students to solve mathematics word problems, and therefore they concentrate on teaching skills, because that will help their students pass examinations. Noridah (1999) found that A-level teachers who participated in her study consciously carried out time-cost-benefit analyses of how topics should be taught in order to maximise student performance. For some topics the teachers decided not to teach for "understanding" because that would take "too much time".

Many Bruneian students have difficulty understanding questions written in English. Consider, for example, the performance of Bruneian tenth-graders on the following question (taken from MacGregor (1991): "The number y is eight times the number z. Write this information in mathematical symbols". Only 11.7% of Form 4 O-level Mathematics students in Lim's (2000) study gave a correct response. 16.2% of the responses yielded the "reversal error" (i.e., $z = 8y$) and 71.2% of the responses were in the "other error" category. Responses in the "other error" category included: $y^8 = z$, $y^8 > z$, $8y > z$, y^8z, $y = z^8$, and $8yz$.

When Lim asked Form 4 teachers why students did so poorly on a question which should not be expected to be difficult for even Form 1 students, the teachers said students tended to forget everything they learned in mathematics classes in earlier years. Data put forward by Khoo (2001), Lim (2000), Noridah (1999), Radiah (1998), Sainah (1999) and Saman (2000) suggest that the teachers' explanation was probably correct. Khoo (2001) showed that most of the O-level students in her Form 5 sample could not answer correctly geometry questions which had appeared on the Form 3 PMB examination.

Some Further Algebra Performance Data

Lim (2000), Noridah (1999), and Radiah (1998) found that most O-level students who had "passed" the end-of-Form 3 PMB national mathematics examination knew little about the algebra taught to them in Forms 1, 2 and 3. When Lim (2000) asked his four O-level classes to attempt 30 algebra questions based on lower secondary algebra, the results were as shown in Table 1. Although almost all of the 30 questions involved only elementary factorisation and equations covered in Forms 1 through 3, the mean number of correct answers given by the students in Form 4B, 4C and 4D was about 2.6. All of the students in those classes had passed the 1999 PMB Mathematics examination.

Table 1
Performance on 30 Algebra Questions by Four Form 4 O-Level Classes (Lim, 2000)

Form	No. of Students	Mean Score/30	Standard Deviation	Lowest Score/30	Highest Score/30
4A	23	6.7	3.4	2	14
4B	22	4.3	3.2	0	13
4C	20	1.6	1.2	0	5
4D	29	1.9	1.9	0	7
Overall	94	3.6	3.2	0	14

Table 2 shows performances, by the different classes, on selected questions. The national mathematics curriculum specifically includes factorisation of the difference of two squares (i.e., $a^2x^2 - b^2y^2$), in the Form 2 syllabus and questions on the difference of two squares are included each year on the PMB mathematics examinations. It is a matter of surprise then, that so few students factorised $x^2 - 4z^2$ correctly. Even the results of students in the top-stream Form 4A were not uniformly good. For example, when asked to solve the equation $x^2 = 9$, no student in Form 4A gave the answer "$x = 3$ or $x = -3$" (or "$x = \pm 3$").

Table 2

Class Comparisons for Questions on Factorisation (from Lim, 2000)

	Percentage Correct for Factorising …			
Form	Q. 19 $2x - 4$	Q. 20 $4xy^2 - 9xy$	Q. 22 $x^2 - 4x + 4$	Q. 25 $x^2 - 4z^2$
4A	65.2%	30.4%	17.4%	26.1%
4B	40.9%	22.7%	4.5%	22.7%
4C	10.0%	5.0%	0.0%	5.0%
4D	10.3%	3.4%	0.0%	10.3%
Overall	30.9%	14.9%	5.3%	16.0%

After the four Form 4 classes had been given lessons specifically on factorisation and solving equations, the means (out of 30) for the classes on the 30 algebra questions became 11.7 (Form 4A), 10.1 (Form 4B), 5.3 (Form 4C) and 3.0 (Form 4D). Although pre-teaching/post-teaching class mean gains were highly statistically significant, that is misleading because these "top" Form 4 students still were unable to do most of the standard skill-based questions which they had just been taught to do. In fact, post-teaching interviews indicated that students had learned little from the drill and practice routines. Even students who could factorise simple algebraic expressions merely followed rules, and did not know why the manipulations they carried out actually worked. The best students could factorise, but hardly any of them knew what the word "factor" meant in the context of algebra. Likewise, students could solve simple equations but did not know what the word "solution" meant in the context of algebraic equations.

Interview Data

This fundamental lack of understanding is revealed in the following excerpt of an interview with a Form 4 O-level mathematics student (taken from Lim, 2000) which took place after the student had attended O-level lessons in which factorisation and equations had been specifically taught. The interview centred around the Form 4 student's efforts to factorise $2x - 4$, a skill which had been taught in Form 2, and again in Forms 3 and 4.

Immediately before the interview the student generated a three-line response to a question asking her to "factorise completely $2x - 4$." She had obtained the answer "$x = 6$." During the interview the following "conversation" occurred (I = Interviewer, S = Student):

Interviewer: Let's try Question 19 please *[requiring the student to factorise completely, $2x - 4$]*.
Student: *[Student tried.]*
Interviewer: Can you please read the instruction?
Student: Factorise each of the following expressions completely. All working must be shown.
Interviewer: Can you read Question 19 please?
Student: $2x$ minus 4.
Interviewer: So, what does the question want you to do?
Student: To factorise.

Interviewer: What does it mean?
Student: To make the number smaller.
Interviewer: Can you explain what you have done?
Student: $2x$ minus 4 equals to x equals to 4 plus 2. x equals to 6.
Interviewer: Can you explain how you get x equals 4 plus 2?
Student: Transfer 2 to the other side. Then I get x equals to 4 plus 2.
Interviewer: How do you get 4 here *[pointing at $x = 4 + 2$]*?
Student: I put 4 here and then minus 4 change to plus 4.
Interviewer: OK. What is your answer?
Student: x equals to 6.
Interviewer: What has $x = 6$ got to do with the original question?
Student: I don't know.
Interviewer: How do you check your answer?
Student: Ask the teacher for the answer.

Major Factors Influencing the Poor Mathematics Performance and Understanding of Many Bruneian Students

Outside-of-Class Time Spent Studying Mathematics

Homework and outside-of school tuition in mathematics. Data generated by Lim's (2000) *Study Habits Questionnaire* revealed that 64 percent of the students involved in the study reported spending not more than 1 hour per week doing mathematics homework. In May 2000, almost all of the students stated that they had not taken extra tuition classes in mathematics that year; half of them indicated they would not take private tuition in mathematics later in the year. In Japan and Singapore, by contrast, almost all secondary students receive supplementary tuition in mathematics (Menon, 2000; Schümer, 1999).

When the students were asked whether they used textbooks other than those provided by the school for mathematics homework, most replied "no". Students said that they got most of their mathematics homework questions from the book which they took to class (i.e., Dyna Publisher, 1999)—which, in fact, was a compilation of past examination papers.

The drill-and-practice, examination-orientation of the lessons on algebra. Khoo's (2001), Lim's (2000) and Noridah's (1999) analyses suggest that in the Bruneian context the strong emphasis on drill and practice, and the strong examination orientation in lessons, actually *causes* poor performance and poor understanding. The students did not learn mathematics relationally, and in test and examination situations they struggled to work out which mathematics was needed for which problems. Many of the students had committed a large number of mathematical facts and skill procedures to memory, but because the volume of the memorised material was so great they often forgot parts of it.

Summary and Concluding Comments

Study habits data reported by Khoo (2001) and Lim (2000) indicated that on average the O-level students (at different schools) spent only one hour per week on mathematics homework. That is well below the amount of time spent on mathematics homework by students in Japan, Singapore, the United States and Germany (Menon, 2000; Schümer, 1999). By contrast with the situation in Japan and Singapore, few Bruneian students obtained private tuition in mathematics, and most did not obtain "quality" help at home.

The teachers in Lim's study were resigned to the likelihood that, with the exception of the top-stream Form 4A students, most students would obtain low grades on the O-level examinations. From the teachers' perspective that was realistic because most O-level classes at the school throughout the 1990s had obtained low grades for Mathematics.

Perhaps cramming for examinations, inherent in the didactical contract identified by Lim, was the root of the performance problem. The drill and practice routines, and low-level use of language, led students to regard mathematical "understanding" as something equivalent to being able to answer examination questions correctly. They had come to believe that the only way to learn mathematics well was through drill and practice.

Data reported in Noridah's (1999), Khoo's (2001) and Lim's (2000) studies indicate that in pressure situations (e.g., in tests and formal examinations) most of the students could not remember which skills should be associated with which problems. That was probably because the students had tended to try to learn a whole host of separate skills by rote, and their short-term memories had been strained to the point of bursting.

The lessons analysed by Khoo (2001), Lim (2000) and Noridah (1999) tended to be very examination-oriented. In each lesson a teacher-led introduction was followed by worked examples (which the students copied), and then seatwork, in which students attempted questions from past examination papers. There appeared to be a powerful didactical contract operating, which defined how teachers and students should behave. The main element of the agreement was that the teacher should provide the students with notes and worked examples, and students should do their best to answer questions from past examination papers. The teacher should not expect too much of students in the classroom, and not too much pressure should be put on them to do homework.

In Lim's (2000) and Khoo's (2001) studies, after the lessons on algebraic equations and factorisation (Lim) and the circle theorems (Khoo), the class means of the students' performances on relevant tests were only slightly higher than at the pre-teaching stage. Even after the lessons, many of the students still knew very little about the topics concerned. In most cases, any learning which had occurred was almost entirely of the instrumental kind, with students struggling to remember the rules they had been taught.

The teachers in Khoo's, Lim's, and Noridah's studies were aware of their students' low levels of understanding of basic mathematical concepts, and their students' imperfect knowledge of associated mathematical skills. The teachers blamed that reality on the fact that the students had failed to learn mathematics satisfactorily in Primary 6 and in Forms 1 through 3. In other words, they thought the students had moved on to upper secondary mathematics without possessing the fundamental mathematical skills they would need at that level. All the teachers agreed that a pass on the end-of-Form 3 PMB mathematics examinations did not guarantee possession of basic mathematical knowledge and skills. They acknowledged that students prepared for examinations by memorising formulae. As soon as the examinations were over the students were happy to forget everything.

O-level teachers felt pressured to move through the O-level syllabus as quickly as possible, to leave plenty of time for preparing students for tests and examinations (mainly by going through past examination papers). From the teachers' perspective, there was no time to teach for relational understanding. Examinations waited for no one.

An important part of the didactical contract was the use of questions from past examination papers in class, and for homework. For students, that guaranteed they were being well prepared for examinations, especially the high-stakes examination. However, this paper has raised the fundamental issue of what "well prepared" means in that context.

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Teacher Perspectives on Derivative

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The role of the teacher in the learning of mathematics is one which continues to be refined. Evidence continues to mount for the relationship between teachers' content knowledge and pedagogical content knowledge, and the way they structure their classroom lessons. In this paper we present data from interviews with four teachers about their teaching of differentiation and derivative. The analysis shows that there are both similarities and differences in the pedagogical approaches emphasised, and that these are related to the teachers' conceptual and representational perspectives.

Background

In order for students to learn mathematics whereby they can formulate, construct models, build understanding of concepts, learn algorithmic skills, and even prove theorems, the teacher must be able to stimulate her/his class through various forms of mathematical presentation and the associated classroom discourse (Brousseau, 1997). In this way students acquire understanding, either conceptually, with improved ability to make sense of what they are doing, and/or procedurally, with the ability to apply what they learn in solving problems, depending on the teaching emphasis, or *privileging* of approaches by their teachers (Kendal & Stacey, 2001). Therefore, while there are other factors that contribute to student understanding, we can certainly say that teacher practices in the classroom have a major influence on the enhancement of student understanding.

It seems reasonable that if a teacher possesses a limited knowledge of a concept and its related subconcepts (Chinnappan & Thomas, 1999) then they will be unlikely to provide the kind of experiences students need to construct rich conceptual thinking in the area. However, the way teachers teach, and the manner in which classroom discourse proceeds are not mediated solely by the mathematical understanding of the teachers (Cooney, 1999). They are also influenced by teachers' pedagogical content knowledge and knowledge of how student understanding can be enhanced through appropriate presentation of content, together with appropriate classroom activities (Cooney, 1999; Chinnappan and Thomas, 2001; Shulman, 1986; Simon, 1995). Thus, as the teacher ponders the design and development of lessons, s/he is guided by her/his understanding of how the teaching can best be executed, along with how the topic can best be taught. Complementing this is the aim of assisting students to construct their own understanding as they participate in the classroom discourse. In this way, these different aspects of knowledge that teachers possess influence their design of teaching-learning activities.

The meaning of mathematical entities does not arise simply from a single representation, but is distributed across a number of interacting representational systems, with each emphasising different characteristics (Lesh, 2000), and supporting one or more aspects of mathematical thinking. Thus, making multiple representations available in teaching should help to improve the capacity for learning, provided the teacher addresses the links between them (Kaput, 1998; Noss & Hoyles, 1996). Thus a key conceptual ability

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which teachers can help students with is the handling of different symbolisations and representations of a given mathematical concept, and construction of the *representational versatility* to be able both to interact with the representations in a meaningful manner (Thomas & Hong, 2001), and to acquire the representational fluency (Lesh, 2000) to be able to translate between them. Hence, in view of what we have said above about the role of teacher knowledge in learning it is pertinent to consider the nature of a teacher's representational perspective on a given concept if one wants to enhance classroom situations contributing to student learning. This is especially true since there is evidence that teachers vary considerably in their representational perspective on concepts such as function (Chinnappan and Thomas, 1999, 2001; Williams, 1998). One aspect considered in this research is teachers' perspectives of derivative and differentiation (we note in passing that we may distinguish between the mathematical *object* of derivative and the mathematical *process* of differentiation) from a number of points of view, including, but not limited to, representational aspects.

Recognising the importance of the nature of teacher knowledge, this paper reports on research investigating teachers' views on differentiation and derivatives; what students should learn and how they should learn it. Our aim is to present a discussion of the teachers' content knowledge alongside their pedagogical perspectives on student learning of derivative. We will consider teachers' understanding of derivative, their views on what prerequisite knowledge students require, for derivatives and differentiation, and the pedagogical approach they employ. We will try to infer the emphases teachers make in teaching derivative, and the varying representational approaches (graphical, symbolic, or tabular) they use in helping students understand. This research is a part of a wider project that aims to document how students formulate their understanding of derivatives and differentiation when graphic or CAS calculators are integrated in the classroom. However, we recognise that without teachers' willingness to use them, and without their openness to modification of their teaching practices, potential benefits will not eventuate.

Method

In this study we have been working with four teachers from New Zealand schools, each of whom has been involved in constructing a module of work on the learning of differentiation which integrates use of GC or CAS calculators. The teachers, who were selected on the basis of their willingness to integrate the use of graphic calculators in their classroom teaching, comprise two males (A and B) and two females (C and D). Three of them are very experienced teachers while one of the males has only been teaching for two years (B). They teach in a range of schools, including two private schools (one for girls and one co-educational) and two state schools (both co-educational, but one from a high socio-economic area and one less so). The teachers were given a semi-structured interview which asked them what they think is important when teaching derivatives and differentiation, how they teach their first lessons on derivatives, and how they think students go through the process of understanding derivatives and differentiation (see sample interview questions in Figure 1). The interviews were all tape-recorded and later transcribed for analysis. Later we will be following up the interviews with classroom observation of the teaching of differentiation using the teacher-generated modules.

Teacher Interview Script
(Selected questions only)

1. **The teaching of derivatives and differentiation**
   - Have you taught derivatives before?
   - What is your first lesson on derivatives and differentiation? What is your follow up lesson?
   - Could you tell me about the factors that you consider important when planning for your lessons?
   - Could you give a brief outline of how you have gone about teaching derivatives?
   - What related ideas do you use in introducing derivatives to your students? Why have you considered them important?
   - What kind of examples do you use in teaching derivatives?

2. **The learning of derivatives and differentiation**
   - What do you think students need to know about derivatives and differentiation?
   - What key ideas should they learn?
   - What skills should they master?
   - What ideas do students need to know before studying derivatives?
   - How do you think students learn or go through the process of mathematical understanding of derivatives and differentiation?
   - What factors contribute to better learning and mathematical understanding of students?
   - What emphasis in the teaching of derivatives will help the students to understand better?
   - How will you measure students' success in learning derivatives and differentiation?

Figure 1. Selected questions from the interview schedule.

We aimed to gather data on the representational nature of teachers' content knowledge, and thinking about differentiation, and to relate this to their pedagogical practice.

Results and Discussion

The responses of the teachers have been interpreted and characterised according to the emphases they give, and how they create links between the graphical, symbolic, or tabular representations of derivative. They illuminate the teachers' views of their current practices, including their beliefs on what students should learn. In particular we aimed to answer the questions: how, pedagogically, do teachers approach derivative, and what is the role of representation in their teaching and what do teachers expect the students to learn about derivative and differentiation?

Prerequisite Knowledge

Clearly, for study of any area of mathematics there will be necessary prerequisite knowledge. Teacher C was emphatic in saying that she doesn't think one "can teach any maths if there is not a previous understanding of what's gone on before. That's just wasting time." When asked what they thought students should know before studying derivative, B replied "Derivatives…well you see, gradient [pause] ah...the idea of function. Ah [pause] yes [pause] I would say that they should have a fairly reasonable algebraic foundation", while A said, "They should have a reasonable algebraic background. They should have and appreciation of some basic functions. So, they should be familiar with linear function, they should be familiar with parabola."

As we can see, the teachers B and A both recognise the importance of function in the study of derivatives. According to A, students should have an appreciation of "basic functions" such as linear functions. Although, from B's experience, students in year 12 are not taught "a formal definition of function," students have encountered functions represented in the form $y=f(x)$, together with their associated graphs, becoming familiar with graphs of "lines, parabolas, cubic, hyperbolas, exponential, logs". C thinks that the idea of functions with their graphs is important because they "always use graphical approach quite so often…[Thus] their understanding of graphs and graphs of functions is fairly fundamental." A shares the same sentiment, highlighting the importance of an appreciation of the graphical representation of functions.

The remarks of A and B quoted above show that they also recognise a 'reasonable' standard of algebraic understanding to be an important prerequisite. However, A thinks that a graphical approach might be more helpful in assisting students towards an understanding: "traditionally, things have been concentrated on the algebraic method (the teaching of derivatives) and I'm quite sure that a lot of students can manipulate the algebra…but they don't have a real feel for what they are doing," and he laments that "there is not enough emphasis on graphics."

In addition, two of the teachers, A and C, consider the informal idea/concept of limit to be important and C also talks about variation and how "The whole idea of one variable changing with respect to another, that's all needs to be fairly sound and solid." While the comments made about prior learning are fairly predictable it can be noted that their comments contain a mix of skills and concepts, and cover both algebraic and graphical representations.

Derivatives: What Should Students Know?

Not surprisingly, all the teachers agree that after a course on differentiation, the basic derivatives and rules of differentiation should be known, with students able to differentiate functions from first principles as well as symbolically. However, only C and D said that the students should also be able to interpret derivative as a rate of change, thinking of graphically presented gradients as well as the algebraic dy/dx as rates of change. As D said, students should know that "differentiating is working out rate of change." It appeared that there was an emphasis on the graphical representation in the manner in which C would approach rate of change, and this may be related to her own preference.

> C: I would hope by the time they would come to looking at change, I handle most of them in a graphical point of view. So, I would hope they would have mental picture and essentially, they're looking at a rate of change at a particular moment that essentially, if they can simply grasp this is constantly changing and have that concept.

This agreed with a strong tendency apparent in the comments of three of the teachers, B and C to refer to the derivative or derived function as a 'gradient function'. This idea, which emerged from an emphasis on referring to slopes at points on graphs, was interesting because the word gradient carries with it a very strong link to the graphical representation. When asked about the gradient function, teacher B, replied that it is a "Function which tells you the gradient at a given point of a given curve." It is clear from subsequent comments though that for B this term carries over to algebraic work too, since when introducing differentiation rules he will say "…given a function, how do we obtain its gradient function? To go here and draw tangents all the time, we don't have to…. there is a short way to obtain the gradient function." This terminology is so strong that the teacher even corrects himself when using the word derivative with reference to a graph, and makes the point that gradient function will be better understood by his students than using the phrase 'rate of change'.

> B: Well, a graph they can see. If you said, if you have a relationship of function…and then you said "sketch the graph of that function" they'll probably do that and there you start talking about the derivative of that. Well, the gradient function of that. Well, probably they'll know what you're talking about then. If you just said, find the rate of change of this with respect to that, then, they don't quite understand it as the same.

Teacher C also refers to the term gradient function for the graphical case, although she has the concepts of derivative and rate of change clearly connected to it.

> C: Overall, I want them to know that derivatives, are basically, of functions which give one the rates of change of something that is changing and that derivative which is not constant and this whole idea of gradient function because, I think, this is by far quite difficult to comprehend that you…your gradient, whereas before that, they'll be talking about gradient of straight line, but the gradient of something is constant, and then all of a sudden this gradient actually has variable in it, and it's that they find quite difficult and this whole idea of gradient function.

The question of whether some teachers see this gradient function in the graphical representation as distinct from the derivative in the algebraic representation is one which arises. Some evidence that teachers' thinking in terms of gradient functions might not be linking the two came from the interviews. The idea of derivative was mentioned by a couple of the teachers. While one of them (C) related the second derivative to "the gradient of the gradient function.", the other (B) linked it to the limit definition "That's what the actual derivative is. We're talking about…it's defined as a limit. So you know the limit of the chord, the gradient of the chord as the horizontal step approaches 0." This raises the prospect of a distinction being made in the minds of some teachers between the derivative defined algebraically as $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ (or a corresponding limit), and the gradient function as reserved for the application of a specific derivative to a graphical representation of a function. On the other hand it may be that the gradient function is the primary concept for some, such as B, who comments that "I had a set parabola to start with, $y=x^2$ and for each person, draw a tangent at a different point. And then work out the gradient of the tangent, and then tabulate it on the board and identify the relationship and the gradient function." This appears to indicate that for B the concept of gradient function is fundamental and crosses graphical, tabular and algebraic representations.

Pedagogical Approaches

When asked how they would introduce the teaching of differentiation three of the teachers spoke about using a standard graphical approach, possibly followed by procedural skills. B states that he would "usually start of talking about tangents to curves and establishing your idea of a gradient function and one or two activities. … So getting them drawing tangents, working out gradients and in looking into the difficult gradient function and then start getting into the rules of differentiation." C describes how she tends "to teach derivative by looking at the gradient of a curve at a point, and then working through to find the gradient function.", while A says:

> I would quite often start with an activity where they would try to develop the notion of the gradient of the tangent being equivalent to the gradient of the derivative at a particular point. … I suppose there will be some skills-based differentiation of some basic polynomials first ….I would have

> introduced … a parabola and you choose two points, and you calculate the gradient between those two points. And if you get those points closer and closer together, you get the line that is closer and closer to the tangent.

However, teacher D has a quite different way of starting. She begins by trying to emphasise limits of functions before approaching graphs.

> D: I normally start off by concentrating on functions, and what the function means and how functions behave in different ways; that they increase, decrease, … I generally move on to thinking about the concept of limit, what the limit means, and … I spend a lot of time concentrating on the limit not only in a symbolic way but in a verbal way… and then after that I get to talk about gradient, gradient as a rate of change…you just do the average rate of change … and make the secant smaller and smaller until it becomes the tangent.

We note that this teacher has strong links in her knowledge schemas between differentiation and a number of other related concepts, including: limit; gradient; rate of change; average rate of change; secant; and tangent. Furthermore she wants students to see these concepts in both "a symbolic way" and graphically. A agreed that the concept of limit was "a crucial idea" and also mentioned the idea of "some derivative sketching activities" early on in his teaching where, given a function, he would "ask them to …draw a graph of what the derivative would look like". He was the only teacher to raise this. Instead of referring to gradient function, teacher A described how "they need to know that derivative represents the slope of a function." In this way he keeps the derivative as a separate concept from a primary application, namely the slope or gradient of a graph.

We can say that the teachers' pedagogical approach to derivative has an emphasis on the graphical representation, relating it to tangents on curves. A even mentioned that "there is an indication of things to improve on emphasising the importance of a graphic approach to calculus and derivatives." This can be supported by D's comment when asked about the value of her use of different colours in representing the moving secant. She said that she "uses colours for the secant because it enhances the link and the connection that can be established between the average gradient and the gradient at a point. Such views may be summarised by B's comment that "visual learning is a powerful way to learn."

Though all of them have used the graphical approach of secants moving towards the tangent, D relates the symbolic representation, *dy/dx*, and the graphical representation of tangents to curves to rate of change. She even mentioned that topics in trigonometry, such as the tangent relation, are related to the gradient of a line and the symbol *dy/dx*, making all those seem to 'flow together'. Likewise, for C the links between representations are crucial, and as she describes it "[I] always want to try and be linking the graphical representation…by looking at symbolic representation…got the idea of basically go through differentiation…then when you go back and tie with the graph."

Applications and Problem Solving

From A's perspective students should be able to apply their skills to solve problems, particularly unfamiliar, non-routine ones. He explained non-routine problems to be ones where they do not apply a direct routine, where information needs to be extracted, and where students interpret what steps they need to do. D and B also see the importance of the contextual use of derivatives, with D saying that in the teaching of derivatives, the teacher should "attach stories to it in teaching." B thinks that students could understand derivatives as gradient functions but they (the students) "do not quite understand derivatives as rate of change. Thus, he "start(s) with kinematics because that is something they can understand…they can relate to the idea of speed with the gradient." C also sees kinematics as a good route in to derivatives and a strong connection to graphs, commenting that "the whole idea of linking [a graph] to it, and is useful when you come to things like kinematics, when you talk about velocity and acceleration." Thus the teachers seem to believe that derivative as a rate of change can be understood better if taught in a familiar problem solving context.

The Role of Student Ability

One factor that the four teachers agree they consider in teaching mathematics, and in particular derivatives, is the ability range of their students. There seems to be a consensus among them that the emphasis on teaching varies depending on the calibre of the group. D describes the high-ability group as just being there and "they're like a sponge, they're sitting there waiting and whatever you throw into them, they're just so happy. Similarly for A, "if you've got high performing students, you've got to develop the concepts as clearly, and as appropriate, as possible.", and for B, "the good students can be challenged with harder problems: problems where more interpretation are required."

For the less-able students, however, B spends a lot of time "reinforcing their skills…If they can master skills, they can pass [since] skills are ones where students are able to do various things whey they don't have to decide…they do not have to think about the problem to know what is required or to know much." D describes how she will "try and keep the…class a little bit more structured so that they don't get lost…because I know that the worst thing for any student is to feel that they're lost…I do not just hurry along…[and] would just stick more to skill and more to rules…keep them comfortable." In a similar vein A admits that for weaker students he will "maybe concentrate on, perhaps, a more routine approach…getting the hang of the [pause] a lot of skill practice, differentiation, calculations of equations of tangent." This distinction between the teaching for different ability groups is interesting. Concentrating on skills rather than concepts for those perceived as of lower ability may be misplaced if a strong conceptual base is a necessary precursor to skills as preparation for good progress in learning.

Conclusion

What do we learn from these teachers? That they do not all have a common way of approaching the teaching of differentiation, and they have differing conceptions and emphases for some of the key concepts and sub-concepts. However, although the teachers involved in this research have emphasised, or *privileged* (Kendal & Stacey, 2001) particular representational ways of presenting derivative, especially the graphical one, they want their students to acquire a broad conceptual understanding of derivative, including understanding it as a gradient function, as a rate of change, as a limit, and symbolically (c.f. Thurston, 1995). In addition, in order for their students to build a fuller appreciation, they make attempts to relate derivative to contexts that the students are familiar with, thus working on their problem-solving abilities. However, these aims are constrained by a recognition of the different abilities of the students with the result that those perceived as less-able are limited to procedural methods, unable to handle challenging problems.

The privileging of the gradient function seen in two teachers' expressions would appear to be stressing a valuable object view (derivative, function) rather than simply a process view (differentiation) of the mathematics. However there is much evidence that students do not follow this but rather form a strong process view (e.g., Delos Santos & Thomas, 2001).

Furthermore, such privileging has the potential to cause cognitive obstacles to the construction of *representational fluency* (Lesh, 2000) and *representational versatility* (Thomas & Hong, 2001) which are essential for a full understanding of derivative. This could occur because, while the graphical perspective is important, it does not extend as easily as rate of change to some other important areas and applications, such as kinematics, certain maximisation problems, related rates, etc. The approach which needs to be encouraged, and which these teachers would surely endorse (C and D already clearly subscribe to it), is that suggested by Thurston's own experience (1995, p. 30), when he writes of his need to be "spending a good deal of mental time and effort digesting and practicing with each [way of conceiving of derivative], reconciling it with the others." This requires experiencing and relating conceptions of derivative across representations. It raises the question of whether the integration of technology, particularly the CAS calculator, may help teachers who are privileging a visual approach but are open to technology use, and modification of their pedagogy, to provide an environment which can improve understanding of derivatives for all students by stressing visual attributes but doing so in a manner which provides strong, dynamic links to other representations.

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Summing up the Education of Mathematically Gifted Students

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Recent government reports identify the importance of producing individuals with high-level mathematical capabilities who can contribute creatively to Australia's intellectual capital. To address the shortfall in such individuals, mathematically gifted students must be identified and nurtured. Hence, teachers' programs need to be well-grounded in research on educating the mathematically gifted. In this paper, we report on the status of research on these students, draw upon our research to inform practice, and highlight myths associated with the education of these students.

At the commencement of the 20th century, Australia was proclaimed to be "the paradise of mediocrity and the grave of genius" (NSW Attorney General, 1901, as cited in Barcan, 1983). A century later we hear of the need for a "clever country", and how we now should value the role of our creative individuals: "we are seeking today to nurture a new generation of young scientific minds capable of achieving great things for their country" (Howard, 2001). However, if we are to enhance our performance in the sciences, and rhetoric become reality, we must address the education of our most able students. Thus given this focus on creative achievement, the need to redress the shortfall in students who choose to pursue mathematical studies, and the importance of mathematics to the new economy (e.g., Miles, 2000; Thomas, 2000), the education of mathematically gifted students must be a high priority.

Neglect concerning the status of the mathematical sciences has been recognised in the context of a national focus on numeracy. Through our concerted attention to the "minimums" of mathematical capability, we have been overlooking the "maximums". MacGillivray (2000), suggests three distinctive levels of mathematical capability:

> (1) the quantitative capabilities of the whole of society; (2) *the mathematical capabilities in the broad spectrum of areas with quantitative links*; and (3) *the high level expertise capability of the discipline of mathematical sciences*. … It is in the second and third level that there have been recent increases in both importance and danger signs. (numbers and emphasis added)

While numeracy is an important starting point, mathematics education must also prepare those potentially most able students to achieve the second and third levels of capabilities. The second concern relates to equitable educational provision for "ALL students ... including the average, above average and *most able students* (emphasis added) (MacGillivray, 2000). The Senate report into the education of gifted children (Collins, 2001) acknowledges that focusing on minimum standards could have a deleterious effect on satisfying the special needs of the gifted, who are already affected by "underachievement, boredom, frustration, and psychological distress, (and) … negative attitudes and mistaken beliefs" (p. xiii). The report argues that equitable provision for these students includes teacher training in the identification of gifted students and gifted pedagogy, the development of appropriate curriculum materials, and curriculum

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B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 219-226). Sydney: MERGA. ©2002 MERGA Inc.

differentiation. It also highlights the inadequacy of current provisions for the unique needs of these children: "Ad hoc enrichment activities, or enrichment for the whole class, are insufficient" (Collins, 2001). Others have recommended fostering high level capabilities by making mathematics "more exciting" and by applying "creative, innovative approaches to problem solving" (Batterham, 2000, p. 51). and by "using extension activities to supplement the normal curriculum" (Thomas, 2001 p. 21). However, such strategies in isolation are inconsistent with the research on gifted education (Collins, 2001), and inadequate because they fail to address the myriad of cognitive (e.g., knowledge), environmental (e.g., education), and personality variables (e.g, motivation) that contribute towards creative achievement (Eysenck, 1995) and which can be fostered from childhood (Torrance, 1994). Thus, to optimise a mathematically gifted individual's chances for creative achievement, his or her needs should be addressed from the early years at school. The purpose of this paper is to examine the status of research on mathematically gifted students and to explore some of the assumptions underpinning the education of mathematically gifted children. First, we provide an overview on the education of mathematically gifted students. Second we examine Australasian literature on mathematically gifted students, and third we suggest key directions based on our research.

The Education of Mathematically Gifted Students

Mathematically gifted students are distinguished from their non-gifted peers by their exceptional reasoning ability (House, 1987; M. Johnson, 1983). These students may also exhibit an exceptional memory, the ability to solve problems in unexpected ways, success in identifying patterns and relationships, enjoyment from posing original problems, a preference for working abstractly, rapid learning, a long concentration span when interested, a capacity for self-directed activity, a preference for mathematical activities, and enjoyment from mathematical puzzles and games (House, 1987). While many theorists propose similar characteristics to these (see Putt, 1998 for an overview), little is known about the cognitive differences that may distinguish a young spatially-gifted potential "Einstein" from a more analytically-gifted potential "Erdos".

Catering for mathematically gifted students requires curriculum differentiation (D. Johnson, 1994; Sheffield, 1999). However, Hall (1997) argues that due to the inadequate curriculum guidance, teachers need to access and interpret research articles to modify the curriculum "to better fit the precociousness of the mathematically talented" (p. 21). The Senate Inquiry (Collins, 2001) also acknowledges the inadequacy of curriculum materials for the gifted. Given the need for teachers to access research information, to differentiate the curriculum for these students, the adequacy and accessibility of the literature base on educating the mathematical gifted is paramount.

Mathematically gifted students are affected in multiple ways by the attitudes of others. These students' interests and capabilities set them apart from other children and identify them as a "marked" group or "deviant" population, due to the generally negative attitudes towards mathematics that are held by the general populace (Damarin, 2000). As society finds an orientation towards mathematics less acceptable for girls than for boys, gifted girls are considered to be "doubly marked" (Damarin, 2000). Negative community attitudes towards the gifted are reflected by the derisive labelling of these students as "little Einsteins" or "nerds". The attitudes of teachers towards gifted students vary substantially. At one extreme, there are teachers who are highly supportive of gifted students or who believe that all students should receive support to achieve their potential (McLeod &

Copley, 1989). At the other extreme, there are teachers who view gifted students as over endowed and react negatively to any special treatment for these students or teachers who are either indifferent or unaware of the needs of these children. The most significant factor affecting teachers' attitudes to gifted students is whether or not they have done any specific study in gifted education (Plunkett, 2000). Gifted students in Australia are further affected by the generally anti-intellectual perspective of the community and an overly broad view of giftedness that would be untenable internationally (Gross, 1993). Thus, although mathematically gifted students have characteristics that predispose them towards high performance and creative achievement, they need considerable support and resilience to overcome negative attitudes and fulfil their potential.

Thus, emerging from the literature about the education of mathematically gifted students are two fundamental questions that need to be addressed to inform teacher education, to underpin classroom practice in Australasia, and to guide future research:

1. *What is the status of research on mathematically gifted students?*
2. *What are the characteristics of mathematically gifted students and how can their educational needs be supported?*

Design and Methods

The two research questions about the education of the mathematically gifted students were addressed using a two-phase design. In *phase 1*, we undertook a document analysis (Hodder, 2000) to investigate the question, *What is the status of research on mathematically gifted students?* The documents selected for review were the past decade of *Mathematics Research Group of Australasia Conference [MERGA] Proceedings* and the past decade of issues of the *Australasian Journal of Gifted Education [AJGE]*. These publications were deemed to be representative of Australasian mathematics education research and gifted education research respectively and relatively accessible to teacher educators and classroom teachers. The abstracts or introductions of all full papers in the MERGA proceedings between 1992 and 2001 inclusive were read and papers selected for analysis in which reference was made to research with mathematically able students. Key words used in the search included "gifted", "talented", "high achievers", "capable", and "able'. Similarly, the papers in AJGE were reviewed and papers selected in which there was reference to "mathematics" or "numeracy" in the abstract or introduction. In *phase 2*, we employed a retrospective analysis of selected studies from our research (Torrance, 1994) to explore the question, *What are the characteristics of mathematically gifted students and how can their educational needs be supported?* Over the past decade, our research has encompassed a variety of studies on mathematically and scientifically gifted students. The research has generally involved teaching interventions of approximately 10 weeks duration that explored the characteristics, behaviours, and needs of gifted students. Typically, the data comprised classroom video recordings, field notes, students' work samples, and interview responses. Although these studies varied in specific purpose, the studies referenced in this paper were underpinned by the latter research question.

Results and Discussion

The Status of Mathematically Gifted Students

The document analysis revealed a paucity of Australasian research on mathematically gifted students as reported in either MERGA proceedings or AJGE. Between 1992 and 2001, of the 794 published papers in the MERGA proceedings only four papers focussed on mathematically gifted students (Clarke & Bana, 2001; Curran, Daniel, & Holton, 1995; Marshall, 1994; Putt, 1998). In contrast, 22 publications focused on low achieving students or students with learning difficulties. An additional four papers addressed high and low achieving students or streaming. In AJGE, six papers of the 105 papers published in the past decade considered mathematically gifted students (Beardon, Jared, & Way, 1999; Hall, 1997; Landvogt, Leder, & Forgasz, 1998; Lowrie, 1995, 1996; Taylor, 1992). Given that mathematically gifted students are the focus of less than one percent of publications in the MERGA proceedings and six percent of articles in AJGE, the research or dissemination of research on mathematically gifted students, has been of low status in the past decade. Although Australasian researchers in this area publish elsewhere (e.g. Diezmann & Watters, 2001), this literature may be inaccessible to many teachers.

Supporting the Mathematically Gifted Students

We explored the second question: *What are the characteristics of mathematically gifted students and how can their educational needs be supported?* through four sub-questions:

(a) What are the differing characteristics of these students? The central characteristic of mathematically gifted students is their advanced capacity to reason either analytically or spatially (Diezmann, 2001; Diezmann & Watters, 1996, 2000c). Analytically gifted students are generally fast and accurate workers, who are able to articulate their chain of reasoning. In contrast, spatially gifted students may underachieve in classrooms due to the typical emphasis on analytical tasks and may experience significant difficulty verbalising their reasoning (Diezmann & Watters, 1996). Spatially gifted students perform best on visual-spatial mathematical tasks and when using spatial tools and techniques, such as diagrams and visualisation. Hence, the identification of these students is dependent on providing relevant opportunities for them to demonstrate their ability. Spatially gifted students also tend to have interests in spatially-oriented activities, such as drawing, map reading, construction or chess. However, despite the tendency for spatially gifted students to underachieve in school, it is acknowledged that high spatial ability rather than high analytical ability generally underpins creative breakthroughs in mathematics and science (Diezmann & Watters, 2000c). The spatially gifted seem to be advantaged by being able to process information simultaneously rather than sequentially (Watters & English, 1995). Hence, they are able to abstract rich networks of connections between concepts and procedures, and capitalise on cognitive tools, such as analogies. Students who process information simultaneously are more successful on reasoning tasks than students who process information sequentially (Watters & English, 1995). Just as there is a need for sensitivity to the types of tasks that will allow spatially-gifted students to demonstrate their ability, there is also a need for appropriate tasks and procedures to identify other known underachieving populations, such as indigenous students (Cronin & Diezmann, submitted).

(b) What types of academic tasks have learning potential for these students? Mathematically gifted students' capabilities and interests in mathematics can be many

years ahead of their age peers. These students seem to benefit most from four types of tasks. First, gifted students require challenging tasks to provide scope for learning and the use of metacognitive skills. Tasks that are too simple for particular students can be modified to increase the level of challenge. A task can be problematised by including more complex numbers in the task, by adding obstacles to solution, by requiring students to engage in novel solution processes, or by requiring students to use particular representations (Diezmann & Watters, 2000b). In contrast to many of their age peers, gifted students express an explicit preference for difficult mathematical tasks (Diezmann & Watters, 2000b). Second, gifted students may need tasks that introduce them to mathematical ideas beyond those typically addressed for their age group. For example, while most children in their first year at school learn about one digit numbers, similarly-aged gifted students may seek to develop multi-digit number sense to understand the quantitative information they encounter about topics such as space travel (Diezmann & English, 2001). Third, open-ended investigatory tasks can provide rich learning opportunities for gifted students. These tasks require the application of mathematical knowledge, can be cross-disciplinary, and provide scope for creativity (Diezmann, English, & Watters, 2001). Fourth, the tasks should be of interest to mathematically gifted students whose interests can differ substantively from their age peers. Young mathematically gifted children's interests can include large numbers, space travel, futures, abstract mathematical explorations, and applications of mathematics (Diezmann & English, 2001; Watters & Diezmann, 1997). Fundamentally, a gifted student's capability should guide the selection and implementation of tasks rather than a predominantly age-focused curriculum.

(c) What constitutes an effective learning environment for these students? There are three main features of an effective learning environment for mathematically gifted students apart from academic tasks and the role of the teacher, which have been addressed separately. First, the environment should provide opportunities for these students to develop the skills to become autonomous learners (Diezmann & Watters, 2000a). This includes knowing how to conduct investigations (Diezmann, English, & Watters, 2001) and how to learn through discourse (Diezmann & Watters, 2001). Productive discourse incorporates evidence, logic, and argumentation and involves students in sharing ideas, building on each other's ideas, and critiquing ideas (Diezmann & Watters, 1998, 2000a, 2001). Second, gifted students' preferences for working individually or in a group should be addressed. These students' preferences for working in a group situation appear to be goal-oriented (Diezmann & Watters, 2001). When gifted students work on relatively easy tasks, they prefer to work independently or side-by-side with another student. However when tasks are sufficiently challenging, gifted students' prefer to work in a group in order to share knowledge and access a support network (Diezmann & Watters, 2001). Third, gifted students should also have opportunities to work with like-minded peers who share their interests and will challenge their ideas (Diezmann & Watters, 2001). This may occur within the regular classroom or through acceleration or enrichment. Acceleration may benefit students who are sufficiently advanced to work full-time at a level beyond that of their age peers (Diezmann, Watters, & Fox, 2001), while enrichment programs enable students to explore topics in depth with like-minded peers (Watters & Diezmann, 2000).

(d) How can teachers support these students? Teachers can support mathematically gifted students in three key ways. First, teachers need to educate themselves sufficiently about gifted students (Diezmann & Watters, 1999). This includes ability to identify mathematically gifted students including spatially gifted students (Diezmann & Watters,

2000c), and indigenous students (Cronin & Diezmann, submitted); being aware of the difficulties of gifted students (Diezmann & Watters, 1996); and knowing effective strategies for catering for gifted students (Watters & Diezmann, 1997). Teachers also need to be aware of mandated policies on the educational provisions for gifted students and their subsequent responsibilities for the education of these students (Diezmann, Watters, & Fox, 2001). As many teachers are untrained in gifted education, strategic professional development support is necessary (Holz, Diezmann, & Watters, 1999). Second, teachers should select challenging tasks for these students and provide the necessary support for students to learn from these tasks. This support includes modifying the difficulty level of simple tasks (Diezmann & Watters, 2000b), providing authentic opportunities for students to benefit from working collaboratively with other gifted students (Diezmann & Watters, 2001), providing the optimal scaffolding to assist students on challenging tasks (Diezmann & Watters, 2000b), enabling them to learn from discourse (Diezmann & Watters, 2001), and providing hands-on materials or pictorial aids where necessary to stimulate or facilitate thinking (Diezmann & English, 2001). Third, teachers should support the development of students' mathematical knowledge and reasoning ability (Diezmann, 2001) because these are the cornerstones of creative achievement in mathematics (Diezmann, 2001). Fourth, teachers should consider how new syllabi impact on gifted students. For example, in science, while the philosophy of outcome-based education has scope for gifted students, but enactment may disadvantage students (Diezmann & Watters, 2002).

Conclusion

Clearly, it is important to educate the mathematically gifted students so they can be fulfilled as individuals and contribute to society. At present, there are considerable funds being expended on the education of these students through special initiatives (e.g., Education Queensland, 2002). However, there appears to be limited research-based literature on educating mathematically gifted students in readily accessible mathematics education research literature or the counterpart gifted literature to inform these initiatives, or to guide classroom teachers and teacher educators. While our work provides some insight into the education of mathematically gifted students, this review has raised our awareness of the need for more strategic research that brings together theoretical frameworks from both mathematics and gifted education. Additionally, the literature base on mathematically gifted students can be developed by researchers reporting the performance of gifted students as a distinct cohort within their studies. Furthermore, there is a need for a concerted research effort on the education of mathematically gifted students particularly, in relation to commonly held views and practices. To adequately cater for mathematically gifted students, we need to draw upon and extend research that informs professionals, curriculum developers and policy makers about educational practices and addresses the assumptions underpinning the education of the mathematically gifted. Our research with these students indicates that many common practices and views about the education of the mathematically gifted are merely myths, and hence, have little educational validity. The myths are:

1. *Mathematically gifted students are high achievers and, fast and accurate workers.* While this is true of some mathematically gifted students, there are also gifted students who underachieve on traditional school mathematics tasks.

2. *Additional exercises or activities for "fast finishers" caters for the needs of mathematical gifted students.* If these tasks are only at the same level as the class work, this work is merely "busy work" and lacks the opportunity for mathematical learning.
3. *Mathematics competitions cater for the needs of mathematically gifted students.* While students can demonstrate their ability and compete at a high level periodically in competitions, they need ongoing opportunities to develop their potential.
4. *Gifted students prefer to work in homogeneous groups.* Students' preferences for working individually or in a group vary according to the challenge of the task and their need for cognitive and affective support.
5. *Gifted students do not need opportunities to work with other gifted students.* Gifted students interests' and abilities can vary substantively from their age peers, hence without the company of other gifted students or older students, they can be very isolated and have limited opportunities to learn through discourse.
6. *Gifted students need little teacher support.* Gifted students need little support on routine tasks. However, to maximise learning they need support on challenging tasks.
7. *There are insufficient funds and time to dedicate to the education of mathematically gifted students.* This view is short-sighted and overlooks the long-term return that a small investment in the education of mathematically gifted students should yield. The current shortage of mathematics teachers may be a consequence of past failures to nurture and support gifted students' interests in mathematics.

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Summative and Formative Assessment: Creating a Tool for Improving Learning

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This paper describes part of a continuing program of research and development into improved reporting of student performance in mathematics. The goal of the work reported in this paper is to derive summative and formative information from a single assessment. Developers of multiple-choice test items typically have a good knowledge of the research evidence about children's errors and misconceptions. These common errors are often employed as item distractors. Descriptions of item distractors were made for a multiple-choice test (scored correct/incorrect) to provide descriptions of errors likely to be made by students of different abilities. Through the linking of formative and summative analyses a 'map' was constructed that provides a way of alerting teachers to possible flaws in students' mathematical understanding.

Introduction

The use of standardised achievement tests for assessing students' mathematical development is well known, and appears to be increasing its dominance in both national and international large-scale assessment programs. Many of these tests use multiple-choice as their response format, a format that critics claim, leaves students' understanding un-assessed. More recently, however, attempts have been made to use summative assessment data from these tests to describe developing understanding and provide some formative information.

Mathematics educators, among others, have been provoked to call for achievement assessment to be replaced with alternatives that provide information that is of value for improving teaching and learning. Harlen and James (1997) for example, argue that

> there is a need to recognise in theory and in practice the differences in function and characteristics between formative and summative assessment and to find a way of relating them together that preserves their different functions (p. 366)

and Helmke (1995), in his review of approaches to the diagnosis of student needs, concludes that

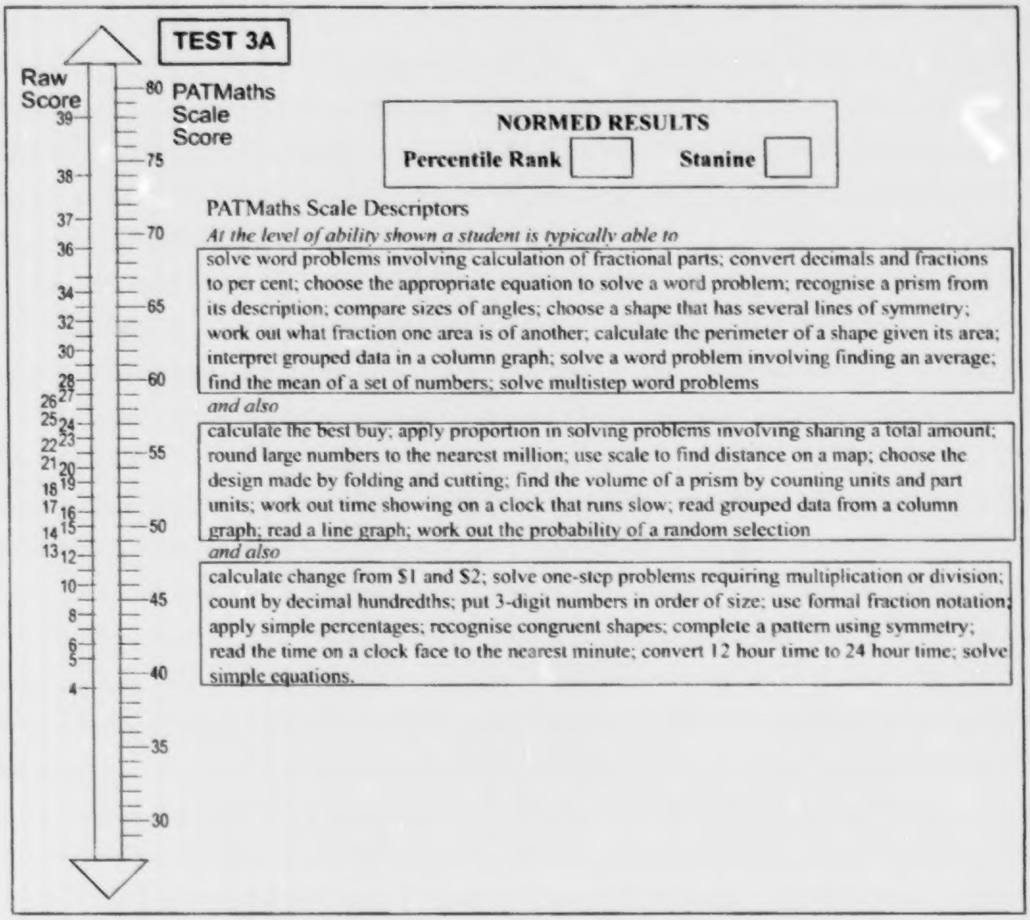
> attempts to analyse students' understanding and their content-specific knowledge structures, including the analysis of errors and misconceptions, are a necessary counter-weight against the widespread use of standardised multiple-choice achievement tests (p. 289).

However, to date, rarely have the summative and formative aspects of students' learning been described simultaneously (see, for example, Doig, Fox, Ryan, & Williams, 1997; Williams, Fox, Ryan, & Doig, 1999). Wiliams's (2001) suggestion that "the same assessment can serve both formative and summative functions, although in general, the assessment will have been designed so as to emphasise one of the functions" (p. 176) echoes Biggs (1998) who argues that

> there is a powerful interaction between FA [formative assessment] and SA [summative assessment] that could usefully be incorporated in an overall synthesis so that both … are conceptualised within the same framework (p. 106).

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 227-235). Sydney: MERGA. ©2002 MERGA Inc.

Attempts to link both summative and formative information can be seen in the descriptive reporting formats offered for many current tests. An example is shown in Figure 1, where descriptions of what students can do and understand are linked to their (summative) scores.


TEST 3A
Raw Score
PATMaths Scale Score
39
38
37
36
34
32
30
28
27
26
25
24
23
22
21
20
19
18
17
16
15
14
13
12
10
8
6
5
4
80
75
70
65
60
55
50
45
40
35
30

NORMED RESULTS

Percentile Rank ☐ **Stanine** ☐

PATMaths Scale Descriptors

At the level of ability shown a student is typically able to

solve word problems involving calculation of fractional parts; convert decimals and fractions to per cent; choose the appropriate equation to solve a word problem; recognise a prism from its description; compare sizes of angles; choose a shape that has several lines of symmetry; work out what fraction one area is of another; calculate the perimeter of a shape given its area; interpret grouped data in a column graph; solve a word problem involving finding an average; find the mean of a set of numbers; solve multistep word problems

and also

calculate the best buy; apply proportion in solving problems involving sharing a total amount; round large numbers to the nearest million; use scale to find distance on a map; choose the design made by folding and cutting; find the volume of a prism by counting units and part units; work out time showing on a clock that runs slow; read grouped data from a column graph; read a line graph; work out the probability of a random selection

and also

calculate change from $1 and $2; solve one-step problems requiring multiplication or division; count by decimal hundredths; put 3-digit numbers in order of size; use formal fraction notation; apply simple percentages; recognise congruent shapes; complete a pattern using symmetry; read the time on a clock face to the nearest minute; convert 12 hour time to 24 hour time; solve simple equations.

Figure 1. Example of a descriptive report from a multiple-choice test.

This style of report employs a form of scale anchoring (Kelly, 1999) that emphasises, rightly, the positive aspects of student performance. However, we believe that there is more that can be done with the student response data from a multiple-choice test.

In this paper we outline a strategy for producing a summative–formative 'map' that we believe points the way forward to achieving the goal of deriving both summative a formative information from a single assessment procedure and reporting the links between these two aspects of assessment.

Methodology

The basis of this procedure is the revised ACER Progressive Achievement Tests in Mathematics (PATMaths Revised) (ACER, 1997; Lindsey, 1998). The PATMaths is a series of three pairs of two tests, linked to the same scale and spanning the mathematics curriculum of the fourth to ninth years of schooling. The tests contain between 37 and 41

239

multiple-choice items in number, space, measurement, chance and data, and algebra. As the name suggest, PATMaths is an achievement test, and student achievement is reported on a PATMaths scale across all six tests. The scale was constructed from data collected in a norming study in late 1997. The data were collected from students in all parts of Australia with the exception of the Northern Territory. In all, nearly six thousand primary students and nearly four thousand secondary students were assessed during the norming exercise.

Data

The data for this present procedure are those from the PATMaths 2A test, which is designed for students in years 5 to 8 (in the Victorian numbering system) and covers the content of the National Profiles in Mathematics Levels 2 and 3. Data were selected from the complete PATMaths Test 2A data-set by eliminating all records that were incomplete; that is, contained items for which the student had not provided a response. This was done in order to provide the same number of response data for all items, thus eliminating the possibility of items being estimated as more difficult, due to lack of responses rather than on incorrect student understandings. After the elimination procedure, 1310 student records remained providing complete information on all thirty-nine items in the test. No elimination was carried out based on a student's sex, age or place of origin.

Analysis

The analysis had two phases, quantitative, conducted first, and qualitative.

The quantitative phase of the analysis was of correct–incorrect responses only. Items answered correctly were scored as 1, and incorrectly answered items as zero. The student's total number of correct items is their raw score. All incorrect responses are taken as indicators of lack of achievement (e.g., they are wrong answers) and are scored zero to provide a summative perspective of the student's mathematical achievement.

These scores were used to establish a scale for reporting student achievement, using the Rasch software Quest (Adams & Khoo, 1993) for a simple logistic model. The raw score scale for the test is at the extreme left of Figure 2. Item order on this map is from easier items at the bottom, to more difficult items at the top, and analogously, the lower achieving students at the bottom, and the higher achieving students at the top, of the student distribution.

The Rasch model used for this analysis places a student whose ability estimate is equal (in logits, the scale's metric) to an item's difficulty estimate (also in logits), has a 0.50 probability of answering that item correctly. For example, in Figure 2, a student whose ability estimate is about 1.5 logits (a raw score of 30), has a 0.50 probability of answering items 3 and 6 correctly, and a greater likelihood of answering items below 1.5 logits in difficulty (for example, items 32, 5, and 7). Items whose difficulty is greater than 1.5 logits (for example, items 20, 25, and 39) are less likely to be answered correctly by this student.

In the qualitative phase, the correct answer was ignored and the item distractors examined for evidence of student misunderstandings. This is the formative analysis of the students' responses. The focus of the analysis was on describing the student understanding that led to the selection of that particular distractor. In some instances it was impossible to establish the underlying meaning of selecting a distractor, and so this distractor did not contribute to our knowledge of the student's understanding. An example of this is distractor C in Item 22.

```
                   XXXXXXXXXXXXX  |
 38    4.0                        |
                                  |
                                  |
                                  |
                XXXXXXXXXXXXXXXX  |
                                  |
                                  |
       3.0       XXXXXXXXXXXXXXX  |
                                  |
           XXXXXXXXXXXXXXXXXXXXX  |
                                  |
                XXXXXXXXXXXXXXXX  |
                                  |      39
             XXXXXXXXXXXXXXXXXXX  |      20
       2.0                        |      25
                   XXXXXXXXXXXXX  |
                XXXXXXXXXXXXXXXX  |
 30                XXXXXXXXXXXXX  |       3       6
                     XXXXXXXXXXX  |
                   XXXXXXXXXXXXX  |      22
       1.0            XXXXXXXXXX  |
                    XXXXXXXXXXXX  |       4      19      26
                       XXXXXXXXX  |
                       XXXXXXXXX  |      32      38
 24                      XXXXXXX  |      15      17      23      31
                           XXXXX  |      18      33
                           XXXXX  |
 20     .0                   XXX  |      16
                            XXXX  |      28      34      35      37
                             XXX  |       5      30
                             XXX  |       9      24
                           XXXXX  |       8      12
                               X  |
                              XX  |      14      21
 12   -1.0                   XXX  |      10      36
                               X  |       7      29
                             XXX  |
                                  |
                                  |      11      27
                                  |       1       2
                                  |
  6   -2.0                        |
                                  |      13
--------------------------------------------------------------------------
 Each X represents    5 students
==========================================================================
```

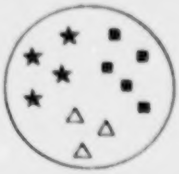
Figure 2. Item map showing student ability and item difficulty on the same (logit) scale.

The 39 items in the test address the following mathematics content: number, geometry, measurement, and chance and data. Examples of items from each of these content areas, and the analysis of their distractors, are set out below.

Item 6 (Number). The student who has selected distractor A has used part of the designated subset as a fraction of the total in the set (i.e., 4 black stars are one-third of 12 objects), or, used an incorrect subset as a fraction of the designated subset as their answer (i.e., 3 triangles are one-third of 9 black shapes).

The student who has selected distractor B has given part of the designated subset as a fraction of another part of the set as their answer. That is, the 'ratio' of 5 squares to the squares and triangles.

A student who selected distractor C has given part of the designated subset as a fraction of the remaining part of the set as their answer. That is, the 'ratio' of 5 squares to the remaining 7 shapes. distractor D is the correct answer. The student who has selected distractor E has given part of the designated subset as a fraction of the remaining part of the designated subset as their answer (i.e., the 'ratio' of 4 stars to 5 squares). As can be seen from the above analysis, students either misunderstand which sub-sets to use to form the required fraction (distractor A), or have a 'ratio' or 'part-to-part' view of fractions (distractors B, C, and E).



What fraction of the shapes in this circle is coloured black?

A $\frac{1}{3}$

B $\frac{5}{8}$

C $\frac{5}{7}$

D $\frac{3}{4}$

E $\frac{4}{5}$

Figure 3. Item 6.

Item 20 (Space). A student who has selected distractor A has given the value of the angle corresponding to the 'minutes' position of the minute hand (i.e., 25 minutes gives 25°). Students who select distractor B give a value to the angle corresponding to the time given (i.e., 5:00 gives 50°). Students who select distractor C have given the value that is half of the correct value, most likely because the size of a 'straight angle' is thought to be 90°. Distractor D is the correct answer. Our analysis suggests that students fall into two categories; those who distinguish between angle and time (distractors C and D) and those who do not (distractors A and B).

What is the smaller angle between the hands of a clock at 5:00 pm?

A 25°

☐ B 50°

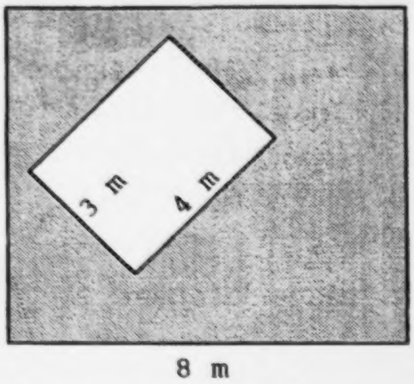
☐ C 75°

☐ D 150°

Figure 4. Item 20.

Item 22 (Measurement). A student who selects distractor A calculates the difference between the two perimeters (i.e., they confuse area and perimeter). Students who select distractor B use the difference between one perimeter (the larger rectangle) and the area (the smaller rectangle). Again, like distractor A, there is a confusion of perimeter and area.

We are unable to reliably interpret the understanding of students who select distractor C, thus it provides no formative information. Distractor D is the correct answer. Students who select distractor E ignore the 'cut-out' section and calculate the larger area only. It is clear that the first two distractors elicit the well-known perimeter–area confusion, while mis-reading the question would appear to lead to the choice of distractor E.



What is the area of the shaded region above?

A 16 m^2

B 18 m^2

C 35 m^2

Figure 5. Item 22.

Item 32. Distractor A is the correct answer. Students who select distractor B work out the chance of choosing one object from a set of objects use an inappropriate attribute (i.e., the ball colour rather than number of balls). A student who selects distractor C uses only part of the total set when working out the chance of choosing one object (i.e., choose one ball out of two). Why they should do this is not clear. We found it impossible to infer the reasoning behind the selection of distractor D.

Robby has four marbles in his pocket: a red one, a blue one, and two green ones. If he takes out a marble without looking, what is the chance that it is blue?

A. $\frac{1}{4}$

B. $\frac{1}{3}$

C. $\frac{1}{2}$

D. $\frac{1}{1}$

Figure 6. Item 32.

In summary, we found that the selection of any one of the incorrect distractors gave little information about the understanding of the students.

Item 38. Items of this type are included in all PATMaths tests and are significant in that students are not permitted to use a calculator for answering them (calculators are allowed to be used for most items).

4560 ÷ 4 =
A 111 and 2 remainder
B 114
C 1115
D 1140

Figure 7. Item 38

A student selecting distractor A performs a simple division without using regrouping and ignores the zero in the units place (i.e., zero has no value, so omit it).

Students who select distractor B perform a division that is correct except for the units place (e.g., four cannot divide zero, so ignore this part of the question). An alternative explanation is that these students may only disregard zeroes. Distractor C is selected by students who perform a division that is partly correct, with regrouping, but only regroup once. Again, as with distractor B, an alternative explanation is that these students may simply disregard zeroes. Distractor D is the correct answer.

It is likely that, in addition to regrouping problems, students have problems when a zero appears in the problem. This is consistent with the problems caused by zeroes in the other three basic operations with whole numbers.

A similar analysis may be applied to other items in the test and their distractors.

Linking the Summative and Formative Analyses

The critical question that arises from the summative and formative analyses of an achievement test is how do we link the results of the analyses. The use of an Item Response Theory (IRT) quantitative analysis suggests a possible strategy for doing this, as the IRT analysis provides, among other statistics, the mean ability of the students who selected each distractor. That is to say, we can find a position on the summative scale where we may place the distractor description from the qualitative analysis. This position is at the mean ability of students who selected that distractor. For convenience we shall term this the mean ability level for the distractor.

The report, shown in Figures 8 and 9 (next page) have the distractor descriptions placed on the scale at the mean ability level for each distractor. Thus, one can see the relationship between a student's score and their typical choice of distractor. As the qualitative analysis of the distractors provides evidence of a student's mathematical understanding, this is a linking of students' scores and mathematical understanding. In other words, the summative aspects of the analyses are linked to the formative information.

Discussion

The value of being able to put student responses to individual item distractors on a scale derived from their achievement data lies in the opportunity to expose the likely mis-understandings of students linked to their achievement score.

Establishing a relationship between achievement and understanding creates a pedagogical framework from which suitable teaching and learning strategies can arise. It has been argued that teachers who are aware of their students' mis-understandings are able to teach more effectively (Ryan, Williams, & Doig, 1998) and the strategy outlined in this paper is intended to give teachers that knowledge.

As stated at the beginning of this paper, this is current work, the next step of which is to complete the distractor analysis to prepare a complete framework for PATMaths users.

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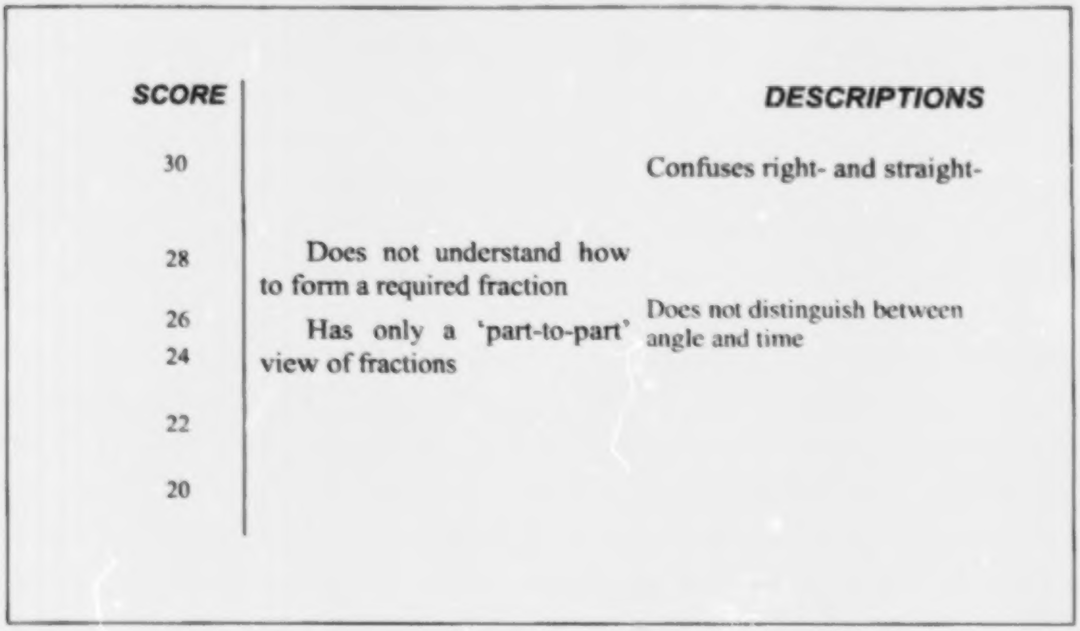
SCORE
DESCRIPTIONS
30
28
26
24
22
20
Confuses right- and straight-
Does not understand how to form a required fraction
Does not distinguish between angle and time
Has only a 'part-to-part' view of fractions

Figure 8. Part of a summative–formative map (fractions and angle)

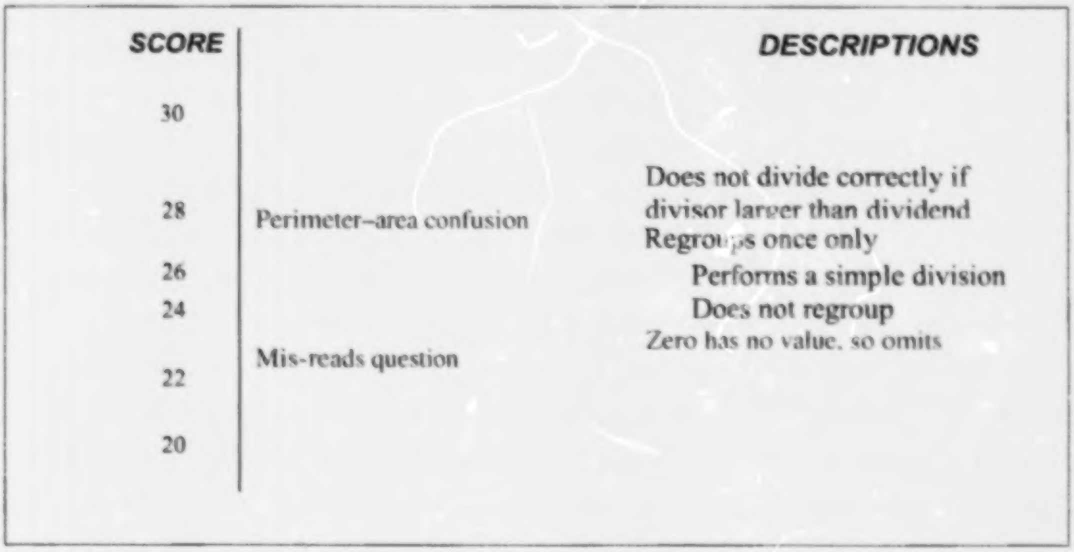
SCORE
DESCRIPTIONS
30
28
26
24
22
20
Does not divide correctly if divisor larger than dividend
Perimeter–area confusion
Regroups once only
Performs a simple division
Does not regroup
Zero has no value, so omits
Mis-reads question

Figure 9. Part of a summative–formative map (perimeter and division)

Maths Anxiety Self-assessment as a Quality Assurance Measure

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Many students come to preservice mathematics teacher education programs with considerable anxiety and fear of mathematics. The challenge is to provide students with the necessary skills, knowledge and understanding for teaching, and to positively impact upon their belief systems. In this study 429 preservice students rated change of beliefs about mathematics through a self-assessment questionnaire. Results indicated a strong positive change, and together with interview data, highlighted key contributing aspects of the program and thus provided a measure of program quality assurance.

Background

The importance of all children achieving appropriate levels of numeracy properly continues to be emphasized by educational policy makers and is the focus of the National Numeracy Strategy (DETYA, 2000). While it is generally accepted that a succinct definition of numeracy is neither possible nor desirable in a constantly changing and increasingly technological society, it clearly encompasses skills that are key to enabling an individual to fully and effectively participate in society (AAMT, 1997). Importantly, it includes not only knowledge of mathematical skills and procedures but also the ability and inclination to apply them in relevant situations (AEC, 1990).

While the teaching of numeracy in schools is, like literacy, the responsibility of all teachers (AAMT, 1997; DETYA, 2000), the foundations of numeracy are located in the key learning area of mathematics. Thus, the responsibility to prepare effective teachers of numeracy falls primarily to teacher educators involved in the planning and delivery of the mathematics curriculum components of preservice programs. Such programs must seek to develop each of Shulman's (1986) types of teacher knowledge, namely a) content knowledge of the subject of mathematics, b) pedagogic-content knowledge of how to teach mathematics, and c) curricular knowledge of mathematics applied to other curriculum areas, as well as addressing the affective factors that are integral to numeracy. Indeed it is arguably in addressing the affective characteristics of preservice primary teachers that the greatest challenge lies.

The Attitudes and Beliefs of Preservice Primary Teachers

It has been well established that students enter preservice primary teacher education programmes with beliefs about mathematics that are not helpful in that they are likely to lead to teaching that emphasizes exposition, practice and memorization (Civil, 1993; Mayers, 1994). In addition research into mathematics anxiety in the 1970s and 80s established that many pre-service primary teachers have negative attitudes to mathematics and regard the subject with considerable fear (Wood, 1988). Later studies by Frank (1990) and Beswick and Dole (2001) suggest that the beliefs held by mathematics anxious individuals persist among significant numbers of current pre-service primary teachers. Research in the affective domain often does not clearly distinguish between beliefs and

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 236-243). Sydney: MERGA. ©2002 MERGA Inc.

attitudes, including mathematics anxiety, however, McLeod (1992) suggests that the two are similar with attitudes having a greater affective and lesser cognitive component than beliefs. Given this blurring of the boundaries between beliefs and attitudes it is reasonable to assume that many of the preservice teachers who hold negative beliefs in relation to mathematics also harbour negative attitudes to the subject and would include those who could be described as mathematics anxious.

Mathematics Anxiety and Pre-service Teachers

Mathematics anxiety has been defined as "the panic, helplessness, paralysis, and mental disorganization that arises among some people when they are required to solve a mathematical problem" (Tobias & Weissbrod, 1980, p. 65). This definition conveys the context specific nature of the phenomenon and suggests that the problem can be simply dealt with by avoiding mathematics and any situation that may require the use of mathematics. Such avoidance of mathematics puts the goals of school mathematics (AEC, 1990) clearly out of reach for such individuals and, when considered in relation to the accepted views of what constitutes numeracy, the mathematics avoidant must be considered, to some degree at least, innumerate.

Both the development of the negative and unhelpful beliefs that pre-service primary teachers bring to their tertiary education, and the development of mathematics anxiety have been attributed to experiences of learning mathematics at school (Baroody, 1987; Bush; 1991; Sullivan, 1989). Baroody (1987) particularly identified teaching that overemphasizes memorization as likely to lead to negative beliefs about mathematics and one's own ability to learn mathematics, while Bush (1991) found that mathematics anxiety among students tended to decrease as teachers spent more time on small-group work. It is ironic, but perhaps not surprising, that it is the very kind of teaching that leads to negative beliefs and attitudes, that those with these negative beliefs and attitudes are likely to practice. The cyclical nature of the problem both explains its resistance to efforts to change the situation and highlights the importance of doing just that.

The fact that those with negative attitudes to mathematics tend to avoid mathematics adds to the difficulty of breaking the cycle. It is arguably these students who most need to engage with a course aimed at providing the knowledge and affective traits that they need in order to change from the traditional teaching paradigms that they are likely to hold. Indeed, unless positive changes in students' attitudes are achieved, preferably early in pre-service teacher preparation, even excellent teaching addressing the knowledge of pre-service teachers is likely to be ineffective.

Teaching approaches that have proven effective in positively influencing the beliefs and attitudes of pre-service teachers have included the use of group problem-solving and discussion, having pre-service teachers actually engage in doing mathematics and encouraging students to reflect upon their own beliefs and practice (e.g. Civil, 1993; Malone, 1995; Schuck & Foley, 1998; Stuart & Thurlow, 2000). All of these imply an increased emphasis on instruction in small group settings as opposed to mass lectures, yet the economic constraints in which university teaching must occur are rendering such an emphasis increasingly unachievable.

The Study

This study followed from an initial investigation of the beliefs of pre-service primary teachers entering a particular Australian university that had shown that the mathematics curriculum course that operated within the usual budgetary constraints may have positively impacted the beliefs of students. The current study sought to investigate a broader range of students' beliefs and attitudes across all of their units in mathematics education, and to identify which specific aspects of the program contributed to any desirable changes. The results of this study will serve as benchmarks against which subsequent student evaluations of the programme can be compared.

The Mathematics Curriculum Program

At the University of Tasmania, in which this study was conducted, students are required to study mathematics curriculum in three semesters in their second, third and fourth year of the education degree course. Mathematics Curriculum Studies is combined with English Curriculum Studies, therefore students study three half units of mathematics curriculum. Each half-unit is conducted over 13 weeks in one semester, delivered via a weekly one hour lecture and a one hour tutorial. Tutorials are conducted in groups of 25-30 students. Instruction in this context is designed to be interactive with students working cooperatively on activities designed to illustrate and explore information presented in the lectures. The lecturers in the program aimed to model an essentially constructivist approach to teaching in the tutorials. In both lectures and tutorials the emphasis of teaching was on promoting students' awareness of broad pedagogical ideas for meaningful learning of mathematics, such as the importance of rich mathematical learning environments for conceptual development, a mathematics curriculum that focuses on problem solving and thinking skills, and appropriate materials for concept representation. In lectures and tutorials, it was the lecturers' intention to communicate these ideas through modeling best practice, using lecture and particularly tutorial times, to engage students in activities designed for such notions to surface. A further objective of the program in total was to promote students' beliefs in the importance of mathematics and its teaching, whilst engendering their confidence, and fostering positive attitudes to the teaching of mathematics. Throughout the program, the issue of teacher beliefs was not overtly addressed in any of the lectures or tutorials, but was implied through careful planning and delivery of all lectures and tutorials.

In this study, of prime interest was data from fourth year students as they had been exposed to the Mathematics Curriculum program in its entirety.

Methodology

The study utilized quantitative and qualitative methods with students program evaluation data gathered through a questionnaire, and structured interviews with selected students.

Subjects

Four-hundred and twenty-nine students across Year 2 (178 students), Year 3 (141 students) and Year 4 (110 students) completed the Mathematics Program Student Evaluation Survey (MPSES). Fifteen randomly selected students from the fourth-year cohort participated in the structured individual interview.

Instrument

The MPSES, devised by the researchers, comprised two sections. The first section contained a list of 18 items pertaining to elements of the program the researchers considered vital for preparing pre-service teachers to become effective teachers of mathematics. The items related to lecture content, activities presented during tutorials, issues in mathematics education, the structure and presentation of the program and attributes of staff involved in the program. Students were required to tick the elements they regarded as valuable aspects of the course.

The second section of the survey was designed to encourage students to reflect upon the course and its impact upon their beliefs about mathematics, mathematics teaching and learning. In this section, students were presented with 21 statements in total from four belief categories, as detailed below:

1. Personal beliefs about mathematics, confidence to use mathematics, feelings (Category S for Self-beliefs - 6 statements)
2. The Nature of Mathematics - what is it? (Category N for Nature - 2 statements)
3. Issues related to teaching mathematics to promote conceptual understanding - sequential planning, structure of mathematics lessons, fostering thinking, problem solving, making mathematics meaningful, teacher's energy and enthusiasm, concrete materials, calculators, assessment (Category T for Teaching -10 items)
4. Children's learning of mathematics – investigating, group work, children's thinking (Category L for Learning - 3 items).

For each statement, two rating scales were provided. Students were required to indicate on the first rating scale the extent to which they agreed or disagreed with the statement prior to commencement of the Mathematics Curriculum program by colouring a dot to correspond with a number between 1 and 5, with 1 indicating least agreement and 5 indicating strongest agreement. They were then required to consider the same statement in relation to their feelings about the statement at their current point of study in the Mathematics Curriculum program, indicating their level of agreement or disagreement on the second rating scale. An example of an item from Section II of the survey is presented in Figure 1.

S1. Mathematics makes me feel uneasy and nervous

pre EPC250	○	○	○	○	○
	1	2	3	4	5
post EPC450	○	○	○	○	○
	1	2	3	4	5

Figure 1. Sample item from Section II of MPSES Survey

Results

Data from this study is considerable, and for brevity, results reported here are confined to an analysis of mathematics anxiety levels and change upon completion of the preservice Mathematics Curriculum program.

Items from Section II of the MPSES specifically relating to mathematics anxiety are those from Category S of the four belief categories. The six statements presented to students were as follows:

S1 Maths makes me feel uneasy and nervous.
S2 I am interested and willing to use mathematics in everyday life.
S3 I find mathematics difficult and confusing.
S4 I would like to further develop my mathematical skills and study this subject more.
S5 Maths is dull and boring.

S6 The study of maths makes me feel anxious and fearful.

Student levels of agreement with each of the 6 statements across the 3 year levels, prior to commencement of Mathematics Curriculum Studies in Year 2 and at the completion of the program in Year 4 are presented in Table 1. For clarity and ease of analysis, scores that indicated students who 'agreed' with a particular statement (that is, they selected 4 or 5 on the rating scale) or 'disagreed' with a particular statement (that is, they selected 1 or 2 on the rating scale) were collapsed together to give a single score for agreement (rather than strongly agree or agree) and a single score for disagreement (rather than strongly disagree or disagree) for that statement. Results therefore reflect three categories of response (D for disagreement, N for neutral, A for agree) for each statement.

Table 1

Percentage of Preservice Students across 3 Year Levels Who Disagreed (D), were Neutral (N) or Agreed (A) with Particular Statements about Mathematics.

	Year 2			Year 3			Year 4		
	D	N	A	D	N	A	D	N	A
S1. Maths makes me feel uneasy and nervous									
pre	39	21	40	44	20	36	37	8	56
post	58	27	15	70	20	10	73	14	13
S2. I am interested and willing to use maths in everyday life.									
pre	22	28	50	28	27	45	18	34	48
post	17	22	61	21	13	66	18	13	69
S3. I find mathematics difficult and confusing									
pre	42	25	33	44	18	39	40	15	45
post	55	26	19	70	16	14	71	13	16
S4. I would like to further develop my mathematical skills and study this subject more.									
pre	39	25	36	29	24	48	21	14	65
post	17	19	64	11	15	74	10	15	76
S5. Maths is dull and boring.									
pre	35	25	40	52	20	28	46	21	32
post	68	20	12	87	6	7	88	9	3
S6. The study of maths makes me feel anxious and fearful.									
pre	39	26	35	39	18	42	43	16	50
post	58	28	14	69	21	10	73	15	13

From Table 1 it can be seen that the extent to which students agreed with particular statements relating to mathematics anxiety changed in a positive sense as they undertook each component of the Mathematics Curriculum program, and that this change was more pronounced at the end of Year 4. For example, for item S1 (Maths makes me feel uneasy and nervous), of the second year cohort, 40% of students indicated that they agreed with the statement prior to study of mathematics curriculum but at the end of the second year, 15% agreed, 27% were neutral, and 58% disagreed. Of the third year cohort, 36% indicated that they agreed with the statement prior to mathematics curriculum study, but after third year maths curriculum studies, 10% indicated that they now agreed, 20% were neutral and

70% indicated that they now disagreed with the statement. Of the fourth year students, 56% indicated that they agreed with the statement prior to mathematics curriculum study, but after completion of the three-half semester program, this changed to 13% agreement, 14% neutral and 73% disagreement. Item S6 (The study of maths makes me feel anxious and fearful) is similar to S1 (Maths makes me feel uneasy and nervous). Students' responses to this item are almost identical to those for item S1 across the three student cohorts.

As a measure of mathematics anxiety, the results show that approximately 40% of the students indicated that they did suffer mathematics anxiety (and 40% who would not rate themselves as maths anxious). Specifically, to items S1 (Maths makes me feel uneasy and nervous), S6 (The study of maths makes me feel anxious and fearful), S3 (I find mathematics difficult and confusing) and S4 (I would like to further develop my mathematical skills and study this subject more) the percentage of disagreement is similar across all year levels. The prevalence of mathematics anxiety is a cause for concern in terms of teacher education and numeracy development of children in schools, but the fact that this drops to around 10% after Mathematics Curriculum studies suggests the positive impact of the program. Interestingly, the fourth year students indicated the highest level of mathematics anxiety prior to studies in Mathematics Curriculum of all the year groups, with 56% and 50% agreement with the statements 'Maths makes me feel uneasy and nervous' and 'The study of maths makes me feel anxious and fearful' respectively. It is also interesting to note that a greater percentage of fourth year students indicated that they wanted to develop their mathematics skills and study mathematics further prior to undertaking their mathematics curriculum studies (65%) compared to the third year students (48%) and second year students (36%). It is extremely heartening that at each point of their studies in mathematics curriculum, the number of students indicating that they would like to continue in mathematics education increased (64% for second years, 74% for third years and 76% for fourth years).

One of the items upon which students indicated the biggest change was item S5 (Maths is dull and boring). Results across the three year levels are similar with approximately 20% of students neutral and 30-40% agreeing with the statement prior to Mathematics Curriculum Studies. After Mathematics Curriculum at each year level, a dramatic change is indicated with 12% of second year students, 7% of third year students, and just 3% of fourth year students agreeing with this statement. Such a positive rating for this item suggests great promise in terms of the teaching of mathematics in these students' future classrooms. Such results provide a tantalising image of lively, energetic, interesting mathematics lessons where students are actively engaged in exploring meaningful mathematics because the classroom teacher holds a strong belief that mathematics is not dull and boring.

The item for which least change is apparent is S2 (I am interested and willing to use mathematics in everyday life). Between 45-50% of students agreed with this statement prior to Mathematics Curriculum Studies. There is a progressive increase in agreement with this statement across the three year levels (61% for second years, 66% for third years and 69% for fourth years), but the number of students disagreeing with this statement is relatively static across the three year levels both before and after Mathematics Curriculum Studies (approximately 20%). The change was in the number of students selecting neutral to this statement, possibly indicating that prior to Curriculum Studies, students had not fully considered the extent to which they could or did use mathematics in their everyday lives. It could be argued that, with a focus on mathematics through their tertiary studies, they became more aware of numeracy and the importance of numeracy for living and this may be one reason for the change of neutral opinion to a positive opinion.

Discussion

Unpacking mathematics anxiety in terms of personal feelings about the study of mathematics, perceptions of mathematics as a subject and personal impetus to use mathematics or pursue further study of mathematics provides a picture of this belief category of a large cohort of preservice teachers. From data gathered during this study, prior to Mathematics Curriculum Studies, approximately 50% of students felt uneasy and nervous about mathematics and were anxious and fearful about the study of mathematics; 35-45% found mathematics dull and boring, as well as difficult and confusing with only 50% willing to use mathematics in everyday life. Approximately 50% of the student population in this preservice course had beliefs about mathematics that would not have been helpful in the planning and teaching of mathematics to promote numeracy and achieve the goals of school mathematics. With students indicating positive change of opinions, the program appears to have influenced such change.

Interview data provides further evidence for the positive impact of the program. The following two responses of students dealing with maths anxiety describe the impact of the program upon their beliefs:

> My beliefs about mathematics before probably weren't so good because I didn't have a very good high school experience with mathematics …but this course has proven my beliefs to be completely wrong. I was a bit scared of doing mathematics in Education.
>
> I just didn't have good experience at school with mathematics …when I got to Uni and had to do maths, I was really nervous…but the course has changed my whole perspective and it has made me feel fantastic about teaching mathematics.

For one student, who was clearly not mathematics anxious, the following response shows a new perspective on mathematics, which appears to be a contrast of her own mathematics experiences:

> …the course hasn't influenced my beliefs about mathematics itself…but obviously it has on how you'd go about teaching it…I think of mathematics more as problem solving and patterning rather than formula memorization.

So what aspects of the course impacted upon this change? The following two comments provide further insight that augment the previous statement in which mathematics was described as patterning and problem solving:

> I've seen how mathematics can be more hands-on and more group work and more talking about it…before I had just text books and chalk and board so to speak.
>
> I've always enjoyed maths, but when I was going through school I can't remember doing the hands-on stuff

The term 'hands-on' was frequently used by students in all interviews indicating that the tutorial structure was a contributing factor to anxiety change.

Conclusion

The necessarily brief analysis of data from this study has provided a picture of a sizable proportion of pre-service teachers suffering mathematics anxiety entering preservice primary education degree courses. Analysis of survey data has indicated that Mathematics

Curriculum Studies programs at tertiary level can positively impact upon anxiety levels, and that this appears to be more positive as students continue with their studies. This preliminary analysis reinforces the need for mathematics curriculum programs in teacher education and emphasises the need for hands-on activities for students in tutorial modes. Given that the survey returned such positive results when the program delivery was via mass lecture and one weekly tutorial, the impact of such a course, for example, delivered in an electronic mode is questioned. Experimentation in other modes of delivery would be an avenue for further research. Further analysis of interview data will assist in unpacking specific aspects of the program for large cohorts of students in a tertiary program operating with limited time and resources. Such analysis will serve to provide a clearer blueprint of necessary components of a tertiary program that positively impacts future-teachers' beliefs about mathematics. What this study has provided is a self-assessment instrument for mapping change in beliefs and attitudes to mathematics, as well as a measure of quality assurance of a particular program within a degree course.

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Using CSCL Methods in Secondary Mathematics

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This study compares computer supported collaborative learning (CSCL) based instruction and traditional pen-and-paper methodologies and is aimed at investigating the effectiveness of using spreadsheets and collaborative learning methods in mathematics instruction for enhancing student learning. Ten Year 11 mathematics classes comprising of 172 students from six high schools across Sydney participated in the study. This study showed that majority of students found the collaborative learning activities very interesting and enjoyable although positive effects of using spreadsheets were not apparent. Statistical analysis of the pre-tests and post-tests showed significant differences in the mean improvement scores between the schools and also between the teaching interventions when only schools 4, 5 and 6 were taken as part of the analysis. Implications of the findings as well as recommendations for future research are discussed.

Following the rapid increase in the use of technology in education over the last decade, one would perhaps expect to find an overabundance of literature regarding the effects of its use (Dix, 1998). However, the number of technology related studies has been surprisingly low (Jones, 1997) especially those pertaining to the curriculum area of mathematics. Morrell (1992) suggests that the availability of quality software, the need for curriculum redesign, and limited research on the effectiveness of computers as a teaching tool, are factors to have retarded the rate of implementation and of subsequent research. This remains true even today. Statements from administrative bodies like the *Australian Association of Mathematics Teachers* (AAMT, 1996) and the *Curriculum Development Centre* (AEC, 1991) have given momentum but little bearing to the integration of technology in the mathematics curriculum. The reality appears to be that the use of computer technology in the mathematics curriculum is not widespread.

Significance of Study

Since educational computing today is still in the decisive stages (Joiner, 1996), its final impact on the structure of education cannot be completely extrapolated from current experimentation. There is clearly a need for further research, so that bounds can be established as the field matures. Although there are a number of studies comparing the effectiveness of traditional instruction with computer based instruction (e.g. Morrell, 1992; Stick, 1997), none have been found that are specific to the area of Australian secondary mathematics. This study is therefore potentially important and of particular relevance in that it responds to the void of qualitative and quantitative data in this domain and also hopes to put at ease the minds of people with significant oppositions toward the introduction of the new *General Mathematics* syllabus which is seen to have ambitious aims and objectives. The new *General Mathematics Stage 6 Syllabus* replaced the 1981 *Mathematics in Society* (MIS) and the 1989 *Mathematics in Practice* (MIP) syllabuses. GM takes on an information processing approach toward learning mathematics, characterised by collecting, organising, interpreting and analysing data (Yen, 2000). The importance of language, technology and the interpretation of graphs and tables are promoted throughout this new course. Technology should play a major role in the form of

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spreadsheets, graphics calculators, computer software and the Internet.

Aims and Objectives

Research indicates that computer technology can help support learning, and that it is especially useful in developing higher-order skills of critical thinking, analysis, and scientific inquiry (Roschelle et al., 2000). However, the mere presence of computers in the classroom does not ensure their effective use. This paper explores the various ways in which computer technology, in particular, spreadsheets can be used to improve how and what students learn in a *collaborative learning* (learning in small groups where students share ideas and work together to complete tasks) environment. We designed a study that compared technology based instruction with traditional pen-and-paper methodologies. We were interested in determining whether or not (1) there was a difference between the achievement of students using spreadsheets to solve algebraic problems and the achievement of students using pen-and-paper for the same unit of work; and (2) students were more interested and motivated to learn mathematics collaboratively using spreadsheets or when using pen-and-paper individually.

Literature Review

Computer supported collaborative learning (CSCL) is one of the promising ideas to improve the teaching and learning of mathematics with the help of computers. CSCL appears to engage students to participate in in-depth inquiry over substantial periods of time and to provide socially distributed cognitive resources for comprehension monitoring and other metacognitive activities (Lehtinen et al., 1998). Meta-analyses on the effectiveness of computers have shown that in the majority of studies, the use of technology markedly improved learning outcomes (Fletcher-Flinn & Gravatt, 1995; Kulik, 1994). These studies do not, however, distinguish between different pedagogical ideas on how computers have been implemented in classrooms. Thus, it is impossible to draw any conclusions about the effectiveness of CSCL on the basis of these general impact studies (Lehtinen et al., 1998). Several empirical experiments offer some evidence that CSCL environments like *Computer Supported Intentional Learning Environments* (CSILE) have proved to be helpful for higher order social interaction and, subsequently, for better learning in terms of deep understanding (Scardamalia et al., 1994). The primary goal of CSILE is to support structured collaborative knowledge building by having students communicate their ideas and criticisms - in the form of questions, statements, and diagrams - to a shared database classified by different types of thinking (Roschelle et al., 2000).

There are numerous studies on CSCL environments demonstrating encouraging effects on the amount and quality of social interaction and other procedural features of teaching-learning processes (Amigues, & Agostinelli, 1992; Fishman & Gomez, 1997; Lamon et al., 1996; McConnell, 1994; Scardamalia et al., 1994). The number of studies on CSCL has dramatically increased during the last decade. There have been many studies aimed at investigating the effects of CSCL on students' achievement. Many studies on small group computer based instruction, published in the late eighties and the early nineties, indicated at least some positive impact on students' learning (Anderson et al., 1995).

Methodology and Design of Study

Mathematics can be learned in a variety of ways - with or without technology, individually or collaboratively. We investigated the effects of CSCL in the area of

Financial Mathematics, one of the five strands in the new *General Mathematics* (GM) syllabus. The design of this study included three methods of data collection - an attitudinal questionnaire administered to students; pre-tests and post-tests; observations of classes and interviews with randomly selected groups of students. Figure 1 outlines the design and methodology of the research study and Table 1 summarises the allocation of teaching intervention to each class.

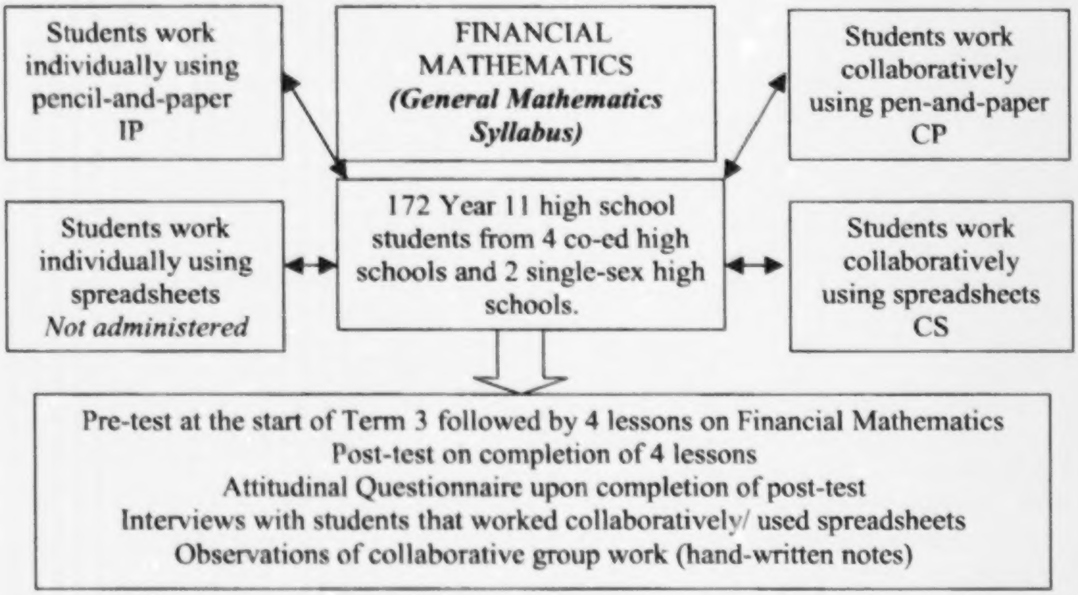
Students work individually using pencil-and-paper IP
FINANCIAL MATHEMATICS (General Mathematics Syllabus)
Students work collaboratively using pen-and-paper CP
Students work individually using spreadsheets Not administered
172 Year 11 high school students from 4 co-ed high schools and 2 single-sex high schools.
Students work collaboratively using spreadsheets CS
Pre-test at the start of Term 3 followed by 4 lessons on Financial Mathematics
Post-test on completion of 4 lessons
Attitudinal Questionnaire upon completion of post-test
Interviews with students that worked collaboratively/ used spreadsheets
Observations of collaborative group work (hand-written notes)

Figure 1. Design of study.

The pre-test and post-test was administered to determine students' knowledge of mathematical concepts like simple interest and compound interest in the area of *Financial Mathematics*, one at the beginning of term and the second towards the end of term, after students had an opportunity to work with the materials. Students under the CP intervention worked in groups of 3-4 students on tasks set out on cards and divided up into parts. Students under the CS intervention working in pairs on similar activities using spreadsheets while students under the IP intervention worked on 'standard-type' problems individually. The pre-test and post-test were based on the content covered in the four lessons under each of the three teaching interventions. Student's feelings and/or opinions towards collaborative group-work using spreadsheets were determined from the responses to questionnaire items administered upon completion of the post-test. The analysis of the questionnaire items, however, is not discussed in this paper. The last method of data collection involved informal observations of students working in each teaching intervention group. Finally, randomly selected students were asked to participate in an interview where responses were handwritten by the researchers, as students felt uncomfortable with the idea of having their responses audio-taped.

A total of ten year 11 classes from six high schools in Sydney, Australia participated in the study and each class was allocated a particular teaching intervention. It is important to note that, although these were arranged to be randomly allocated, some arrangements changed in order to accommodate teachers' preferences for particular treatments. Some teachers for instance, requested that the CS intervention be administered to their students because they wanted to learn about using spreadsheets in the classroom. Also, due to lack of computer resources, the *individual work using spreadsheets* (IS)' teaching intervention did not get administered in this study.

Table 1

Allocation of Teaching Interventions

School	No. of Classes	Teaching Interventions Administered
School 1	1 class	Collaborative work using spreadsheets (CS)
School 2	1 class	Collaborative work using spreadsheets (CS)
School 3	1 class	Individual work using pen-and-paper (IP)
School 4	2 classes	Collaborative work using pen-and-paper (CP) Individual work using pen-and-paper (IP)
School 5	3 classes	Collaborative work using spreadsheets (CS) Individual work using pen-and-paper (IP) Collaborative work using pen-and-paper (CP)
School 6	2 classes	Collaborative work using spreadsheets (CS) Collaborative work using pen-and-paper (CP)

Although there may be value in studying a range of year levels within the secondary school, this study was limited to one-year level. Year 11 students were selected for the reasons such as: (1) the new General Mathematics syllabus for Year 11 students; (2) the recommendation for the use of spreadsheets in different areas of the syllabus; (3) the non interference with end of year examinations during the course of the study; (4) students below Year 11 not generally requiring access to computers for mathematics; and (5) research on year 11 students not having been conducted often.

Results and Analysis

Analysis of Pre-Tests and Post-Tests

Out of the 172 students that participated in this study, 150 did the pre-test, while 116 did the post-test. It is important to note that fewer (and different) students sat for the post-test. In some schools it was difficult to get the students to do the post-test seriously. They enjoyed the activities and participated well in class but did not enjoy being tested. Also, the fact that the study was voluntary in nature, implied that students did not have to do any component of the study if they did not want to. An *Analysis of Variance* (ANOVA) using the *general linear model* for improvement scores (difference between pre-test and post-test scores) and *sex*, *school*, and *intervention* as explanatory variables showed very significant differences in effects between schools (n.b.** denotes $p<0.001$) The highest mean improvement score was recorded for school 2 (11.7 out of 40) while the lowest (negative) score was recorded for school 3 (-7.2). This may have been due to the 'holiday mode' of the students as the study was carried out in the last week of term in some schools, but may be a real effect. A similar analysis using only schools 4, 5, and 6 showed differences between the three teaching interventions, with the greatest improvement being in the IP intervention and the least (negative) improvement in the CS intervention. The researchers wanted to investigate any differences in the mean improvement scores between schools that had more than one class in them, namely, schools 4, 5 and 6. In both analyses, no differences between genders were observed. There were difficulties in using computers in the teaching intervention, which will be discussed later.

Observations of Classes

The informal observations of the classes were impressions gained over the four lessons, as opposed to minuting them lesson-by-lesson. These observations were analysed using Bell's (1993) indicators of learning: (1) interest and enjoyment; (2) involvement and engagement of students in the learning activities, (3) ownership of work and (4) on-task behaviour. Overall, in the CP intervention groups, there was a large degree of interest and enjoyment during the four collaborative group lessons. Over the course of these four lessons, noise levels became quite high at times, and students were frequently observed in animated and enthusiastic discussions. Engagement levels and involvement in the activities seemed to be high, especially as measured by task completion. On-task behaviour was relatively high, and there were no instances of obviously distracted behaviour seen for whole groups. Students under the CS intervention showed reasonable interest at the start, which slowly appeared to 'dwindle off'. The same was observed with engagement and involvement levels where students appeared to become disinterested later in the study. Shared ownership of work was high in these groups, and off-task behaviour was also high with students getting distracted and going off to check their emails *et cetera*. Students under the IP intervention appeared to do the work set out as they would have done during any of their normal mathematics classes, engagement and involvement levels were very high in this group as measured by task completion, as was on-task behaviour.

Limitations of the Study

A number of constraints served as limitations in this study. These include the duration of the study, the timing of the study, sample size, access to computers, and computer resources in schools. However, steps were taken to minimise these affects. In cases where students were involved in the CS intervention, computers with *Microsoft Excel* (at least) were required. In most schools, problems were experienced with being able to book laboratories for a particular time or day and also, many computers in the schools were non functional which meant that in most cases, more than two students had to share a computer, which got a bit tricky in terms of workload, and could account for some of the non participation in this study. During the first lesson, most students experienced difficulty with operating and working with the software and so got exasperated. However, as the lessons progressed and students became familiar with the software, they found the activities enjoyable and interesting.

As mentioned above, access to computer laboratories was a considerable restriction. For a number of reasons the preferred method of comparing whole units of work, requiring several weeks, was reduced to activities, achievable in four lessons. Firstly, time constraints and access to resources did not permit the design and development of whole units. Secondly, with computer rooms fully booked terms in advance, securing several weeks of class time was not possible. The request of one week in the computer laboratory for a certain class was more readily accommodated. Finally, the expectation that a clear result would emerge from a study of such short duration was considered optimistic. It is hoped that this research may provide direction and insight for future studies.

Implications

A majority of students in this study indicated that they found the collaborative activities interesting and enjoyable. However, it must be noted that some students did not agree with these statements, and some even expressed an extreme dislike to collaborative learning (CL) methods. A variety of learning strategies is thus called for, to best meet the needs of all students. It was also seen that collaborative groups do not always function effectively, and instructors employing CL methods must pay constant attention to minimising factors that contribute to such ineffectiveness, and addressing problems where they occur. Despite this, there is a clear indication that we should include some opportunities within mathematics classes for students to work collaboratively and talk about mathematics.

This study also highlights the need for adequate computer resources, support and training when implementing new curricula. The use of software such as *Microsoft Excel* requires time to learn. The package initially interferes with mathematics learning. However the start up time in this study was only about 2 lessons. Over time, and with adequate technological and pedagogical support and training, implementation of technology into the mathematics classroom would receive less resistance from teachers and students. Without proper resources, adequate training of teachers in using computer technology, on-going technological and pedagogical support, and funding, implementation of CSCL methods in the mathematics curriculum is going to continue to be a challenge for both students and teachers. If teachers are not going to receive the support they require to implement technology in their classrooms, they are not going to make the effort to learn the software to be able to teach it. Not many teachers have 'spare' time on their hands during which they could learn how to use computers in the classrooms. This in turn affects students and their opportunities for different learning styles. This is a vicious cycle that needs to be broken sooner than later. Collaborative learning without computers was quick and easy to implement in this study, and students could learn the mathematics almost immediately. However, the design of good collaborative materials is time-intensive and needs to be at an appropriate level. Preference for learning using individual pen-and-paper methods by majority of students implies that students feel that they learn better using traditional teaching methods, having being taught in this way all their lives. An implication for this is that changing teaching and learning methodologies to include CSCL methods may receive initial resistance from students as well as teachers.

Recommendations for Future Research

This study and other reports of studies using computer technology indicate the potential for spreadsheets to dramatically influence the way students learn and the way mathematics is taught in the classroom. The potential for all students to explore powerful mathematical ideas is exciting. Recommendations from this study include giving teachers support to learn about this software package and to find interesting and comprehensive methods for exploring mathematics. The new mathematics curriculum *General Mathematics*, supports the recommendations of the NCTM of promoting the infusion of the mathematics curriculum with technology. As this research study shows, there are problems with the implementation. The syllabus is new and curriculum implementation takes time. More research into teaching and learning with spreadsheets is recommended.

Future studies, therefore, may want to consider the following: (1) This study centred on the use of computer spreadsheets in a collaborative learning environment. Are there similar results when using technology other than spreadsheets, for instance, when using tools such as graphics calculators? (2) This study was completed in 12-weeks, with each class (in the six schools) being administered a post-test at the end of a six-day study. Would results be the same following a study that exposed students to spreadsheets for a longer period of time? (3) This study did not acquire data on student access to computers outside of the classroom. Do students with computers at home perform better than students who only have access at school? (4) The post-test and attitudinal questionnaire was administered in most cases near the time of Year 11 examinations. Would significant changes in attitude be apparent if the timing was different? (5) Attitudes of teachers toward CSCL methods and their teaching approaches were not tested. A future study may want to include actual testing of teacher attitudes and approaches. (6) This study centred on Year 11 students. Would results be the same with tertiary students - those who might already have prior experiences with spreadsheets and/or collaborative learning methods? (7) This study has not considered any assessment issues. What kind of assessment guidelines could be put into place in a CSCL mathematics classroom?

Concluding Remarks

The review of the literature has provided many pointers and suggestions into ways in which the researchers can improve the use of collaborative learning in their own teaching as well as increasing confidence in continuing to use such methods.

The results of this study itself have suggested ways in which the researchers might structure collaborative lessons, in order to further increase the learning opportunities for all students. The study also highlights problems with access to computer technology in real classrooms, which is a serious issue in mathematics teaching and learning. The use of computer spreadsheets can be a valuable tool for secondary school students as they study mathematics. Students who made the greatest gains after the unit of study were not necessarily those who were the most successful, as measured by the post-test. Computers are likely to bring about major changes in the way we teach secondary school mathematics. We need to examine objectively the effects of teaching and learning with or without technology as well as different learning styles. As computers become accessible to all students, changes can be made in the classrooms that enrich students' experiences in mathematics.

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Assessing the Potential Suitability of "Show That" Questions in CAS-Permitted Examinations

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Examples of "show that" examination questions highlight the various ways CAS can impact upon question design, possible solution approaches, documented responses and marking schemes. A classification scheme used on some recent non-CAS NSW and Victorian examinations found that CAS and the role of automatic simplification can influence the solution process in a variety of ways. Assessment issues such as control over CAS and the nature of solution accounts are also discussed using student responses from a pilot CAS subject in Victoria.

The construction of examinations that allow students to use CAS pose direct *technical* and *pedagogical* challenges for external examination setting panels, and hence, for teachers setting assessment tasks and textbook authors. In a technical sense, CAS influences many facets of examination question design practices such as structure, language, explicitness, marking scheme protocols and equal access for users of different permitted machines. On a pedagogical level, careful consideration needs to be given to the nature and bandwidth of mathematical abilities we want (and are able) to assess in CAS-permitted examinations. In particular, consideration must be given to the viability of testing students' abilities to prove mathematical results when CAS could be used to construct the proof. In this paper, the suitability of current non-CAS "show that/prove" examination questions for use in future CAS-permitted examinations are scrutinised by examining the impact CAS has on intermediate steps and the implications for question design, the nature of student responses and student control of CAS. A first example is used to demonstrate the need for scrutiny of "show that" questions and the problems that specification of form of results can cause for CAS users. A second example reports on students' responses to a "show that" question designed to avoid common pitfalls. Finally, a survey of "show that" questions from recent NSW and Victorian examinations provides a catalogue and classification system of the many ways in which these questions can be affected by CAS.

Relevant Literature

Flynn & McCrae (2001) and Flynn (2001) categorised graphics calculator-permitted examination questions in Victoria according to the degree of CAS impact. These schemes classified questions as those where CAS has either (i) no impact; (ii) some impact but remaining useable as the question's conceptual difficulty is preserved (although slight adjustments in structure, wording or marking schemes may be needed); or (iii) sufficient impact that the question would be omitted from a CAS-permitted examination because it only tests knowledge of a procedure that can be done with CAS. These schemes demonstrated that CAS use can render some traditional examination (algebra and calculus) questions redundant for either technical and/or pedagogical reasons and cause a shift in what mathematical knowledge a question is actually testing. Included in this subset were questions where CAS use eradicated all possible intermediate steps and hence a student

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would be unable to provide a written solution account unless some intermediate steps were deliberately done by hand.

Several authors (for example Lagrange, 1999; Pierce & Stacey, 2001) outline other mathematical abilities that will increase in importance when using CAS. Students must possess algebraic insight including being able to convert conventional mathematics to the required syntax for entry into CAS and to recognise equivalent expressions when interpreting CAS outputs and checking the reasonableness of results (Pierce & Stacey, 2001). Algebraic insight is needed for doing mathematics by hand, but it is a greater part of work with CAS, partly because the CAS often throws up unexpected results that would never arise by hand. Lagrange argues that while traditional algebraic techniques will become less important, students using CAS will need to master new techniques to be able to reflect on and demonstrate algebraic equivalence of expressions generated either by-hand and/or by-CAS in addition to understanding the form of CAS outputs.

In addition to shifts in emphasis of required algebraic knowledge, shifts will also occur in how solution accounts are documented. Ball & Stacey's (in press) RIPA rubric suggests a framework for recording solutions with CAS, to help teachers ensure that students learn good mathematical practice for presentation of solutions assisted by technology. The framework stresses that the overall solution plan must be apparent and there must be clear recording of mathematical reasoning, inputs to CAS and selected CAS outputs following correct mathematical syntax and conventions. Hence, it is anticipated that whilst there will be a decreased emphasis on presenting manipulative intermediate steps, this needs to be compensated for by enhanced written evidence in choice of solution approach.

There are many instances in 1995-2000 Victorian VCE and New South Wales' HSC Examiners' Reports that comment upon current student difficulties in addressing non-CAS "show that" examination questions successfully. See, for example, these quotes firstly from the 1998 VCE Specialist Mathematics Report for Teachers (Board of Studies, VIC 1999) and secondly the 1998 NSW HSC Examination Report (Board of Studies, NSW 1999):

> This year there were a number of questions that required students to establish results given on the paper. The answers to such 'show that' questions are often poorly presented and it needs to be stressed to students they must show, clearly and logically, every step that leads to the result. (p. 22)
>
> It is also worth emphasising that, in questions where the answer is given, it is the responsibility of the candidate to ensure that sufficient lines of working are given to convince the examiners that the result has been shown. (p. 61)

Both examination bodies see that students find difficulty adequately demonstrating reasoning with by-hand solutions. Coupled with automatic simplification, the influence of CAS could have important implications for the structure of "show that" examination questions, the nature of student responses, and their ability to control CAS successfully.

"Show That" Examination Questions

Dual Functions in Examination Assessment

It is a relatively common practice to structure examination questions around instructions such as "show that" or "prove" (words which we believe have the same intention). These questions perform two important functions in examination design. One is to assess students' ability to construct and present a logical well-reasoned mathematical argument using a sequence of intermediate steps. These intermediate steps, rewarded through marking schemes, can incorporate the testing of many facets of mathematical knowledge including use of mathematical definitions, facts and theorems and employment of various techniques, procedures and algorithms. Through these questions, students can demonstrate their knowledge of the connections between mathematical ideas. For this reason, it seems important that "show that" questions be retained in new assessment.

A second function of "show that" questions is to increase accessibility by supplying a result required in later question parts. In addition to assisting students through these gateways, the supplied results are usually in a form which enhances the likelihood of student's successfully completing later parts of the question (e.g. in a form easy to integrate, interpret etc). This function of "show that" questions is therefore to make difficult questions easier by providing implicit guidance along the solution path.

A First Example

This section raises some of the issues with "show that" questions by discussing an example. Flynn and McCrae (2001) and Flynn (2001) found that CAS use can render some traditional examination (algebra and calculus) questions unusable for either technical and/or pedagogical reasons and cause a shift in what is tested in a question. A subset of these unsuitable questions can be completed by one direct input into CAS (without any formulation or abstraction), with CAS providing an identical output to the required result, meaning no intermediate steps can be documented.

The diagram (not reproduced here) shows part of the graph of the curve with equation $y = e^{2x} \cos x$.

(a) *Show that* $\frac{dy}{dx} = e^{2x} (2 \cos x - \sin x)$.

(b) Find $\frac{d^2y}{dx^2}$.

(c) There is an inflexion point at P(a, b).Use the results from (a) and (b) to *prove that*

(i) tan $a = 3/4$ and

(ii) the gradient of the curve at P is e^{2a}

Figure 1. IBO (2000) Mathematical Methods Standard Level Paper 2, Question 7.

Such a situation occurs in the "show that" examination question from the 2000 International Baccalaureate Mathematical Methods Standard Level Paper 2 (IBO, 2000) shown in Figure 1. A student using a TI-89 would be unable to provide any account of their solution to part (a) because the required factored result is provided immediately as an output. Note this is not necessarily the case for all CAS, an issue which is taken up in the next section. Part (a) also loses its value in a CAS-permitted examination because it is no longer testing whether students can use the product rule of differentiation. This is a clear example of the potential influence of CAS to cause shifts in the mathematical abilities assessed.

Challenges in Setting "Show That" Examination Questions for Different CAS.

Interest in structuring fair questions for users of different CAS is precipitated by the introduction in Victoria of VCE Mathematical Methods (CAS) which permits different CAS in its examinations. Figure 2 shows the potential difficulties in setting "show that" CAS-permitted questions for users of different CAS (labelled CAS 1 and CAS 2 in the figure).

CAS 1	CAS 2
(I) Differentiate(*expression*, *variable*)	(I) Differentiate(*expression*, *variable*)
(O) $\frac{dy}{dx} = e^{2x}(2\cos x - \sin x)$	(O) $\frac{dy}{dx} = 2e^{2x}\cos x - e^{2x}\sin x$
	(I) Factor(*Answer*)
	(O) $\frac{dy}{dx} = e^{2x}(2\cos x - \sin x)$
(I) Differentiate(*Answer*, *variable*)	(I) Differentiate(*Answer*, *variable*)
(O) $\frac{d^2y}{dx^2} = e^{2x}(3\cos x - 4\sin x)$	(O) $\frac{d^2y}{dx^2} = 3e^{2x}\cos x - 4e^{2x}\sin x$
	(I) Factor(*Answer*)
	(O) $\frac{d^2y}{dx^2} = e^{2x}(3\cos x - 4\sin x)$
(I) Solve(*equation*, *variable*)	(I) Solve(*equation*, *variable*)
Solve $e^{2a}(3\cos a - 4\sin a) = 0$ for a.	Solve $e^{2a}(3\cos a - 4\sin a) = 0$ for a.
(O) $a = n\pi + \tan^{-1}(3/4)$ where $n \in J$.	(O) $a = 2k\pi + 2\tan^{-1}(1/3)$ where $k \in J$.
	(N) $t = \tan(a/2)$ and $\tan a = 2t/(1-t^2)$, $\tan a = 3/4$.
(I) tan(*answer*)	(I) Apply tan *to both sides* in *approx*. mode
Apply the tan function *to both sides*	(O) $\tan a = 3/4$.
(O) $\tan a = 3/4$.	(I) Substitute into derivative. $f'(\tan^{-1}(3/4))$ is $e^{2\tan^{-1}(3/4)}(2\cos(\tan^{-1}(3/4)) - \sin(\tan^{-1}(3/4)))$.
(I) Substitute into derivative.	(I) $2\cos(\tan^{-1}(3/4)) - \sin(\tan^{-1}(3/4)) = 1$,
(O) $f'(\tan^{-1}(3/4)) = e^{2\tan^{-1}(3/4)}$ which is e^{2a} as $a = \tan^{-1}(3/4)$.	Hence $f'(\tan^{-1}(3/4))$ is e^{2a} in terms of a.

Figure 2. Solutions to IBO 2000 Question 7, using two different CAS.

The solutions illustrating CAS input (I), CAS output (O) and non-CAS (N) address the proof question 7(c) (Figure 1) which require use of results obtained or given in parts (a) and (b). A CAS 2 user needed to demonstrate a little more knowledge for parts (a) and (b) to obtain the factored forms, and greater knowledge of trigonometric identities than a CAS 1 user to show the results in part (c). A CAS 2 user may need greater flexibility moving between by-hand and CAS techniques due to interruptions caused by unexpected CAS output. The solutions provided for a ($a = n\pi + \tan^{-1}(3/4)$ and $a = 2k\pi + 2\tan^{-1}(1/3)$) are equally valid, but the form of the specified result privileges CAS 1 over CAS 2. A CAS 2 user can also show the last result by using the approximate function that operates less directly than CAS 1 to simplify the trigonometric expressions. This shows that different CAS can offer different solution pathways, intermediate results and can be mode sensitive.

These sample solution records illustrate that "show that" questions can be problematic because calculator outputs are dependent upon the internal simplification code and the capabilities of the particular CAS. On the other hand, if intermediate answers are not specified, the gateways to direct solution paths are absent so the question would become more difficult. This example shows that challenges will confront students when using CAS in examination assessment. Students will be required to present logical well-reasoned solution accounts when using CAS and they must be able to exert a strong degree of *control* over their CAS in these situations. This control must be maintained throughout any input/output interaction with CAS. The predisposition of CAS to simplify expressions automatically and its sensitivity towards syntax of entered expressions could present difficulties for students controlling and responding to CAS inputs and outputs.

Example of a "Show That" CAS-Permitted Examination Question

Setting/Methodology

We wanted to investigate the viability of setting "show that" examination questions influenced by CAS and to commence looking at whether students could (i) employ the capabilities of CAS with enough control to address such questions successfully; (ii) present logical and sufficiently detailed solutions; and (iii), move flexibly between by-hand and CAS techniques. One of the three participating schools in The University of Melbourne's CAS-CAT project (http://www.edfac.unimelb.edu.au/DSME/CAS-CAT) used a "show that" examination question (Figure 3 below) in their internally reported 2001 Unit 1 VCE Mathematical Methods (CAS) Examination. Forty two Year 11 students sat the examination and were permitted to use a Casio FX 2.0. The students had five months experience using CAS and hence were still developing strategies for effective CAS use. CAS impacts upon this question by decreasing the number and nature of intermediate steps and provides a natural solution approach because of the CAS's facility to factor the expression directly. Alternatively, CAS could be used to expand and then factor the expression. The marking scheme awarded one mark for a correct substitution; one method mark for obtaining $(x^2 + 3y^2)^3$ or a partially factored variant of this and a third mark for explicitly relating the factored result to being a perfect cube. The third mark was only awarded if the obvious perfect cube form was specified. The analysis below is based only on the written records of the students. Each student's solution was assessed in accordance with the marking scheme, various solution approaches categorised and interesting features in solution accounts noted.

Let $a = x^3 - 9xy^2$ and $b = 3x^2y - 3y^3$. Show that $(a^2 + 3b^2)$ is a perfect cube.

Figure 3. Unit 1 VCE Mathematical Methods (CAS) Examination Question

Results/Implications for CAS-Permitted Examination Assessment

Table 1 below describes the approaches used and a measure of their success. Despite five months exposure to CAS, a significant proportion of students (22/42) did not use CAS. Students using by-hand methods had to undertake lengthy and highly error-prone approaches to obtain the perfect cube form and, not surprisingly, were not successful, with familiar errors in substitution, expansion and factorisation exhibited. Twelve out of 18 students who used CAS were successful in establishing the perfect cube result.

Table 1

Profile of Student Responses to 'Show That' Examination Question

Response Mode	Not Attempted	No CAS	CAS	Mixture
Responses	3	19	18	2
Correct Answers	0	1	12	0
Mean score out of 3	0.0	0.8	2.2	1.5

These students typically demonstrated the result by explicit recording of the mathematical operations (eg using the words factor, simplify or substitute) rather than making use of these mathematical operations implicitly in manipulative intermediate steps such is done in non-CAS situations. The successful CAS students invariably informed the marker of their solution approach, the mathematical operations used and they recorded selected outputs. This is shown in Figure 4, illustrating features of CAS-style "show that" responses recorded by two students.

Few students amalgamated CAS and by-hand techniques. This weak interplay was signified by students almost universally selecting either CAS or non-CAS at the start of the solution process and continuing with one approach. There was little evidence of students moving flexibly between non-CAS and CAS approaches even after experiencing algebraic difficulties with the former.

Student A	Student B
$(x^3 - 9xy^2)^2+3\,(3x^2y - 3y^3)^2$	*Factor* $(x^3 - 9xy^2)^2+3\,(3x^2y - 3y^3)^2$
Factor gives $(x^2 + 3y^2)^3$	*Factor* $x^2(x-3y)^2(x+3y)^2+3(3x^2y-3y^3)^2$
This is in perfect cube form	$(x^2 + 3y^2)^3$ (The power 3 means a perfect cube).

Figure 4. Sample student CAS responses to the "Show That" question.

A tension was noticed amongst some students with inputting the correct algebraic expressions accompanied by correct syntax, detecting the presence of syntactical errors and having the facility to deal with unexpected CAS outputs. Five students omitted an important set of surrounding brackets and entered *Factor (expression* instead of correctly entering *Factor ((expression)*. (Note that their calculator does not require a final closing bracket.) This resulted in a partial factorisation of the first part of the expression but the second part was left untouched. Two of these five students (e.g. Student B in Figure 4) exhibited excellent algebraic expectation (Pierce & Stacey, 2001) by considering the output and again factoring to obtain the perfect cube form. The other three students confronted with this situation did not deal with this unexpected output and left the partially factored result as is, hence not obtaining the perfect cube form. Generally, students using CAS were reasonably successful with their control of CAS and were able to demonstrate the required result with well-reasoned solutions. In attempting questions in this manner, students must become accustomed to checking their inputs (syntax) and outputs for any unexpected results.

Recent Non-CAS "Show That" Examination Questions

Analysis of the "show that" examination question described in the previous section highlights the possible decrease in required intermediate steps and greater emphasis placed on explicit naming of mathematical operations in solution accounts. This encouraged us to look at the effect of CAS on "show that" examination questions more widely. A classification scheme adapted from previous schemes employed by Flynn & McCrae (2001) and Flynn (2001) was used to categorise 'show that' examination questions from the 1997 to 2001 VCE Specialist Mathematics Examination 2 and the 1997 and 2000 NSW HSC 3 Unit (Additional) and 3/ 4 Unit (Common) Examinations. Each question was answered using a TI-89 and classified according to the *degree that CAS impacted upon the number and nature of required intermediate steps*. The percentage of marks of "show that" questions affected was noted and are shown in Table 2. The table illustrates the marks allocated to "show that" questions as a percentage of the total marks available in the examination and the distribution of the three categorised types of "show that" questions expressed as a percentage of the total marks allocated to "show that" questions. The last column in the table provides the percentage of marks allocated to "show that" question parts where the result was needed for a later question part.

Table 2

Distribution of marks allocated to "Show That" Examination Questions by Type

Examination	Total Marks (%)	Types			Result Needed (%)
		1	2	3	
(1997-2001) VCE Specialist Mathematics Exam 2	17	23	65	12	85
1997 HSC 3 Unit (Additional) and 3/4 Unit (Common)	32	19	74	7	67
2000 HSC 3 Unit (Additional) and 3/4 Unit (Common)	33	25	75	0	46

Type 1 questions contain no CAS-affected intermediate steps. They typically include geometry and other questions not involving algebra; formulation/abstraction of mathematical expressions from worded or diagrammatic information; or interpretation of properties of unspecified functions. Type 3 "show that" examination questions are unsuitable in CAS-permitted examinations because using CAS devours all the intermediate steps and provides the desired result without giving students the need to display any intermediate mathematical reasoning. Part (a) of the question in Figure 1 is an example for a TI-89 user. For CAS-permitted papers, these questions need to become "find" type questions.

Type 2 questions are somewhat affected by CAS. These may still be useable in CAS-permitted examinations, with modification, but the emphasis of what is being tested may change. Some Type 2 questions would not be asked in a CAS-permitted paper as these results are provided currently to assist students answering questions that otherwise could not be done, for example, the direct technique is outside the syllabus. CAS would influence these questions to become "find" style questions. In some cases, solution pathways are parallel, where CAS use compresses the number of required intermediate steps (e.g. question in Figure 3). In others cases, the automatic simplification of CAS takes you on a different solution pathway (e.g. Figure 1 part (c) for CAS 2 user) or it forces the interplay between CAS and by-hand techniques. There are also occasions where CAS can be used in

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a by-hand like manner to construct the required solution. There are many variants belonging to this type, and space does not permit the variety to be demonstrated here.

Conclusion

The examples discussed here and the analysis of selected non-CAS examination papers demonstrate the care needed when setting CAS-affected "show that" questions in CAS-permitted assessment and the variety of ways CAS can impact upon them. "Show that" CAS-permitted examination questions should have their suitability scrutinised in terms of the impact of CAS (including different CAS) on question structure and wording, intermediate steps, anticipated solution responses and marking scheme frameworks. These questions can be strongly affected by CAS use, particularly when construction of the proof is achieved mainly by symbolic manipulation and/or use of algebraic or trigonometric identities. Automatic simplification, differences in brand capabilities and internal simplification codes between various CAS and appropriate CAS settings can also affect responses. However, replacement of 'show that' examination questions with 'find' type questions in CAS-permitted examinations could unduly raise the overall level of difficulty and lower the accessibility of examinations.

Retaining "show that" examination questions is important because they are one pointer to the central role that proof plays in mathematics. Demonstrating how concepts are linked and displaying a logical argument is possible in CAS-permitted assessment, but achieving it fairly will require some imaginative recasting of current question types.

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Computers For Learning Mathematics: Equity Considerations[1]

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Mathematics and computing have been stereotyped as 'male domains'. Contemporary mathematics teachers are encouraged to use computers in class and it is believed that students' learning will thus be enhanced. In this paper, findings from a large survey tapping students' attitudes towards computers for mathematics learning are presented. The results were analysed by several equity factors – gender, student socio-economic background, ethnicity, school type and location (rural/metropolitan, and socio-economic location) – to explore the gender stereotyped beliefs of these groups of students.

Introduction

Over time, the disciplines of mathematics and computing have been viewed as *male domains*, that is, both have been considered more suited to males than to females. Compared to females, males have generally been found to hold more functional (likely to lead to success) beliefs and attitudes towards mathematics and about themselves as learners of mathematics (see Leder, 1992). More recently, it has been reported that Australian students' gendered patterns of beliefs associated with the stereotyping of 'mathematics as a male domain' appear to be changing (e.g., Forgasz, 2001a, 2001b). The extent of these changes has been found to differ among ethnic groups (Barkatsas, Forgasz, & Leder, 2001).

Research on secondary students' attitudes towards computers reveals that compared to males, females are generally found to be less positive about computers, to like them less, to perceive them as less useful, to fear them more, to feel more helpless around them, to view themselves as having less aptitude with them, and to show less interest in learning about and using them. Females are also less likely than males to stereotype computing as a male domain, to have received parental encouragement, to use computers out of school or to own one (e.g., Busch, 1995; Colley, Gale, & Harris, 1994; Durndell, Glissov & Siann, 1995; Levin & Gordon, 1989; Makrakis & Sawada, 1996). Makrakis and Sawada (1996) also found that male and female students from both Japan and Sweden perceived the usefulness of mathematics and computers and boys' and girls' aptitudes in these fields as male stereotyped. Miura (1986) reported that undergraduate males were more likely than females to consider being able to use a computer to be important in their future careers. Andre, Whigham, Hendrickson, and Chambers (1999) found that among older primary level children, boys perceived jobs requiring life science, computer science and team sports as more male dominated than did girls and that parents held stereotyped perceptions of the importance of school subjects and of their children's competencies. Shashaani (1993)

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concluded that gender differences were influenced by socialisation and, as a result, females "have low expectations for success in computing" (p.179). Schools policies and practices may also contribute to these expectations (Ogbu, 1992).

Computers are now commonly found in mathematics classrooms and there is much pressure to use them. Maintaining that it was crucial to know whether using computers for mathematics learning would exacerbate or challenge previously identified gender differences in mathematics education, Forgasz (2002) examined students' gendered perceptions of mathematics, of computers, and of computers for the learning of mathematics. The students' beliefs about computers were found to be the most traditionally gender-stereotyped and their beliefs about using computers for learning mathematics seemed to fall somewhere between their beliefs about the stereotyping of mathematics and of computing.

In this paper, results are presented of an examination of students' beliefs about computers for the learning of mathematics by a range of equity factors – student gender, socio-economic background and ethnicity (first language, and aboriginality), school type and location (rural/metropolitan, and socio-economic location). Within the Australian context, these dimensions of equity have been identified as contributors to educational disadvantage in government policy (e.g., Ministerial Council on Education, Employment, Training and Youth Affairs [MCEETYA], 1999), and in research into factors associated with differences in performance levels (e.g., Teese, Davies, Charlton, & Polesel, 1995). The extent to which students in the various equity groups hold gender-stereotyped beliefs about computers for the learning of mathematics are compared and discussed.

The Study

Aims

The findings reported here are based on data gathered in the first year of a three year study[1]. The aims of the main study are: (i) to determine the effects on students' affective and cognitive learning outcomes of using computers for mathematics learning, (ii) to identify factors which may contribute to inequities in these learning outcomes, and (iii) to monitor how computers are being used for the learning of mathematics in grades 7-10 (see Forgasz & Prince, 2001). Students' attitudes and beliefs about using computers for the learning of mathematics are the focus of the findings presented in this paper.

Sample, Instrument and Methods

Students in grades 7-10 from 28 co-educational schools in Victoria participated in the study. There were 15 metropolitan and 13 rural schools from the three educational sectors — Government (17 schools), Catholic (4 schools), and Independent (7 schools). The total sample size was 2140. Frequencies and percentages of students in the various equity categories of interest in this study are shown in Table 1.

A survey questionnaire was administered to the students in semester two of the 2001 academic year. As well as gathering information on several student background characteristics (e.g., gender, ethnicity, and socio-economic status [SES]), the survey included a set of ten items (*Who & computers for mathematics*) tapping students' beliefs about using computers for learning mathematics (see Table 2 for the items used). The ten items were developed and adapted to reflect dimensions that have been associated with the gender stereotyping of mathematics: ability, teacher, classroom, general attitude, career (see Leder, 2001). For each item, students were required to select one of the following responses with respect to the behaviour or belief reflected in the wording of the item:

BD	boys definitely more likely than girls	GP	girls probably more likely than boys
BP	boys probably more likely than girls	GD	girls definitely more likely than boys
ND	no difference between boys and girls		

Table 1

Frequencies and Valid Percentages of Grade 7-10 Students by Grouping Categories

Category	Total N=2140[a]		
School type	Government	Catholic	Independent
(Valid N=2140)	1316 (61.5%)	339 (15.8%)	485 (22.7%)
School SES location[b]	High	Medium	Low
(Valid N=2140)	683 (31.9%)	1047 (48.9%)	410 (19.2%)
Student SES[b]	High	Medium	Low
(Valid N=2066: 96.5%)	500 (24.2%)	1185 (57.4%)	381 (18.4%)
Gender	Female	Male	
(Valid N=2127: 99.4%)	1015 (47.7%)	1112 (52.3%)	
School location	Metropolitan	Rural	
(Valid N=2140)	1211 (56.6%)	929 (43.4%)	
Ethnicity: (Non-)English speaking background	NESB[c]	ESB	
(Valid N=2134: 99.7%)	491 (23.0%)	1643 (77.0%)	
Ethnicity: Aboriginality	ATSI[d]	Non-ATSI	
(Valid N=2121:99.1%)	42 (2.0)	2079 (98.0)	

[a] Missing data account for 'Valid Ns' not being equal to 2140

[b] The socio-economic status [SES] of a school's location was determined from its postcode; the SES of a student was based on home postcode. The categorisations were derived from: Australian Bureau of Statistics (1990). *Socioeconomic indexes for areas*. Catalogue No.1356.0. Canberra: AGPS.

[c] NESB = Non-English speaking background; ESB = English speaking background

[b] ATSI = Aboriginal and Torres Strait Islander

Analyses, Results and Discussion

In order to determine an overall (average) directional response to each of the then *Who & computers for mathematics* items, mean scores were calculated based on assigning scores to each response as follows:

BD = 1   BP = 2   ND = 3   GP = 4   GD = 5.

Mean scores less than 3 thus indicate that, on average, respondents believe that "boys are more likely than girls" to behave or hold the belief reflected in the wording of the item; means greater than 3 that they believe that "girls are more likely than boys" to do so. For mean scores close to 3 (no difference between boys and girls), one-sample *t*-tests were used to determine if the score was significantly different from 3. In Table 2, the predicted response direction for each item, based on the literature in the field, as well as the actual response direction for the entire sample of students are shown as follows:

F = "girls are more likely than boys to…"
M = "boys are more likely than girls to…"

ND = "no difference between girls and boys" (mean not significantly different from 3)

From the data in Table 2, it can be seen that the results for the whole sample are consistent with six of the predicted response directions (Items 1, 2, 3, 6, 8 and 9) and generally confirm traditional gender stereotypical expectations about computer use for mathematics learning: males are more competent than females, females need more assistance from the teacher, males take control of the computers in the classroom setting, and that being able to use computers for mathematics is more relevant to males' than females' future careers. Counter to predictions, the students perceived that there was no difference between males and females in liking to use computers for learning mathematics (Item 5), in giving up when things get difficult (Item 7), nor for whom parents think that using computers for mathematics is important (Item 10). They also considered males more likely than females to find using computers for mathematics to be boring (Item 4).

Table 2
Who & Computers for Mathematics (10 Items), Predicted (Pred) and Actual (Act) Response Directions

	Items: Who & computers for mathematics	Pred	Act
1	Are good at using computers for learning mathematics (Ability)	M	M
2	Mathematics teacher gives them more help when using computers in class (Teacher)	F	F
3	Think it important for their future jobs to be able to use computers for mathematics learning (Career)	M	M
4	Find using computers for mathematics to be boring (General attitude)	F	M
5	Do not like using computers for doing mathematics (General attitude)	F	ND
6	Tease kids who are good at using computers for their mathematics work (Classroom)	M	M
7	Give up when they find using computers for mathematics to be difficult (Ability)	F	ND
8	Like to take control of the computer when students work together in mathematics classes (Classroom)	M	M
9	Distract others as they work on computers in mathematics classes (Classroom)	M	M
10	Parents think it is important for them to use computers for learning mathematics (Parents)	M	ND

A summary of the directions of student responses for the whole sample and by equity groupings is shown in Table 3. The item numbers (and total frequencies in square brackets) on which response directions were consistent with the traditional gender-stereotype are indicated. With respect to responses inconsistent with the traditional stereotype, two categories have been used: ND (no difference between girls and boys), and OPP (the opposite direction to the traditional stereotype).

Table 3
Beliefs by Equity Categories

	Traditional stereotype	Not traditional stereotype	
	Items [N]	ND – no difference [N]	OPP - Opposite direction [N]
Whole sample	1, 2, 3, 6, 8, 9 [6]	5, 7, 10 [3]	4 [1]
School type			
Government	1, 2, 6, 8, 9 [5]	3, 5, 7, 10 [4]	4 [1]
Catholic	2, 6, 8, 9 [4]	1, 3, 7, 10 [4]	4, 5 [2]
Independent	1, 2, 3, 4, 5, 6, 7, 8, 9 [9]	10 [1]	-
School SES location			
High SES	1, 2, 3, 5, 6, 7, 8, 9 [8]	4, 10 [2]	-
Medium SES	2, 6, 8, 9 [4]	1, 3, 5, 7, 10 [5]	4 [1]
Low SES	1, 2, 3, 6, 8, 9 [6]	4, 5, 7, 10 [4]	-
Student SES			
High SES	1, 2, 3, 5, 6, 7, 8, 9, [8]	4, 10 [2]	-
Medium SES	1, 2, 6, 8, 9 [5]	3, 5, 7, 10 [4]	4 [1]
Low SES	1, 2, 6, 8, 9 [5]	3, 5, 7, 10 [4]	4 [1]
Gender			
Female	2, 4, 5, 6, 8, 9 [6]	1, 3, 7 [3]	10 [1]
Male	1, 2, 3, 6, 8, 9 [6]	7, 10 [2]	4, 5 [2]
School location			
Metropolitan	1, 2, 3, 5, 6, 7, 8, 9 [8]	4, 10 [2]	-
Rural	2, 6, 8, 9 [4]	1, 3, 5, 7, 10 [5]	4 [1]
Ethnicity: (Non-) English speaking background			
NESB	1, 2, 6, 8, 9 [5]	3, 4, 5, 7, 10 [5]	-
ESB	1, 2, 6, 8, 9 [5]	3, 5, 7, 10 [4]	4 [1]
Ethnicity: Aboriginality			
ATSI	1, 6 [2]	2, 3, 4, 5, 7, 8, 9, 10 [8]	-
Non-ATSI	1, 2, 3, 6, 8, 9 [6]	5, 7, 10 [3]	4 [1]

The data in Table 3 reveal that:

1. Students in every equity category agreed with the traditional gender stereotype that 'boys tease kids who are good at using computers for their mathematics work' (Item 6)
2. Except ATSI students (conclusions drawn are very tentative due to the very small sample size), students from every other equity category held two traditional gender stereotypical views about boys:
   - Boys like to take control of the computer when students work together in mathematics classes (Item 8); and
   - Boys distract others as they work on computers in mathematics classes (Item 9) and one traditional gender stereotypical view about girls:
   - Mathematics teachers give girls more help when using computers in class (Item 2).

3. Except females, students from all sectors, backgrounds and locations believed that parents think it is equally important for girls and boys to use computers for learning mathematics (Item 10); females believed that parents think it is more important for girls.
4. Students at Independent schools held the most traditional gender stereotypical views (9 items), closely followed by students attending schools located in high SES areas (8 items), students form high socio-economic status backgrounds (8 items) and students attending metropolitan schools (8 items). The similarity in the pattern of items on which these groups reflected traditionally gender stereotypical beliefs is not inconsistent with the close relationships between these factors.
5. ATSI students appear to hold the fewest traditional gender stereotypical views (2 items), followed by students attending Catholic schools (4 items), schools in medium-level SES locations (4 items), and rural schools (4 items).
6. Of all of the equity categories, the patterns of beliefs of students from NESB and ESB backgrounds were the only ones that were virtually identical. Student ethnicity (whether of ESB or NESB) would therefore not appear to be a major factor associated with gendered beliefs about computers for learning mathematics.

Table 4
Statistically Significant Differences in Mean Scores by Equity Categories

Equity group	Statistically significant differences
School type	7 Items 1, 2, 3, 4, 5, 6 & 7
School SES location	7 items: 1, 2, 3, 4, 5, 6 & 7
Student SES	5 items: 2, 4, 5, 6 & 7
Gender	8 items; 1, 2, 3, 4, 5, 6, 9 & 10
School location	6 items: 1, 3, 5, 7, 8 & 9
Ethnicity: (Non-) English speaking background	2 items: 4 & 6
Ethnicity: Aboriginality	2 items: 1 & 9

NB. The large disparity in sample sizes may account for the apparent inconsistency in the number of items with different response directions and the number with statistically significant differences in mean scores.

Within each of the equity categories, *t*-tests (for two groups) or univariate ANOVAs (for three groups) were conducted to examine for statistically significant differences at the .05 level in the mean scores for each of the 10 *Who & computers for mathematics* items. The results are summarised in Table 4. Space constraints do not allow for the recording of the mean scores nor for a discussion of the post-hoc tests conducted to understand better the statistically significant differences found as a consequence of the ANOVAs.

It would appear that within each equity category, group members differ considerably in their gendered views about the use of computers for learning mathematics. Gender was the equity category with the largest number of items (8) with statistically significant differences in mean scores. The groups within each of the two categories of ethnicity (ESB/NESB and ATSI/non-ATSI) appeared to hold the most similar beliefs (only 2 items with statistically significant differences in mean scores).

Using the information on Tables 2 and 4 together, some of the statistically significant differences in mean scores are self-evident. For example, for Item 5 (do not like using computers for doing mathematics) by school type, it can be seen that students in the three school types responded quite differently: Government (ND); Catholic (OPP, opposite to traditional stereotype); and Independent (traditional stereotype). In many cases the statistically significant differences in mean scores were due to differences in the extent to which views were held. For example, there was a statistically significant difference in mean scores for Item 2 by gender. Although both males and females believed that girls received more help from the mathematics teacher when using computers in class, males believed this more strongly: mean score (M) = 3.26; mean score (F) = 3.06.

Conclusions

With respect to views on computer use for mathematics learning, there were six items on which the entire sample's beliefs were consistent with the traditional gender stereotype (Items 1, 2, 3, 6, 8 and 9 – see Table 2). When examined by equity groupings, there were fairly consistent traditionally gender stereotyped views associated with four of these six items (Items 2, 6, 8 and 9). For Item 6, all groups agreed with the traditional gender stereotype, and all groups except ATSI students responded in the direction of the traditional gender stereotype on Items 2, 8 and 9. Thus, it was generally believed that boys:

- tease kids who are good at using computers for their mathematics work (Item 6);
- like to take control of the computer when students work together in mathematics classes (Item 8); and
- distract others as they work on computers in mathematics classes (Item 9)

and that girls:

- receive more help from the mathematics teacher when using computers for mathematics (Item 2)

For the remaining six items (Items 1, 3, 4, 5, 7, 10), views varied across the equity groupings.

Interestingly, it was students from higher SES backgrounds, those attending schools in high SES locations, and those enrolled in Independent schools who held the most traditional gender-stereotyped beliefs about using computers for mathematics learning. Paradoxically, these are the students most likely to have wide access to computers at school and at home and who, historically, are disproportionately represented among the highest achievers in mathematics (see Teese et al., 1996).

At the other extreme were ATSI students, considered one of the most disadvantaged groups within Australia, who held the least stereotyped views about using computers for learning mathematics. As the ATSI students constituted only a very small sample (42 students), any conclusions drawn must necessarily be very tentative. It was therefore of interest to look at the groups that held the next least traditionally stereotyped views: students attending Catholic schools, schools in medium-level SES locations, and rural schools. More research is needed to understand why these groups, as well as ATSI students, are less stereotyped in their beliefs about computer use for mathematics learning. Could it be that students in rural areas do not consider it likely that their career paths will be dependent on using computers for mathematically-related activities, that is, the outcomes of using computers for mathematics learning are irrelevant to them? In non-metropolitan areas, are there fewer career opportunities in mathematics-based, computer-related professions and trades than in the big cities? Why is it that students attending Catholic schools hold less traditionally gender-stereotyped beliefs than their peers attending Government and Independent schools? Similarly, why are the views of students attending schools in medium level SES locations different from their peers who attend schools in high and low SES areas?

While clearly indicating that beliefs about the gender stereotyping of computer use for mathematics learning differ across and within the various equity groups examined, the findings reported here give rise to many more questions. Answers are needed if the envisaged enhancements in students' mathematics learning as a consequence of using computer technology are to be achieved equitably.

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An Investigation of Disjuncture between Graphing in School Mathematics and School Physics

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This paper reports an inquiry into approaches to graphing required in Physics and Applicable Mathematics tertiary entrance examinations (TEE), in Western Australia. The focus in the paper is on the different graphing practices in Physics and Applicable Mathematics in relation to gradient and transformed data, and in regard to use of graphics calculators. The differences mediate against transfer between the subjects and are one explanation for students' poor performance in graphing in the Physics TEE.

Introduction

> Candidates have always encountered difficulty with graphical questions (Examiners' report, 1995). Candidates have always had problems with graphical questions, and it was not any different this year (Examiners' report, 1998).

Examiners' comments like the above are common in relation to Physics Tertiary Entrance Examinations (TEE) in Western Australia and they prompted me to investigate if there was disjuncture between graphing practices in mathematics at the secondary level and those called on in the Physics TEE. By graphing practices, I mean actions associated with the production and interpretation of graphs. This paper reports selected findings of the inquiry, which is ongoing.

Assumptions of situated cognition theory underpin the analysis. Literature that I drew on included Cobb and Bowers (1999) critical comparison of various situated views and Wenger's (1998) formulation of learning within communities of practice. Literature on graphing as social practice (e.g., Roth & McGinn, 1997; Roth & Bowen, 2001) also informed the analysis.

Inquiry showed there is high call on gradient in solving Physics TEE questions. Gradient as 'continuous rate of change' and 'ratio' need to be brought to bear, where regularly (a) gradient is defined algebraically in terms of multiple parameters, (b) graphs involve transformed data and (c) values for calculating gradient need to be read off graphs. In secondary-school mathematics in Western Australia, the use of multiple parameters is uncommon and students have limited exposure to transforming data. Use of gradient, multiple parameters and the transformation of data are the foci of the paper.

The issue of students' apparent failure with graphing in the Physics TEE is significant, for graphical questions frequently attract substantial mark allocation. For instance in 2001, 12.5% of the total marks in the examination were allocated to drawing graphs or interpreting them. The investigation of graphing is relevant also to jurisdictions beyond Western Australia. Calls on slope are common in university entrance physics examinations, for instance, in the Victorian Certificate of Education, Advanced Placement in the United States and A-Levels in the United Kingdom; and data transformation appears in the A-Level examinations.

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Interpretative Framework

Definition of the Inquiry in Terms of Situated Views of Learning

From a situated viewpoint, inscriptions, including graphs, function as *boundary objects* (Roth & McGinn, 1998) in the discourses of people from different communities. As such, they can mediate *translation* of talk in mathematics classrooms and talk in physics classrooms. In discussing a graph in physics, students might realise that approaches to graphing in mathematics are relevant, and vice-versa. *Transfer* is another construct which is aligned with the translation. From a situated viewpoint, transfer requires that the social practices (in a broad sense) which are called upon in a problem situation be recognised from other task settings (Cobb & Bowers, 1999).

However, students do not necessarily translate or transfer practices between subjects. Furthermore, my initial premises in the investigation were that graphs are treated in different ways in school mathematics and school physics so that possibilities for translation and transfer are exceeded, and sometimes the different ways are inconsistent so that it is inevitable many students wallow in a sea of confusion.

Research Method

Cobb and Bowers (1999) distinguish that, from a situated viewpoint, the unit of analysis can be the individual in the social setting or group practices. The focus in the inquiry reported here is graphing practices that students, as a group, needed to bring to bear in the Physics TEE. I discerned these practices from working through the physics examinations for 1989-2001 and by comparing my solutions against the examiners' solutions. My over-riding task was to identify how changes in the TEE Physics syllabus in 1994 were reflected in the examinations, according to a range of question attributes. The analysis below relates to the 1994-2001 examinations and to graphing.

Thus, the analysis does not address approaches to graphing enacted by students but approaches that would have yielded them successful solutions. Furthermore, a key aspect of the inquiry was how the graphing approaches for the Physics TEE varied from those taught in mathematics and I founded the comparison on my recent twelve years experience of teaching upper-secondary mathematics in Western Australia.

Situated Perspectives on the Nature of Learning and Graphing

Wenger (1998) theorises learning and acting within *communities of practice* and highlights two key aspects: *participation* and *reification*. "Participation refers to a process of taking part" and "the term reification means 'making into a thing'" (p. 55-56). Students participate in graphing processes in an examination and some aspects of these processes are reified in the relationships that they recognise and the graphs they produce. Or, their processes of interpreting a given graph are, at least to some extent, reified in the explanations and numerical values, etc., which they write down. Moreover, participation produces reification and reification influences future participation.

Roth and McGinn (1997) provide alternative views on participation and the artefacts which result. In particular, they identify graphing practices and characterise graphs. They cast graphs as (a) semiotic objects, where the object and the event it re-presents are constructed by the observer and this involves adjustment and readjustment; or graphs might be (b) objects that don't signify events, for instance a plot of points with given coordinates. Graphs can also be (c) rhetorical devices, where they are used to present an argument, point to a relationship, support a claim. So, students might choose to provide a graph in an examination in support of an answer or as an answer, even if a graph is not required. Last graphs can be (d) conscription devices and enlist students in activity.

Citing Leinhardt, Zaslavsky and Stein, Roth and McGinn (1997) put forward the view that physical situations and algebraic rules are afforded different emphases, in relation to graphs, in mathematics and science. "Mathematics educators are, depending on the curricular topic, interested in algebraic rules, graphs, and the movement within and between these spaces. Most science educators, however, are more interested in the relationship between graphs and situations and . . . the relation between algebraic rules (such as motion equations, optical equations) and situations" (pp. 92-93).

Bowen and Roth (1998) further explore graphing in science and identify reasons for students' experiencing difficulty in learning to graph and with interpreting graphs, including lack of experience in a field and lack of specificity in teachers' narratives about graphs. The need for specificity arises because of students' lack of experience and, at least in science, "interpretations of a graph lie not in understanding the representation itself as a static object but rather in understanding the social actions through which a graph was originally constructed" (p. 86).

Roth and McGinn (1998), citing Latour, identify other attributes of graphs and inscriptions, in general. They can be moved, copied, rescaled, combined, translated (into other inscriptions) and incorporated in different contexts. Further, Roth and Bowen (2001) point to two major difficulties that students experience with graphs: "slope/height confusions and iconic interpretation" (p. 161). They give the instance of students' judging relative speed of two objects on the basis of the height on distance-time graphs instead of on the basis of the gradients of the graphs. They identify also that (a) because graphs have arbitrary relations to the things they represent, students need to know the conventions of graphing; and (b) situations which graphs represent are inherently under-determined by the information that is available on the graph.

Other Situated Views and Their Relevance to the Inquiry

Nespor (1994) articulates a 'big picture' of networks of communities of practice and describes how people join communities, are mobilised along trajectories of practice within them, then move to other parts of the network. The context of the analysis was the passage of physics students through an undergraduate course during which they assumed the routines of physicists. Parallels can be drawn between the circumstances of the undergraduates and the Physics TEE students. The TEE is a rite of passage to university. Students must comply with the graphing practices expected by the examiners. So, views of the 'university community of physicists' as to what are appropriate practices in graphing underpin the practices called on in the Physics TEE: the examination panel comprises two university representatives and one teacher. Furthermore, past examination papers and examiners' solutions provide the best guide to what might be required, for the syllabus makes only brief mention of graphing. It specifies: "students should be able to . . . present and interpret experimental data in graphical and tabular form" (Curriculum Council, 2000, p. 125); "practical work is intended to . . . promote the development of practical skills, including . . . recording, analysis and graphing of data" (p. 128). As well, four types of graphs are listed: displacement/time and displacement/distance (or position) for progressive waves; displacement-distance graphs to illustrate standing (stationary) waves; and the stress/strain curves for typical brittle and ductile materials.

Another facet of students' graphing is the technologies that they have at their disposal and these, too, originate in sources outside school physics. Technologies, including tools, change how people go about an activity (Wenger, 1998). The tools Physics TEE students have which structure their graphing include, since 1998, non-CAS (computer algebra system) graphics calculator. Examiners' responses to the use of the calculators in the Physics TEE are quoted in the analysis.

Analysis

A question taken from the year 2001 Physics TEE (Curriculum Council, 2001) illustrates the call on gradient and the use of transformed data:

> Geraldine was investigating the speed of waves along stretch strings. She generated these waves by plucking a 0.760 m length of guitar string. She knew the speed was given by
>
> $$v = \sqrt{\frac{T}{\mu}}$$
>
> where T is the tension in the string and μ is the mass per unit length. She plotted her results in the graph as shown [see Figure 1].

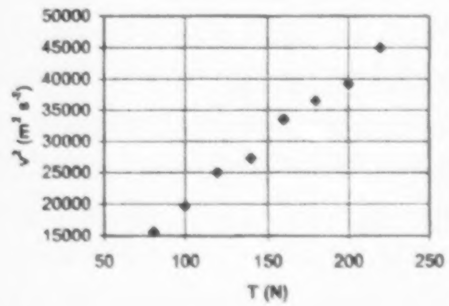

Figure 1. Graph included in the examination question.

> (a) (i) Why did Geraldine plot v^2 against T and not just v against T?
> (ii) What are the units of μ .
>
> (b) Use the graph to determine the best experimental value of μ for this string. Show your working clearly.
>
> Parts (c) and (d) followed. They did not require use of the graph so are not included.

The official solutions in the examiners' report showed the following:

> (a) (i) This makes the graph a straight line. (ii) μ = ... the units of μ are kg m^{-1}
>
> (b) Slope = (45000 – 15000) ÷ (210 – 75)
> Since $v^2 = (1/\mu)T$, slope = $1/\mu$ = 3000/135, from which $\mu = 4.50$ g m^{-1}. [3000 is in error and should be 30000, giving $\mu = 4.50 \times 10$ g m^{-1}]

In each of the years 1994-2001 there has been a question in the Physics TEE of this type--asking for calculations based on a graph of transformed data. Topics and specification vary. For instance, besides the text of the question, in 1994 the equation $s = X + \{\tau / 2rm\} t^2$ was given, data for *s* (distance) and *t* (time) were tabulated and the parameters τ , *r* and *m* were defined. Students were asked to use the data to draw a straight line, calculate τ from the gradient, and calculate X, the y intercept. In 1995 the topic was sound waves with frequency f in a tube length l, the length to produce a standing wave. The equation $l = (v/4)(1/f) - e$ was given and the parameters v and e were defined. Students were to transform l, f data given in a table, in order to plot a straight line; then, determine the parameters v and e using the gradient and y intercept.

Transformations Required and Comparison with Transformations in Mathematics

The examples illustrate that in some questions data was already transformed and plotted (the 2001 question), and in other questions students needed to decide what transformation was required (see the 1994 and 1995 questions). Two types of transformation have been requested in the period 1994-2001: squaring in 1994, 1996, 1997 and 2001, and taking the reciprocal in 1995 and 1999.

In Applicable Mathematics (in Western Australia), which the majority of Physics TEE students study, students are expected to be able to transform data in exponential relationship into log-linear form and to plot and interpret log-linear graphs (Curriculum Council, 2000). As well, they would encounter adding a constant to co-ordinate values and multiplying co-ordinate values by a constant as an introduction to the effects of 'change of origin' and 'change of scale' on summary statistics. Transformations of function equations, $y = f(x)$ to $y = 2f(x)$ etc., and function graphs are included, but the processes bear little resemblance to those required in the above physics questions. They constitute movement within and between the algebraic and graphical spaces that Roth and McGinn (1997) identify, whereas in the above examples movement was between the physical situation and numerical, graphical and algebraic objectification of it.

So, in regard to the transformation of data in the above type of physics questions, there is minimal overlap with transformations in Applicable Mathematics. There was opportunity for overlap until 1993 with the log-linear transformation of exponential relationships, which relate to nuclear decay in physics. But questions about radioactive decay, at least from 1989-1993, relied on raw, untransformed data. Radioactive decay is not now in the Physics TEE course.

Line of Best Fit

Instructions and the official solutions for the Physics TEE indicate that students are expected to hand draw lines of best fit. As well, fitting the line by eye seems expected because marks are not allocated for specific points on the line and because 'latitude' is allowed in values students read from the graph (e.g., Examiners' report, 1999).

In Applicable Mathematics, lines of best fit or least squares regression lines might be introduced through hand methods and mean values are used to position the line. But, in my experience, from 1994-1997 determination of linear regression lines would be passed to a scientific calculator and, since 1998, to a graphics calculator. Both types of calculator produce the gradient and y intercept and, if a hand drawn graph is needed, it is drawn using these values or by retrieving points on the line from the calculators.

Moreover, students are expected to use their calculators in the Applicable Mathematics TEE to calculate the line before plotting it by hand (e.g., in 1997). Sometimes the graphical form of the line is not required and only the algebraic form is called for (e.g., in 1998). The graph produced on a graphics calculator can be relied on, for example, to identify outliers.

Students are expected, as well, to use their calculators for exponential regression, with raw (untransformed) data (e.g., in 1999).

Only one examiners' report in physics for the period 1994-2001 has acknowledged students could use their calculators for lines of best fit or, more particularly, use them to obtain the *x* intercept and gradient of a line after it had been drawn: "The wording of the question included an 'otherwise' to specifically permit the use of a statistical calculator to obtain both the intercept and gradient of the line of regression. Unfortunately this was frequently interpreted as pick any two points to obtain the gradient, throwing away the benefits of drawing a best straight line in the first place" (Examiners' report, 1996). However, the wording "From the graph **or otherwise** find . . ." (bolding included) didn't give any strong hint to use the technology; and students might not have recognised to use their calculators because the format of previous examination questions didn't invite the use. In another instance, an examiner's comment suggested he/she didn't realise the technology might be used for gradient: "Many did not show how they found the gradient in part b) but obtained the answer anyway" (Examiners' report, 2000).

So, it is not necessarily the case that the presence of a technology alters significantly how people go about an activity. Moreover, changes in assessment are fundamental to how students adopt new technologies in a subject (Berger, 1998). Following this line of argument, because the responses to the use of calculators in mathematics and physics have differed in tertiary entrance examinations, it is likely that classroom graphing practices are different, at least for the line of best fit. Also, the terminology 'line of best fit' predominates in the Physics TEE and 'least squares regression line' predominates in the Applicable Mathematics TEE; and the object associated with the 'line' in physics is likely to be a graph, while the reification of 'line' in mathematics is frequently an algebraic equation. All these factors contribute to disjuncture between the two subjects, to the possible detriment of learning, for connections between subject domains are known to contribute to learning being meaningful (Hiebert & Carpenter, 1992).

Furthermore, data and graphs are commonly transformed by scientists in the field (e.g., Roth, Hawryshyn & Haimberger, 2001) and this is only practical via the use of computer-technologies. On the other hand, some dispute that student understanding of graphs and graphing is enhanced by the use of computers (Berg & Smith, 1994).

Gradient

Having hand-drawn the line of best fit, solutions in the official TEE reports indicate that physics students are to read the co-ordinates of two points on the line and calculate gradient using rise/run. This approach is also widely used in mathematics, although with data analysis at upper-secondary level the gradient is commonly retrieved from a calculator, possibly leading to deskilling in regards the hand method. The *y* intercept is also retrieved from the calculator. In addition, the graphics calculator provides the equation for the line so that students don't need to formulate it.

Furthermore, calculations which follow finding the gradient differ in the two subjects. Physics TEE questions usually ask for values for parameters that make up the gradient (e.g., value of τ where the gradient was $\tau/(2rm)$, in 1994). In Applicable Mathematics, once the gradient and intercept have been determined, typically questions ask students to predict values for the dependent variable for given values of the independent variable (e.g., predict *s* given *t*, in relation to the 1994 question); and, in the instance of log-linear equations, the task is to change them to exponential form.

Examiners' comments indicate that, in physics, deducing values of parameters from the gradient is a point at which difficulties occur. For instance: "Candidates . . . are still not comfortable with calculating quantities from the gradients of graphs" (Examiners' report, 1999). The parameters often are associated with properties of the physical setup. However, the straight-lines produced by transformed data bear minimal if any physical resemblance to the setup. In other words, they grossly under-determine it. So, in 1994, the graph of distance s against t^2 did not indicate visually that the rate of change of distance increased over time, and did not signify visually anything about τ, the torque applied to the wheels of a car, which students were asked to determine.

One purpose of manipulating artefacts of an activity is to reveal features of the concrete situation (Hershkowitz, Schwarz & Dreyfus, 2001; Roth et al., 2001) but this regularly has not been the case with lines of fit in the Physics TEE questions Instead of features on the graphs being able to be linked clearly with characteristics of a phenomenon, plotting transformed data has been a technology for producing straight lines, leading to the computation of gradient and y intercept. The abstract nature of the exercise explains students' difficulties.

Multiple parameters arguably also exacerbate the difficulties: students' mean scores were 33% for the1994 question (5 parameters): 44% in 1995 (2 parameters) and 55% in 2001 (1 parameter). Another aspect is the variables are not consistently shown in the same font (regular or italics) in the questions and neither are the parameters (e.g., v and e, in 1995), which could be another source of confusion for students. Another inconsistency is the way the y intercept is treated. In 2001 (see above) the equation of the line of best fit matching the gradient given in the solutions was $\mathrm{v}^2 = (30000/135)\mathrm{T} - 5000/3$. So, in saying the relationship was $\mathrm{v}^2 = (1/\mu)\mathrm{T}$, the y intercept '-5000/3' was ignored. In contrast, in the 1995 question the y intercept had significance in relation to the physical situation that the line of best fit was representing. How to treat the constant term is potentially another source of confusion for students.

Concluding Discussion

In regard to graphing in the Physics TEE, the issues of transformation of data, line of best fit and gradient have been touched upon. The practice of transforming data is encountered in Applicable Mathematics, but there has been no intersection between the specific transformations that are included in the syllabus for that subject and the ones encountered in the Physics TEE for the period 1994-2001. Moreover, the terminology 'line of best', the processes of the production of the line, its reified form and actions upon its gradient have been identified as differing from those in Applicable Mathematics.

The differences limit the scope for transfer of graphing practices from mathematics into solving physics problems of the type that have been discussed. One reason for the differences is utilisation of scientific calculators then graphics calculators in Applicable Mathematics and minimal recognition of them for fitting lines by examiners of the Physics TEE. So, the graph, rather than being a boundary object and site for translation of practices between school mathematics and school physics, stands as an object marking disjuncture between the subjects.

The analysis highlights that inclusion of technologies can compound lack of connection between subjects and might burden students in that they are called on to produce hand methods in one subject and technology-based methods in another. An implication is the need for dialogue between communities in the network of communities participating in education, in particular, in relation to practices expected of students as they attempt the rite of passage from one community to another and when technologies are introduced.

There is not room in this short paper to discuss to any great extent other aspects of graphing in the Physics TEE. However, there are domains of overlap with graphing in mathematics. For instance, in the year 2000 Physics TEE students were asked to estimate acceleration and distance from a velocity-time graph. Most Physics TEE students would have encountered drawing a tangent to the curve and estimating area under it in Year 11 Introductory Calculus. The mean score for the population was 58%. So, it is not that students will be able to graph in physics if they can transfer practices from mathematics, but that transfer between subjects is desirable, and the significance of the inquiry in this paper is identification of an area where more overlap is possible.

Acknowledgment

I thank the Curriculum Council of Western Australia for their permission to copy the 2001 question from the Physics TEE. Copyright for the question belongs to the Curriculum Council.

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The Sherpa-Student Role with a Graphics Calculator: Empowering or Disempowering?

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This paper explores student empowerment in the context of graphics calculator usage. The setting is revision and a student has her calculator attached to an overhead projection panel, hence the Sherpa-student role. She is working according to the teacher's instruction. Problematically, she lags the teacher's instruction and another student preempts her answers. In addition, the teacher uses tripartite questioning which can mediate against student empowerment, but a mitigating factor is students disrupt the questioning repeatedly. Use of the overhead panel and the conditions that worked towards and against student empowerment are discussed in the paper.

Introduction

A Sherpa is "a member of a Tibetan people . . . noted for their skill in mountaineering" or, alternatively, "a (mountain) guide or porter" (Shorter Oxford Dictionary). This paper explores the role of a Sherpa-student, which is the term Guin and Trouche (1999) and Drivjers (2000) use for a student who has her graphics calculator attached to an overhead projection panel so that her calculations are available for all to see. Albeit, Sherpas are usually male, but the inquiry in this instance took place in an all-girls' Calculus class, hence the feminine pronoun. Issues are whether a Sherpa-student is likely to demonstrate that she is skilled and guides others through the terrain; or whether she is guided and finds her role a burden. Conditions for positive and negative outcomes are identified in a classroom episode and cast in terms of empowerment and disempowerment of the Sherpa-student and of students in the class as a whole.

The inquiry is part of a classroom-based study over twenty-one lessons. The Sherpa-student role was enacted regularly, at the teacher's request. Moreover, in one instance a student took the initiative and asked for her calculator to be linked to the over-head panel. Other example of practices in the class are reported in Forster and Taylor (2001) and Forster, Taylor and Davis (in press).

The main epistemic referent for the inquiry was ethnomethodology which is founded on the assumption that "the world we live in is established by the mutually related acting of the members of society" (Jungwirth, 1996, p. 5). People constitute and are constituted by the practices of the groups to which they belong. The analysis was informed also by literature on classroom interaction (e.g., Mehan, 1979), use of graphics calculators (e.g., Goos, Galbraith, Renshaw & Geiger, 2000) and student identity and empowerment (e.g., Klein, 1999).

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The Literature: Empowerment and Graphics Calculators

Perusal of research publications and conference proceedings shows that increased attention has been given to social aspects of mathematics learning over the last ten years and some studies have specifically addressed the notion of empowerment or, more particularly, disempowerment of students (e.g., Klein, 1999; Zevenbergen, 2001). One view of empowerment is that students come to "know themselves as competent and numerate persons" (Klein, 1999, p. 310). Furthermore, "numeracy is not a gift but a social practice always in process; it is contextual and always constituted by, and constitutive of learners" (ibid). So, student empowerment resides in how students perceive themselves and how others perceive them. It depends on assuming authority and being accorded authority by others. It is one aspect of students' identity.

Empowerment also develops in relation to cultural artefacts, including the graphics calculator. Goos et al. (2000) suggest students interact with the technology as though it is a *master* (the student vests authority in the calculator, without question); *servant* (the student monitors the outputs of the calculator for reasonableness); *partner* (the student balances the authority of mathematics with the authority of calculator outputs) and *extension of self* (where students incorporate technological expertise as an integral part of their mathematical repertoire--the partnership between student and calculator merges to a single identity). In brief, students' empowerment with the tool, in this analysis, depends on their technical expertise and the extent to which they exert critical attitudes to the outputs.

Goos et al. (1999) identify also that students' attitudes towards graphics calculators and practices with them emerge through the role-model the teacher provides. Nevertheless, a teacher making explicit the limitations of the technology and encouraging students to be critical does not guarantee the same critical and questioning attitudes in students (Boers & Jones, 1994). Non-critical attitudes to calculator results, per se, seem to be the norm (e.g., Zbiek, 1998).

Classroom-based studies elucidate, as well, that instructors can contribute directly to students' feelings of disempowerment in a curriculum where graphics calculators are included. For instance, Povey and Ransom (2000) report that there was widespread feeling amongst students against technology-first approaches in undergraduate courses which emphasised learning with computer technologies, including graphics calculators. Students preferred to do mathematics first by hand, in order to understand the mathematics and to understand what the technology was doing. In *blackbox use*, where students do not have the background to understand calculator outputs, students can feel out of control (i.e., powerless) and, as well, experience frustration and discomfort, and become resistant to using the technology (Drijvers, 2000; Povey & Ransom, 2000).

On the other hand, the introduction of graphics calculators to the curriculum commonly heralds more student inquiry and less expository teaching (e.g., Farrell, 1996). That is, introduction of the calculators can result in students' exerting more self-determination and empowerment in regard to their mathematics, provided suitable activities are provided to support them (excluding, e.g., blackbox calculator use).

The inquiry reported in this paper is about student empowerment/disempowerment as constituted in whole-class work, in relation to use of a graphics calculator attached to an overhead projection panel. Goos et al. (2000) identify student empowerment when *teachers* use the panel to demonstrate calculator operations and problem solutions, examine alternatives, and enlist student participation; and when *students* use the panel to demonstrate calculator operations, present findings and share partial solutions.

In the instance presented in this paper, the context was revision. A student was operating her calculator with it linked to the projection panel while sitting in her normal seat, which was a normal practice in the class, made possible by a long lead. The student worked in tandem with the teacher's instruction. Drivjers (2000) describes a similar practice, where a pair of students in their seats used a calculator attached to an overhead panel for an entire lesson. The setup was put in place for research purposes and different student-pairs were selected each lesson. A "side-effect of this was that other students could also see what 'today's victims' were doing" (p. 196). We wonder who/what were they victims of? This issue is a thread in our inquiry in the revision setting.

Research Methodology

The main epistemic referent for the inquiry of which this paper is part was ethnomethodology (e.g., Jungwirth, 1996). Thus, patterns of interaction in the Year 12 Calculus class were sought, as evidenced in the twenty-one consecutive lessons during which the first author was an observer-participant. She observed whole-class work and, as well, set a video-recorder to record continuously and included the display from the overhead panel in the field of view. During seatwork she acted as an assistant teacher, which gave her opportunities to observe and note students' individual use of their calculators. Other data were collected but are not relevant to this paper.

The epistemic status of the analysis is that it was conducted by the first author with critical advice by the second author and was checked by the third author, the teacher, to establish that he did not object to it. The episode presented in the paper is anomalous because the Sherpa-student did not appear competent, whereas having their calculators linked to the panel was usually a means through which students displayed competence (e.g., Forster, Taylor & Davis, in press). The episode is reported on pragmatic grounds, to elucidate the problems that can occur in the Sherpa-student role.

Importantly, and consistent with the assumptions of ethnomethodology, it is assumed (a) the Sherpa-student determined the classroom action, as did all students and the teacher, (b) participants were not necessarily aware of the nature of their actions and (c) the research method did not require that participants personal responses be ascertained. Ethnomethodology is an interpretative methodology and the research account is known to be personal those who write it (Roth, 1998).

A Classroom Episode

The teacher, Mr D, is leading a review of compound interest, prior to introducing continuous growth relationships. He asks for the amount in a bank account at the end of the first year if $100 is invested at an interest rate of 100% per annum, with yearly compounding. A student provides the answer ($200). Mr D writes it on the board.

Next, he restates the question with six monthly compounding and hands Emily the lead to the calculator panel. Tanya gives the answer ($225). Mr D asks Tanya to explain her method and writes it on the board. The questioning and writing up continue for quarterly compounding (see Figure 1).

p.a	Amt = 200
6 mth	100×1.5^2
3 mth	100×1.25^4

Figure 1. Written solution on the whiteboard

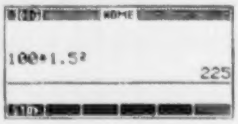
Mr D turns to Emily and asks:

1. Mr D How much do we get there Emily when we do it every six months?

Emily starts the calculation and Tanya responds:

2. Tanya 225.

Mr D writes up the 225 and the screen display from Emily's calculator appears on the whiteboard, to the left of the written solution (see Figure 2).



```
100*1.5²
                 225
```

p.a	Amt = 200
6 mth	$100 \times 1.5^2 = 225$
3 mth	100×1.25^4

Figure 2. Screen display and written solution to the problem.

3. Mr D And what about when we do it by 1.25 for every quarter?

4. Emily $244.14. [The screen display relayed from her calculator changes on the board]

5. Mr D $244.14 [writing it up]. So what is starting to happen here?

6. St It's getting bigger.

7. Mr D It's getting bigger.

Mr D asks for the amount with monthly compounding. He nominates a student to answer and after prompting she gives the expression $100(1+1/12)^{12}$. Mr D asks:

8. Mr D So, how are we going there Emily [looking to the overhead display]?

9. Tanya It's 261.30.

10. Mr D Yes [as Emily finishes]. Emily has it there. 261.30 [writing it as he speaks]. So it has increased again. Is this always going to keep getting bigger? If I go to a really generous bank, will they have to pay out an infinite amount of money to me?

11. Chorus Yes [gleefully].

12. Mr D Okay. Let's go to the next bank and say they calculate the interest daily, and this is how most banks tend to operate. So, how will I calculate it this time, Alex?

13. Alex 100 x 366/365.

14. St What?

15. Mr D Ohh. Hang on. So, here it was 100 [%], so, we divided by twelve. And then it's?

16. Alex 1+1/365.

Widespread chatter erupts.

17. Mr D Yes. So, you are one step ahead of me. So, 1/365 [writing $100(1+1/365)^{365}$ on the board]. Okay. So you are correct in what you are saying there. Now, because we are going from 12 times to 365 wouldn't we get a huge big jump because we are doing it a lot more times?

18. St But the fraction inside is getting smaller.

19. Mr D Okay, but what do we get [looking at the overhead display]? 271 point

20. St 46 [reading it off the display].

21. Mr D 271.46. So, what is starting to happen here? Why isn't it starting to get proportionally bigger here? It is approaching?

22. St An asymptote.

23. Mr D Better than that, say? [pause] It's approaching a limiting value. Anyone know what the limit is? [pause]

Students talk to each other: "e divided by 100", "100 times e", "e times 100" . . .

24. Mr D Okay. Let's do it by the minute. By the minute we do . . . [writing and speaking $100(1+\frac{1}{365\times 24\times 60})^{365\times 24\times 60}$]. What do we get when we do that?

25. Tanya 271.83. [Emily is keying in the expression to her calculator]

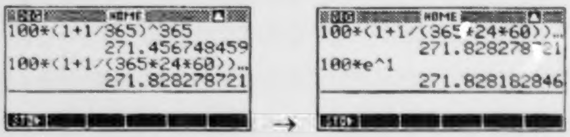
Mr D looks at the overhead display, waits for it to change and writes 271.83 on the board.

26. Mr D Well, it certainly is approaching a limit. Let's say we go to a super generous bank and say they are going to compound your money all the time. [students laugh] . . . What will the amount be?

27. St 100 times *e*.

28. Mr D Yes, 100 times *e*.

He discusses the definition of *e* and the use of it for continuous growth situations; and the screen display changes, see Figure 3.



```
100*(1+1/365)^365
            271.456748459
100*(1+1/(365*24*60))…
            271.828278721
```

```
100*(1+1/(365*24*60))…
            271.828278721
100*e^1
            271.828182846
```

Figure 3. Compounding by the day and minute, changing to show the result for 100*e*.

29. Mr D . . . Yes Anna.

30. Anna The one above is bigger

31. St Yes, that is what I was about to say.

32. Mr D So, the top one is slightly bigger, is it [looking at the overhead display]?

Widespread chatter erupts.

33. Mr D Oh, I know why. It's probably because

34. Tanya Oh, Mr D. It isn't bigger on mine.

Students laugh and the noise level rises.

35. Mr D No, the definition of *e* is this formula . . .

36. Tanya Well, I will do the calculation again, what I did was calculate $365 \times 24 \times 60$ first.

37 Mr D Did anyone else do that?

38. Sts Yes.

39. Mr D Well that's interesting. I will have to think about that. Can you guys think of a good reason?

So, was Emily--the Sherpa-student--a victim; and, if so, what were the circumstances of her victimisation? We consider these issues below and the social conditions amongst the class, in general, that worked towards students' empowerment/disempowerment.

Emily

Having her calculator linked to the panel gave Emily an opportunity to demonstrate her abilities and have them recognised by the class. However, generally she appeared to lag rather than lead the calculation. Lagging arose because, first, she didn't start calculating until the teacher prompted her (turn 1). She did not follow the normal interaction pattern in the class of starting soon after being given the lead. The delay in starting, therefore, could be classed as an instance of *interactional incompetence* (Mehan, 1979).

She caught up temporarily (turn 4), then, fell behind again (turn 8), or appeared to fall behind, for Emily's slowness was relative to the speed with which Tanya stated the answer (turn 9). However, the teacher did not pay attention to Tanya's response (turn 10). Ignoring or not paying attention to students who speak out of turn is a way of signaling that interruptions are unacceptable (Mehan, 1979). However, Tanya did not seem to take the signal on board for, later, she again preempted Emily's calculation (turn 25).

What can explain the difference in the two students' speed with the calculation? Maybe a barrier to Emily keeping pace with the monthly compounding was that she couldn't predict the method. It required the use of a fraction (1/12) instead of the decimals used earlier, and not recognising this could be said to indicate *academic incompetence* (Mehan, 1979), for the subject matter had been met previously. However, this analysis is perhaps too harsh, for the student who was nominated to answer stumbled in articulating the changed pattern.

Then, in regard to compounding by the minute, it became apparent (turns 36-38) that Tanya and others used their calculators more efficiently than Emily. Calculating $365 \times 24 \times 60$ first and then copying it down each time saves several key-strokes and so saves time. By keying the product twice, Emily showed *less technical expertise* than the other students.

Hence, the episode illustrates that a Sherpa-student faces interactional, academic and technical demands. The episode illustrates also that students' personal competence/ incompetence, which translate to empowerment/disempowerment, are socially determined, in relation to and in relationship with other students. Moreover, relative competence and incompetence evolve and are not established in a single short episode.

Other salient aspects of the episode are that Emily appeared calm throughout (and not as though a victim), her relative slowness benefited others in that she moderated the pace of the solution, and she finished by showing initiative and calculating $100e$, in a timely way. Furthermore, Mr D had a central role in the episode which is discussed below, where attention is given to conditions for student empowerment in general.

Mr D

In this episode, Mr D used a tripartite questioning style: teacher elicitation, student response (usually short), and teacher evaluation leading into the next question (Mehan, 1979; Young, 1992). According to the literature (Young, 1992), it is a pattern whereby students produce the answers the teacher intends upon framing the questions and typically, at least in middle-class groups, students know and play the game (Zevenbergen, 2001). The mode of interaction is consistent with tight teacher-control and was the style Mr D consistently used for revision.

Alex's response in turn 13, however, caused disjuncture in the questioning. It fitted the pattern of a *too complete description* (Jungwirth, 1996). Alex didn't keep to the pattern that was in place, $100(1+1/365)^{365}$, but instead went to a simplified version. Teachers typically unpick the condensed (or simplified) answer with the student's assistance, as Mr D did here (turn 15-17), and then reinstate control and order in the questioning. In the process, the student who answered can be left looking incompetent, as though she couldn't articulate the solution properly, and the solution can appear to be the teacher's. However, here Mr D finished by acknowledging and affirming Alex : "So, you are one step ahead of me . . . So, you are correct in what you are saying there" (turn 17). He regularly did return to give students credit and when he didn't, students sometimes insisted, on behalf of each other, that credit be given.

Anna's observation about the value for $100e$ (turn 30) disrupted the proceedings again. It is an example of the critical questioning that is required towards calculator outputs (Boers & Jones, 1994) and can be cast as a student exhibiting mastery (Goos et al., 2000) over the calculator instead of the calculator being master to the student. Furthermore, the teacher afforded the discrepancy high recognition (turn 32-33, 35, 39), which potentially drew more attention to it and fostered critical attitudes towards calculator outputs amongst the class. However, other than Tanya who mentioned the order of calculation, no other student offered an explanation, but Mr D did. The difference was due to rounding. The Hewlett Packard calculator that most students were using rounded to fewer places than Tanya's Texas calculator. The intermediate calculation '$1+1/(365\times24\times60)$' (see the full expression in turn 24) would have been rounded and when raised to the power of $365\times24\times60$ the rounding error was amplified.

In relation to Emily, Mr D's acceptance of Tanya's early response (turn 2) gave Emily time to catch up. Then, after the first interruption, Mr D's waiting, looking to the display and obtaining the answers from it (turns 10, 19, 26, & 32), were consistent with Emily having the responsibility and right to provide the results to support the class work. In other words, the teacher's actions accorded Emily the role of chief-Sherpa.

Another notable aspect of the episode is that six students, in addition to the four who have been named, contributed to the discussion: at the beginning a student questioned the compounding period and others answered in turns 6, 14, 18, 22, 27, 30, and 38. Turn 18 was insightful and turn 30 was further evidence of critical thinking, while turn 38 showed students were performing the calculations themselves and not relying on the projected display. As well, the episode was punctuated by chatter between students (after turns 16, 23 & 32). This was a regular and potentially empowering practice in the class for more students expressed their views through it than was possible during whole-class discussion. As well, the chatter fed into whole-class discussion and, at times, advanced it (e.g., turn 27).

Thus, a large majority of students in the class of 13 participated actively in the episode and their participation wasn't all low quality, which can happen with tripartite questioning. Mr D's acknowledgement of students' unexpected answers (interactively demanded by them), support for the Sherpa-student and pursuit of the source of error can be seen as enabling aspects of the revision in as much as the actions are consistent with encouraging students' active and critical participation; and, so, are potentially empowering.

Concluding Discussion

So, did Emily--in the Sherpa-role--guide others or was she guided? Generally, she appeared to lag rather than lead, partly because of being unusually slow to start and partly because she was pre-empted in calculation by another student. Near the end, though, she took the initiative and entered 100*e*, which revealed the inconsistency with the calculator processing for all students. The public display led also to the strategy 'multiply $365\times24\times60$ first' being identified, for the potential benefit of other students.

Emily's delay in starting calculation was cast in terms of interactional competence; and interactional competence, as such, was shown to be determined also by the actions of others. In addition, Emily's apparent academic competence and technical expertise were identified in relation to other students' actions. In summary, the analysis according to these three dimensions of competence portrays her as being guided more than being a guide in the episode.

Drivjer's (2000) suggestion that students in the Sherpa-role might be victims also needs consideration. Was Emily a victim? Was she subjected to *cruelty, oppression, or other harsh or unfair treatment* (Shorter Oxford Dictionary)? Emily was not unusually singled out by the teacher for a student having her calculator attached to the panel was a normal practice in the class, and the teacher supported her in various ways during her performance. She was, however, subservient to the revision agenda.

In other episodes, having their calculators linked to the overhead panel seemed a powerful and empowering means of communication to students. Empowerment more than victimisation seemed operational in the Sherpa-role in the Calculus class, particularly when Mr D moved from tripartite questioning, to discursive, open questioning (see Forster & Taylor, 2001). Aspects of the classroom practice evidenced in the episode discussed here, including critical questioning of calculator outputs, were also consistent with empowerment.

In conclusion, it can be argued that the appearance of competence /incompetence in a single episode has low significance. However, repeated instances of a student performing in public and the class having to wait could adversely effect how the student perceives her own competence. Our micro-analysis of the episode had the purpose of raising awareness of how this negative outcome might occur.

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Convergence or Divergence? Students, Maple, and Mathematics Learning

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We continue to report on research into outcomes that emerge when symbolic manipulator software forms a central component of undergraduate teaching. Preliminary data led to the identification of 7 categories that encompassed the types of student questions and blockages arising during interactive laboratory sessions. Examples illustrative of the categories are provided and implications discussed. Additionally we report and discuss the results of successive applications of test items designed to assess the impact of complexity (defined in terms of syntax, and function choice and specification), on task demand.

Over a decade ago James Fey surveyed developments in the use of technology in mathematics education to that date. In noting that there was no lack of speculative writing on the promise of the revolution that was expected to follow he drew attention to the paucity of data available to back extravagant claims.

> It is very difficult to determine the real impact of those ideas and development projects in the daily life of mathematics classrooms, and there is very little solid research evidence validating the nearly boundless optimism of technophiles in our field. (Fey, 1989)

This comment seems as relevant today as it was over a decade ago, even if the questions have become more refined. The literature confirms the existence of diverse factors that impact on the development and testing of theoretical frameworks, and on the conduct of practice. Among these sources, papers addressing the use of technology in undergraduate mathematics make for interesting and varied reading. For example:

> The impact upon educational practice of powerful software like Mathematica has been less profound than optimists hoped or pessimists feared … (there is a) tendency to begin by looking for electronic ways of doing the familiar jobs previously done by textbooks and lectures. (Ramsden, 1997).

> Of all the flaws in our mathematics training this seemed to us to be the most dangerous and insidious, for as we removed mathematics from our courses in response to 'student failings', the need for mathematics to do real science was in fact increasing … firstly there was the pious hope that a computer assisted approach would require less staff … problems arose from attempts to use Mathematica in two ways-which were incompatible. Was software an arena for exploration of mathematical ideas, or a channel for their transmission? (Templer et al, 1998)

> There is growing evidence (in the UK and elsewhere) of a general decline in the mathematics preparedness of science and engineering undergraduates … one response has been to simply reduce the mathematics content and to rely on computer-based tools to do much of the mathematical computation … difficult questions (emerge) at the intersection of cognitive and epistemological domains; to what extent must the structure of mathematics be understood in order for it (technology) to be used effectively as a tool? (Kent & Stevenson, 1999)

Such are some of the challenging and problematic issues that have emerged, and continue to challenge, undergraduate mathematics education.

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 285-292). Sydney: MERGA. ©2002 MERGA Inc.

Technology-aided Mathematics Programs

The form of computer-based instruction indicates a range of beliefs among program designers and instructors - both about mathematics, and the nature of mathematics learning. Olsen (1999: 31) discusses one of the most extensive examples of technology used to provide automated instruction. She describes how politicians visiting Virginia Tech's Mathematics Emporium, a 58 000 square foot (1.5-acre) computer classroom:

> see a model of institutional productivity; a vision of the future in which machines handle many kinds of undergraduate teaching duties-and universities pay fewer professors to lecture ... On weekdays from 9 am to midnight dozens of tutors and helpers stroll along the hexagonal pods on which the computers are located. They are trying to spot the students who are stuck on a problem and need help.

This factory model assumes that mathematics is something primarily to be delivered and consumed. By contrast Shneiderman et al (1998) describe a model, in which courses are scheduled into electronic classrooms following a competitive proposal process, and in which active engagement with an interactive, multimedia environment. is mandated.

In between the extremes exist a variety of models of instruction, concerned in varying degrees with factory production on the one hand, and student understanding and engagement on the other, and it is instructive to read comments describing the characteristics of such programs. Here are some selections:

Templer et al (1998) noted problems accompanying efforts to provide meaningful learning that were perceived to arise as a direct result of a symbolic manipulator (*Mathematica*) environment. They noted that typically having mastered the rudiments;

> the majority of students began to hurtle through the work, hell bent on finishing everything in the shortest possible time.

The following comment, or a close relative, was noted as occurring frequently among the students:

> I just don't understand what I'm learning here. I mean all I have to do is ask the machine to solve the problem and it's done. What have I learned?

Kent and Stevenson (1999) in elaborating on their concerns about student quality (see above), question whether mathematical procedures can be learned effectively without an appreciation of their place in the structure of mathematics. They argue that unless some kind of breakdown in the functionality of some concept or procedure (say integration) is provoked, students do not focus on the essential aspects of that concept or procedure. On the other hand they observe that the formal demands of a programming environment serves both to expose any fragility in understanding, and to support the building and conjecturing required in the re (construction) of concepts by learners. This comment engages a debate about whether computer technology should be employed following prior understanding of mathematical concepts and procedures (Harris, 2000), or as a means integral to the development of such understanding (Roddick, 2001). Clearly this matter is not yet resolved.

As noted previously, increasing uses of technology have been linked with perceived decreases in the prior mathematics preparation of undergraduates, and in some quarters at least, there seems to be an implicit belief in the existence of 'good old days'. In fact learning problems have always existed (see Gray (1975) for an Australian example), and their recent escalation is a dubious justification for the introduction or increase of computer-assisted learning. Ultimately the success or failure of any teaching approach resides in the quality with which students engage the learning mediums provided, and the extent to which mathematical integrity, rather than medium specific properties is the arbiter of understanding and quality. Mathematically related problems specifically associated with the introduction of screen-based learning are increasingly being documented (Templer et.al., 1998; Kent & Stevenson, 1999; Guin & Trouche, 1999; Lagrange, 1999; Doerr & Zangor, 2000). Doerr & Zangor refer to student preference for calculator output over contextual reality, whereby students insisted on working with 6 decimal places on problems whose data involved using crude measuring devices. Guin & Trouche report on the erroneous impact of graphic representation of functions. Left alone to experiment students inferred results that were wrong-for example the number of solutions to the equation tan x = x deduced from a screen display featuring six intersections. This gap between real mathematics and the image of mathematics depicted by a screen suggests the emergence of a new 'tyranny of the screen' as an authoritative source, replacing, and possibly more insidious, than the traditional 'tyranny of the text'. Lagrange identified problems associated with the interpretation of output from symbolic manipulator (Derive) software. Specifically, rather than building mathematical meaning from screen feedback as expected, students' perceptions were invaded by the properties of the software. For example, in noting the output of the 'Expand' command on the square of algebraic sums, rather than focusing on mathematical regularities such as the number of terms in the sum and in the expansion, students instead focused on the order of terms in the expansion, a regularity linked only to the software, and of no mathematical significance.

The Study Context

The first-year undergraduate mathematics course, forming the context of this study provides for a population of several hundred students studying within Science and Engineering degree courses. As presented in 1999 and 2000 the courses comprised a lecture series complemented by weekly workshops, in which approximately 40 students were timetabled into a laboratory containing networked computers equipped with *Maple* software. The lecture room was fitted with computer display facilities so *Maple* processing was an integral and continuing part of the lecture presentation. During laboratory workshops two tutors and frequently the lecturer also, were available to assist the students working on tasks structured through the provision of weekly worksheets, and to answer their questions. Unscheduled additional access to the laboratory was available for approximately 5 hours per week. Preliminary observation of students, taken together with issues raised in the literature, led to the formulation of three research questions.

1. What is the distribution, and characteristic of questions raised by students while engaged in learning mathematics within a symbolic manipulator environment?
2. What types of blockages cause progress to be stalled?
3. What factors contribute to the demand of tasks in which *Maple* and mathematics interact?

Data Sources

On the basis of a review of 1999 data (Galbraith & Pemberton, 2000) a set of categories (see Table 1) was defined to structure the 2000 data collection. For each laboratory session the course tutors recorded the range of questions asked of them by students in terms of these categories, together with examples of each type. Analysis of the collated responses enabled the first two research questions to be addressed. A second source of data was derived from a test given 7 weeks after the program started. This test, a voluntary exercise, comprised a series of questions to be addressed with the assistance of *Maple* in its laboratory context. It provided formative feedback to the students on their performance, and was relevant to preparing for their formal assessment. Questions ranged from simple school level manipulations to new material introduced in the tertiary program. The set of basic test questions (13) remained substantially the same with three of the original questions replaced in 2000. Sample questions are included in the appendix.

Complexity

In addressing question 3 we note the importance firstly of accurate syntactical representation of common elementary operations and terms, and secondly the demand imposed by the correct selection and specification of functions necessary to achieve identified mathematical ends. We seek to relate performance and hence task demand to the influence of these two factors (labelled respectively *Syntax* and *Function*). For use in analyses these descriptors were redefined with greater precision than the more generic interpretations adequate for the descriptive levels appropriate for questions 1 and 2.

Syntax: refers to the general *Maple* definitions necessary for the successful execution of commands. These include the correct use of brackets in general expressions, and common symbols representing a specific syntax different from that normally used in scripting mathematical statements (such as *, ^, Pi, g:=).

Function: refers to the selection and specification of particular functions appropriate to the task at hand. Specific internal syntax required in specifying a function is regarded as part of the *Function* component, including brackets when used for this purpose. Complexity is represented by a simple count of the individual components required in successful operation. We illustrate how these definitions work, by applying them to two of the examples given in the appendix.

Q1. *Syntax*: Incidence of ^ [4] *plus* ( ) [2]; total=6.
Function: General structural form of evalf (argument); evalf [1] *plus* ( ) [1] *plus* argument [1]; total=3.
Q11. *Syntax*: Incidence of*[2] *plus* ( ) [3]; *plus* y [1] *plus* := [1]; total=7.
Function: General structural form of plot(function, domain); sub-total [5] *plus*
domain specification [1]; General structural form of int(y, integ interval); sub-total [5] *plus*
(subtraction) [1] *plus* integration interval specifications [2]; total=14.

Similar pairs were assigned to each of the questions forming the test samples. Scoring was on a correct/incorrect basis, with the success rate on the questions defined by the fraction of students obtaining the correct answer, given that the question was attempted. This differs from the approach to the preliminary data described in an earlier paper (Galbraith & Pemberton, 2000) where the absence of an attempt was defined as failure. The success rate may be interpreted as a measure of the probability of success, of a student on respective questions. We note that for purposes of analysis the population is the set of mathematical tasks (questions) in which *Maple* is invoked as the means of solution. The student responses are used to assign the probabilities of success for the sample of questions used in the study.

Results

A summary of responses relevant to research questions 1 and 2 is contained in Table 1, showing the distribution of student questions and comments; and illustrated in the structured collection of substantive examples (Figure 1). A degree of judgement is occasionally involved in allocating questions and comments to one category rather than another, but the distribution (generated from approximately 1300 items) is believed to be a robust representation.

Table 1

Student Question Distribution (2000 data)

Question Category	Number
1. Identify problem caused by a typo (TYPO)	109 (8.4%)
2. Resolve syntax error (SYNTAX)	333 (25.7%)
3. Problem with function choice (FCHCE)	60 (4.6%)
4. Problem specifying function (FSPEC)	192 (14.8%)
5. Stuck on mathematics (MATHSTUCK)	197 (15.2%)
6. Stuck on *Maple* (MAPLESTUCK)	254 (19.6%)
7. Interpreting aspects of output (INTERPRET)	152 (11.7%)

It is significant that both mathematical issues and software issues feature prominently (Table 1), a characteristic made explicit through the examples illustrating the respective categories (Figure 1). It is clear from these examples that both mathematics and *Maple*, separately and together, feature prominently in the questions that students ask, and in the blockages that cause progress to stall.

TYPO:
Spelling, o for 0, Missing parts (= from :=; one of a bracket pair)
SYNTAX:

- *Case sensitivity e.g., pi for Pi;*
- _Missing characters e.g., symbols (* in 2*x etc); brackets (ln3 for ln(3) etc);_
- *Confusion with calculator syntax e.g., e^x for exp(x);*

FUNCTION
Function choice:

- *Confusion between alternatives; e.g., 'simplify' vs 'expand' vs 'factor', 'evalf' vs 'convert', 'fsolve' vs 'solve'; 'angle' vs 'dotprod'.*
- *Problems within a selection field; e.g., left or right or general 'limit', using 'subs' with 'function'.*

Function Specification:

- *Form; not honouring function syntax x=0...1 etc, specifying vector as a matrix, specifying sequences, using hybrid notations e.g., y (x): =x-7 then Plot (y, x=0..8) etc.*
- *Problems within functions; non-matching dimensions in matrix, evalf in function*

STUCK ON MATHS

- Basic Maths: *e.g., order of operations, difference between significant figures and decimal places, common denominators, floating point vs exact form, difference between function and expression.*
- *Advanced Maths (definitions)* e.g., meaning of continuity, meaning of lim x->0+, What is a second order derivative? What is a partial derivative? Classifying stationary points, formal definitions (e.g. continuity and limits), What are orthogonal vectors? What is an AP? What is a Taylor series?

STUCK ON MATHS (CONT)

- *Advanced Maths (procedures)* e.g., Finding partial fractions, how to solve simultaneous equations by hand, how to find a function value to ensure continuity at a point, How does a (Taylor series) expansion around a point work? What does "justify your answer mean?" Finding approximations for errors in series, calculating percentage change, setting up dot products, matrix multiplication.

STUCK ON MAPLE

- *Syntax Blockages* e.g., What does '^' mean? What is the difference between pi and Pi? How do I put a square root sign in Maple? How do I get 5 dec places using %? Mixing together functions, expressions, diff and D; Specific woes e.g. Taylor series syntax.
- *Procedural Issues* e.g., How to insert Maple input and delete previous output? How do I delete lines? Get cursor back? Write text? Print? Cut and paste? Must I use Maple if I can do it in my head?
- *Operational Blockages* e.g., Trying to use already assigned variables; How do I (in Maple) solve 2 simultaneous equations? Do definite integrals? Find first and second derivatives? Approximate Pi? Develop sequences? Do absolute value? Set up systems of equations? Find sums of terms of sequences?

INTERPRETATION

- *General issues* e.g., How do I know if the Maple answer is right? What does this result mean? Is this what I'm meant to be getting? Where did this answer come from? What does undefined mean?
- *Specific Maple issues* e.g., What did Digits:=3 do? What does the 5 in 'evalf (%, 5) do? What does 'collect' do"? What does it mean if 'solve' returns nothing? Is that output of solve (F1, x) right? What does 'zip' do? What is the 'order' term? Many questions about graphical output.

Figure 1. Student response summary (2000).

For research question 3, sample data for the year 2000 are contained in Table 2, with regression analyses for both years summarised in Tables 3 and 4. The analysis sought the impact of *Syntax* and *Function* on probability of success (*Success rate*).

Table 2
Illustrating Complexity and Success Rate (2000)

Question	1	2	3	4	5	6	7	8	9	10	11	12	13
Syntax	6	3	9	5	3	4	6	8	5	10	7	5	2
Function	3	3	3	5	6	11	6	10	5	11	14	6	6
Success rate	0.89	0.92	0.67	0.89	0.96	0.63	0.81	0.63	1	0.56	0.59	0.93	1

Table 3
Regression (2000 & 1999.)

Regression Statistics	2000	1999
Multiple R	0.904	0.884
R Square	0.817	0.782
Adjusted R Square	0.781	0.738
Standard Error	0.078	0.112
Observations	13	13
F value	22.374	7.906
Significance of F	$p<0.001$***	$p<0.001$***

Table 4
Regression Continued (2000 & 1999-in brackets)

	Coefficients	Standard Error	t Statistic	P-value
Intercept	1.202 (1.206)	0.063 (0.078)	15.424 (19.005)	<1E-7 (<1E-7)
Syntax	-0.040 (-0.050)	0.010 (0.013)	-4.108 (-3.754)	0.002** (0.004)**
Function	-0.025 (-0.028)	0.007 (0.010)	-3.705 (-2.641)	0.004** (0.025)*

We note the significant impact of the variables *Syntax* and *Function* on success rate is confirmed for both years. *Syntax* errors penalise those who lack sufficient care in expressing their work symbolically, while the demands imposed by *Function* are proportional to the principles and sophistication of the associated mathematics. Note also the substantial amount of variance accounted for (above 70%).

Reflections

Reviewing the scope and nature of symbolic manipulator supported learning, we note the overt purpose of reducing procedural load though the use of automatic processes. Thus we identify a reduction in the demand for procedural knowledge that could ostensibly lead to improved performance. However we note that achieving mastery of procedures also enables them to be encoded as conceptual knowledge, adding to the network of information available for recall and action in future situations. So is a potential gain on procedural swings lost on conceptual roundabouts? The setting up of *Maple* solutions requires not only that all relevant mathematics concepts be activated, but that *Maple* functions also be appropriately selected and accurately specified using precisely defined syntax. It might be argued then that for students with incomplete learning schemas, the conceptual load, as distinct from the procedural load, is increased in comparison with traditional written approaches? The questions and blockages identified from student workshop activity indicate that the mathematics and software are closely entwined, with problems emerging from each separately and from both in combination. In relation to the literature the students do seem to interrogate the screen more carefully than in some reports. Their responses certainly challenge beliefs in factory models of delivery that imply that computers essentially transmit knowledge for consumption and absorption by learners. By their own expressed reactions via the medium of comments and questions, the students also challenge thinking that would see technology as a means of compensating for poor or absent background knowledge. This inference is strengthened by the analysis relevant to our third question, through which we note that while achieving more rapid and efficient closure to algorithmic procedures, the use of *Maple* has not reduced the need for the mathematical attributes of understanding and attention to detail. This can be observed in the significant impact of the variables *Syntax* and *Function* on success rate. So when used as a learning tool our experience suggests that symbolic manipulator software, far from simplifying demands throws into relief learning issues that add substantially both to our understanding of student problems, and to the challenge of meeting them. On the other hand for those students who possess sound understanding and due regard for precision, the *Maple* environment has provided a means to progress rapidly and successfully at a greater rate than could otherwise be achieved. These latter use the software as a *power tool* rather than a *learning tool*. That is they are able to use its capacity to extend the boundaries of their capability in much the same way that experts use programs in pursuit of their own high level goals. Perhaps one of the immediate challenges in technology-aided learning is to distinguish between such alternative approaches, capabilities, and learning needs. This is where real individual differences may lie, rather than in the more popular assumptions associated with the freedoms provided by individual access to machines and flexible scheduling.

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Appendix

Sample Maple Test Questions

The ordered pairs following the sample questions provide measures of: (*Syntax*, *Function*).

1. *Express* $\dfrac{12^5 + 23^4}{7^8 + 5^9}$ *first as a fraction, and then as a decimal.* (6,3)

3. *F ind a simpler form of* $(x^2+1)^3 + (x^2-1)^3 - 2x^2(x^4+1)$ (9,3)

10. *If* $f(x) = (x^2+1)^{1/2} + (x^2+4)^{1/2} - x$, *for* $x \geq 0$, *find where* $f'(x) = 0$. (10,11)

11. *Plot* $f(x) = (x-1)(x-2)(x-3)$.
Hence find the physical area under the graph from $x = 1$ *to* $x = 3$. (7,14)

12. *Evaluate* $\int_0^1 \sqrt{1 + 4\sin^2 x}\; dx$ (5,6)

Matching the Hatch – Students' Choices and Preferences in Relation to Handheld Technologies and Learning Mathematics

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This paper reports on students' choices of and preferences for different hand held technologies when provide with options within a classroom that integrates technology into the teaching and learning of senior secondary school mathematics. Observations were drawn from video and audio data as well as a whole class interview of the focus group in which the video of a previous lesson was used for stimulated recall. This report notes a number of emergent categories of student preference and choice based on this data and comments on the potential impact of such choices and preferences on student learning.

Introduction

We have moved, over the past five years, to a state where the use of higher end technologies, such as graphing calculators or computers with mathematically enabled softwares, is either mandatory, or at the very least, access is assumed, for the teaching, learning and assessment of mathematics, within the post compulsory years of secondary schooling, in seven out of Australia's eight states and territories. This is evident in curriculum frameworks and study designs (for example Queensland Board of Senior Secondary School Studies 1992, 2000; Victorian Board of Studies 1999). At the same time, the developing availability of this type of technology has also brought with it, at least in some schools, a divergence in the technology available to students to engage in learning mathematical ideas and processes. While there is now a significant body of knowledge, in relation to the use of technology in the teaching, learning and assessment of mathematics, in general, there is also a limited literature base related to the issue of students' choices of technological options when learning or doing mathematics. This paper seeks to extend knowledge in this domain through the analysis of interviews that targeted students' preferences when a choice of graphics calculators, with different levels of functionality, are (IS) available. A number of classes of preference emerge from the analysis.

Literature Review

Advocates for the transformation of school mathematics classrooms into technology rich environments have argued that the use of technology in teaching/learning offers: the promise of better learning outcomes (Vonder Embse, 1992; Portafoglio, 1998; Weber, 1998); the potential to offer more varied approaches to instruction (Dance, Nelson, Jeffers & Reinthaler, 1992); and enhanced student/student and student/teacher interaction in mathematics classrooms (Burrill 1992; Geiger and Goos, 1996; Galbraith, Renshaw, Goos and Geiger, 1999). Further, Kutzler (1999) has argued that mathematically enabled technologies have the potential to scaffold student learning in such a way that gaps in prior

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learning can be managed so that they do not interfere with the acquisition of new mathematical ideas and concepts. Overall findings, however, concerning the "value adding" to students' learning through the use of technology have been somewhat inconclusive (Penglase & Arnold, 1996; Kuchler, 1998; Maldonado, 1998). Despite mixed conclusions reported in these reviews, it has been noted by Asp and McCrae (2000) that this is merely indicative of the need for further research in this area.

An area that has received little attention to date is that related to the choices students make when they have the opportunity to select from more than one type of technology, and what processes they engage when making decisions about how to best match available technologies with the mathematical task at hand. Geiger (1998) explored the issue of students making use of technology in different modes, for example, to work through computationally intense tasks quickly, to explore an unfamiliar situation or to enhance collaborative work with others. This theme was further developed by Galbraith, Goos, Renshaw and Geiger (1999), who provide a sociocultural perspective on teaching and learning mathematics in technology rich environments by identifying categories of student expertise - *Master, Servant, Partner and Extension of Self* - based on the way in which graphing calculators, or other mathematically enabled technologies were used. While these studies indicate that students are aware that they make use of available technology in different ways they do not explore the issue of making use of different technologies. Their findings are suggestive, however, of preferences to work with technology in different ways, which in turn should lead to different choices being made when provided with the opportunity to match their mathematical needs with a choice of different technologies. While our earlier theorising has drawn substantially from Vygotskian notions of the ZPD, Valsiner (1987) has described two additional zones of interaction, the Zone of Free Movement (ZFM) and Zone of Promoted Action (ZPA), that can help to account for the activities that students pursue in learning contexts. According to Valsiner (1987), the ZFM represents environmental constraints that limit freedom of action and thought, For students in secondary mathematics classrooms. elements of the ZFM might include:

1. perceived abilities that may constrain teaching actions and hence learning opportunities;
2. curriculum and assessment requirements, which influence choice of topics and teaching methods;
3. resources, in the form of teaching materials, computers or calculators, or specially equipped rooms;
4. the time available to teach required content;
5. the images that students hold about what mathematics is, and what constitutes mathematical activity.

While the ZFM suggests which learning actions are *possible*, the Zone of Promoted Action (ZPA) represents the efforts of a teacher to *promote* particular teaching skills or approaches. Our interest here is in the ways that students initiate and sustain the use of technology in a classroom where the students' ZFM is subjected to the fewest possible constraints, while the ZPA encourages a maximum use of available devices.

The Study

Classroom Context and Data Source

This study took place within an independent, coeducational secondary school in the state of Queensland. This paper seeks to describe one aspect of a larger two-year study which aim(ED) to investigate the teaching/learning practices of students and teachers in senior secondary school mathematics courses in relation to the use of technology. This paper describes preferences and behaviours that emerged when students used technology to investigate and solve a combinatorics problem which was part of a course designed for tertiary bound students with aspirations for further specialist studies in mathematics or related disciplines. The 15 students (5 female, 10 male) were in their final two years of study and were typically 16 – 17 years old and by nature of their choice of subjects generally displayed a positive disposition to the study of mathematics.

There are no restrictions on the form of educational technology that students are allowed to employ in the learning, teaching or assessment of mathematics in the school based curriculum framework within which this study is embedded. It had become a matter of policy, in the school involved in this study, that technology be integrated into student learning experiences whenever it is appropriate and possible. The type of technology utilised ranged from computers equipped with mathematically enable(D) applications through to graphics calculators. The student group that is the focus of this paper made use of two types of graphics calculator. These are listed below with a brief outline of the features available on each device:

TI-92 Calculators (Texas Instruments)

1. Graphical, numeric representations of functions
2. Statistical capabilities
3. CAS capabilities - including matrix editor and complex number manipulation
4. Dynamic geometry module

TI-83 Calculators (Texas Instruments)

5. Graphical, numeric representations of functions
6. Statistical capabilities

Matrix editor and complex number manipulation

Students in this group also displayed a positive predisposition to the use of technology (Galbraith, Renshaw, Goos and Geiger, 2001). A student statement, taken from a class survey, in relation to the use of technology in the learning of mathematics, illustrates the enthusiasm displayed by students in relation to the use of technology in the mathematics classroom

> Because my calculator has become my best friend. His name is Wilbur. Me and Wilbur go on fantastical adventures together through Maths land. I don't know what I'd do without him. I love you Wilbur.

While it might be suggested that this statement is a little tongue in cheek, there are sufficient other corroborating comments from students to confirm that the essence of the above statement was a prevalent attitude.

The data sources for this project consisted of: audio and video records of classroom observations; individual, group and whole class structured and semi-structured interviews; and questionnaires. The data on which this particular paper is based were drawn from three 45 minute lessons spaced over a two week period. These lessons were video taped, and the dialogue subsequently analysed.

Teaching Episodes and Observations

A Problem in Combinatorics

The style of teaching/learning in the focus classroom had a problem solving orientation. Topic or concept sequences generally began with a problem that was judged to be of a level of challenge just removed from what students could address procedurally. The aim of this approach was to provoke the exploration of the idea and as a consequence to stimulate discussion of the idea rather than simply provide instruction that led to content acquisition.

This paper reports on a series of episodes that were initiated when students were asked to solve the combinatorics problem below:

$$^nC_3 = \frac{5}{18}\,^nC_5 \qquad \textit{Find } n$$

Students were observed to use both types of available calculators to assist them to investigate and solve the problem but it was also noted that they took different approaches to how they made use of a chosen device. This session was video and audio taped for later use with the class.

Student Revelations

The next day students were shown video segments of the whole class working on the combinatorics problem from the previous lesson in order to stimulate students' recall about their observed behaviours during the taped session. A whole class interview was conducted to gain insight into what students believed were the reasons for these decisions. Approximately one week after this session, students undertook a course work focused assessment task. During this assessment task it was again noted that students were using different calculators and using them in different ways. A second whole class interview was conducted during the next available lesson in relation to the students' use of calculators during assessment. Four categories of calculator use emerged from the two discussion sessions.

Getting to Know You: Some students commented that they preferred to use one calculator over another simply because they used it more often (eg in other courses) and were more familiar with its operation. While this is not a surprising result, familiarity with a device has a greater effect that a willingness to use it. One student commented that the TI-83 was a more efficient device for typing in instructions which preferenced their selection toward this model, unless they needed any of the specific facilities of the TI-92 that were unavailable on the TI-83. This directly contradicts the preference of a different student who felt the TI-92 calculator offered more direct access to the commands via the qwerty keyboard than the TI-83. In both cases the students' level of familiarity and expertise with their chosen device allowed them to focus on how to use the device as efficiently and effectively as possible. This was despite the fact that they had chosen different devices for the same reason!

> Teacher. But Stephen, why would you have picked the 83 rather than the 92
>
> S1. Because I don't think there is anything more the 92 could have done that the 83 couldn't have…
>
> Teacher. So Stephen, the idea, what you are telling me is that unless you really need the facilities of a 92 you'd prefer to use the 83?
>
> S1. Yeah.
>
> Teacher. Is there any reason why…that you prefer to use it?

S1. Well it types faster.

Other students

S2. 'cause we use it (TI-83) more often, it's easier to know where all the functions are

S3 We're used to it.

Teacher. Whoa, whoa…let's slow down…. So you think the nice thing about the 92 …'cause you (talking to Student D) tend to use it as a first preference, don't you, is that because you know the command you can just type in and do things straight away, whereas on the 83, you have to go through the menus to get the commands?

Student agrees.

Matching the Hatch: Students were asked directly why they choose one calculator type over another (TI-83 vs TI-92). One student argued that you should only use technology as powerful as is needed for the task at hand

Other students indicated that the choice of calculator depended on the task and the particular device's suitability to that task.

G. Well it depends on what kind of work you're doing…with permutations and combinations, I find it easier on this (signalling TI-83) but with complex numbers and things (another student adds matrices) yeah, matrices - it's so much easier on the 92.

In fly fishing terms they were saying you should "match the hatch" – choose the tool that is most appropriate for the prevailing conditions and environment.

This attitude was taken to the extreme by some students who used both calculators on a particular task – matching sub-tasks with the particular facilities of each calculator based on perceived suitabilities to the job they were attempting to complete.

S4. Whenever I was doing normal calculations, I'd use the little one because I could look at my matrix and do the calculators at the same time, that sort of thing.

Teacher. You're using both at once because…

S4. By then, you could see the matrix really clearly on the big calculator

Teacher. You use that as just a way of looking at it, but you do all the manipulation on the other one.

S4. Well I like using the big one for complicated stuff like "i" and stuff like that, but the little one is just easier to use when you are doing all that little basic stuff.

The Look: Students also commented on differences between the way the calculators displayed certain inputs and outputs. They indicated a preference for a visual display that might be considered closest to how mathematics is written with a pad and pen, or when properly type set. They found this more useful for checking their inputs and looking for mistakes.

S5. It (the TI-92) gives you what you typed in. It actually writes out the matrices you are multiplying together instead of giving just "a" or "b".

Teacher. Okay, so you can see it….you like seeing the matrix, not just a symbol for it?

S6. Visually this (the TI-92) is a bit better.

Lean on Me: Students were also asked to comment on their choice of methods when solving the combinatorics problem that had been the focus of the initial lesson. The responses were varied and ranged from "trial and error" approaches using the basic arithmetic facilities of either calculator, through to methods which required a more sophisticated understanding of the capacities of their chosen device such as the "equation solver". As part of this discussion the teacher sought clarification on what a particular student (absent on the day of the interview) was observed to be doing on the video. Students working with her replied as part of the interchange below:

Teacher. You know what she (N – absent student) was doing E.

S9. She was expanding

Teacher. So she (E) wasn't solving like D, she was expanding using it like an "Algebra Assistant". She was using it as a way of checking her algebra.

S9. She used it to *do* her algebra

This student appears to have used the calculator as a means to "scaffold" her way around an area of mathematics in which she may lack confidence – algebra – so that she can still address the problem she was asked to solve. The student that replies to the question makes it clear that student E was using the TI-92 to do her algebra not merely to check work she had completed with pen or paper. This was confirmed with a follow up question to student E when she returned to class. She indicated that she would have been far less confident about finding a solution if had to rely on her own algebraic skills without the assistance of the calculator.

Discussion

The observations categorised above indicate that students will not only make different choices when provided with options in the range of technologies they can access when working on mathematics but they also base these choices on individual preferences and perceptions about a device's suitability for a task.

While familiarity with a particular device is an important influence on the choices students made about which calculator they used, they were prepared to override established preferences when they could clearly see that another calculator offered a more efficient or effective means to complete a task. This was most strongly evident in students who were observed to use two calculators while completing a single task. These students still like to have the calculator they were most comfortable using as a default but made use of the alternative device when they inferred that the specifics of a sub-task demanded the use of facilities that were only found, or were best implemented, on the alternative device. For example, a number of students commented on how they preferred to work with matrices on the TI-92 rather than the TI-83 and would change to this calculator whenever making use of matrices.

Students also reflected on the differences between visual displays when making a choice between calculators. A number of students commented on the importance of seeing the whole of their input or expected outputs while working with a calculator. They felt less comfortable in working with a problem or detecting errors when some part of the input or output was off screen because of the length of the input or output. Further, students also expressed a preference for displays that represented mathematics in a similar fashion to the way they wrote it - with a pen on paper, or how it appeared in a book. Clearly they found unfamiliar styles of symbolic representation at best distracting and at worst less intelligible. Thus the "visual" aspect of students' interaction with mathematics and technology should not be underestimated.

It was observed that at least one student made use of the CAS capabilities of one the calculators to support her use of an algebraic approach to solving the combinatorics problem. This illustrates the potential that exists for CAS technology to provide the scaffolding needed by students who are less proficient with algebra to access other aspects of mathematics. Further work is needed in this area to determine the potential support this technology offers students, who are less proficient in terms of traditional algebraic capabilities, to find success in other type of mathematics.

The four different categories of calculator use identified indicate that students moved comfortably within a technology ZFM within which they did not feel constrained to a set of 'approved' methods. This was particularly evident within the 'match the hatch' category, with different preferences not only selected but articulated and defended.

Conclusion

While students clearly indicated preferences for using a particular calculator from those available, these choices were influenced by a balance between familiarity with a particular device and the students' perception of a match between the demands of the task and facilities available on different devices. Thus students' predispositions and expertise with particular devices has the potential to influence how they learn and do mathematics within technology rich classroom environments.

It would appear that the provision of choice has allowed students to match their needs as learners with particular facilities offered by the different devices. This can range from what might appear to be a superficial preference for the way inputs and outputs are represented on a screen, as was the case for a number of students in relation to working with matrices, through to the opportunity to scaffold around areas of mathematical weakness such as the case of the student who used the CAS facilities of a TI-92 to deal with her lack of confidence in her algebra skills. It is perhaps also worth reflecting on the potentially negative effects of a mismatch between the needs of the learner and the technology available in situations where only a limited choice is possible.

If the influence of student learning orientations and preferences in relation to choice of technologies is as influential as is suggested here then there is clearly a need to further investigate, identify and delineate such effects in classrooms that integrate the use of technology into the learning of mathematics. In terms of theoretical structuring the ZPA provides a conceptual framework for structuring and presenting opportunities for the creative and productive use of technology in learning mathematics. Application of ZFM notions then provides a reality check that conditions in a classroom are indeed consistent with teaching intentions and opportunities for learning. Taken together with reasonable expectations of performance (students' ZPDs) the respective zone frameworks provide a basis for a self-consistent and challenging design of teaching approaches, learning contexts, and performance criteria for technology enriched classrooms.

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Intervention in Mathematics:
Is Assistance More Effective in Grade 1 or Grade 2?

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It is an accepted practice throughout Australia and New Zealand for schools to implement *Reading Recovery* Programs for Grade 1 children. In contrast, intervention programs in mathematics are less well established. As part of the *Early Numeracy Research Program* (1999-2001), an intervention program was introduced in 21 schools in both Grade 1 and Grade 2. Children's progress was compared to determine whether the program was more effective in either grade. Further perspectives were gained from 21 specialist teachers and the classroom teachers and parents of two children participating in the program. Although Grade 2 children made more progress in two of the four domains considered, the parents and classroom teachers believed that the program is best implemented in Grade 1.

In most classrooms there are children who experience difficulty learning mathematics. Knowing how best to assist these children is often a dilemma for school communities. On one level, this dilemma can be resolved by improving mathematics learning and teaching for all children and through ensuring that children receive the best possible opportunities to learn mathematics successfully. Indeed, school systems in both Australia and New Zealand have focused attention on improving learning and teaching in the early years of schooling in anticipation that students who begin school successfully will continue to do so in later years. In Mathematics, initiatives such as *Count Me In Too* (Stewart, Wright, & Gould, 1998), the *Early Numeracy Research Project* (ENRP, Clarke, 2001) and the *Count Me In Too Project* (Higgins, 2001) have each resulted in improved mathematics learning outcomes for young students. However, despite excellent classroom experiences and curriculum provision, some children continue to experience difficulty and need assistance beyond that which the classroom teacher can provide.

A review of the effects of programs designed to prevent early school failure concluded that optimising learning for young students depends on providing good pre-school and kindergarten experiences, improved curriculum and instruction at school, brief tutoring at critical junctures, and family support (Slavin, Karweit, & Wasik, 1992). Projects such as the *ENRP* and *Count Me in Too* have focused on some of these areas, and have lead to improved kindergarten experiences for children and improved mathematics curriculum and instruction. However, one area that continues to be problematic for schools is providing children with brief tutoring at critical junctures in their mathematics education.

As part of the ENRP, 21 schools introduced a structured mathematics intervention program for children who were having difficulty learning mathematics. Some schools chose to provide this assistance in Grade 1 (six year olds), and other schools chose Grade 2 (seven year olds). This paper considers both quantitative and qualitative data to explore whether the specialised assistance provided was more effective at one or other grade level.

Critical Times to Intervene in Mathematics

Reading Recovery (Clay, 1993) has gained widespread international support from teachers and parents as an effective program for identifying and assisting 'at risk' students

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in the area of literacy. Advocates of the *Reading Recovery* Program are unequivocal in stating that the critical time to intervene for the purpose of improving *reading* is in a child's second year of school. This age is also recommended by Wright, Cowper, Stafford, Stanger, and Stewart (1994) as the most appropriate time to intervene in mathematics. For this purpose, Wright developed the *Mathematics Recovery* Program for Grade 1 children. *Mathematics Intervention* (Merrifield & Pearn, 1999) is another program that focuses on intervention for Grade 1 children, but has been used also with older children. In contrast, eight year olds were the target of a study by Askew, Bibby and Brown (1998) that investigated the effects of a mathematics intervention program on students' learning. Participants were those who were operating below or just below the expected level of attainment for their age as specified by national tests, and who were considered to be low attainers in mathematics rather than having special education needs in mathematics. Although these programs focus on children of different ages, they all intervene with children during the first four years of schooling, and are therefore providing assistance at a relatively early point in children's education. However, there is a lack of research data comparing whether a particular grade level is more critical than another for mathematics intervention. Research in this area tends to focus on comparing the effect of a particular approach to assistance with a comparison group, or on describing the learning gains of children who participate in a particular program. However, the effect of grade level was explored as part of the *Early Numeracy Research Project* in respect to the intervention program *Extending Mathematical Understanding*.

The Extending Mathematical Understanding Program

ENRP trial schools were invited to nominate specialist teachers to be trained in 2000 to implement a specialised assistance program, *Extending Mathematical Understanding* (EMU) for their students who were low attaining in mathematics. Twenty-one of the thirty-five ENRP trial schools accepted for 2000. Schools decided whether to implement the program using an individual or small group structure in students' second or third year of school (Grade 1 or Grade 2). This meant that a comparison could be made between the effectiveness of the different program implementations. This report concentrates on some outcomes of the EMU program for the 42 Grade 1 students and 63 Grade 2 students who participated in the program in 2000.

The *Extending Mathematical Understanding* (EMU) Program comprised daily 30 minute sessions for between 10 and 20 weeks, depending on the progress of students. Teachers worked with groups of three or four students or with individual students. The program is not remedial in nature, but is built upon constructivist learning principles (see, e.g., von Glasersfeld, 1989). It focused on children developing powerful mathematical understandings. Children were engaged in experiences that required 'hard' thinking, and children were required to reflect upon their activity and articulate what they had learnt and how they had learnt. The EMU Program provided different learning experiences for children than were possible within the classroom setting during mathematics lessons. In particular, the specialist teachers were trained to provide intensive instruction and feedback that was directed to the particular learning needs of each child. Observations of more than 30 EMU sessions in 2000 showed that, within each 30 minute session, children and teachers engaged in more than 100 interactions focused on the mathematical ideas investigated during a session. This high level of interaction between the teacher and a child is not possible in a classroom setting. The specialist teachers were able to constantly focus children's attention on key ideas and facilitate their construction of knowledge and understanding. They helped children develop the language that would facilitate communication about mathematics, and provided manipulatives to support children's thinking at critical moments in their learning. Teachers constantly encouraged children to describe their thinking and strategies for solving problems, and provided *wait time* to encourage student's thinking and ability to provide extended responses.

The EMU program included further diagnosis of individual difficulties using the *Extending Mathematical Understanding (EMU) Assessment Interview* and learning experiences focusing on the counting, place value, addition and subtraction and multiplication and division aspects of the ENRP framework. Typically, each EMU session was structured to include 10 minutes of activities focusing on counting and place value, 15 minutes of rich learning activities focusing on problem solving (often with an addition and subtraction, or multiplication and division focus), and 5 minutes reflection about the key aspects that were covered in the session.

Comparison of Intervention at Grade 1 and Grade 2

An issue that teachers grappled with when deciding how to implement the EMU Program was whether to intervene in Grade 1 or Grade 2. This was a difficult choice for many schools. The idea of intervening as early as possible before children experienced failure was foremost in most teachers thinking. However, there were other factors to consider. For example, in Victoria most schools have *Reading Recovery* Programs for Grade 1 children, and some teachers anticipated that Grade 1 might be the most appropriate grade for a specialised mathematics intervention program also. Other teachers considered that intervention programs should be spread over two years, with Reading Recovery in Grade 1 and the EMU program in Grade 2. Other teachers believed that while it may be best, in the long run, to offer the EMU program in Grade 1, when faced with a number of Grade 2 children who were 'at risk' in mathematics, they thought it would be best to begin the program with Grade 2 children before they "fell" further behind. An advantage of implementing the EMU program at both Grade 1 and Grade 2 was that data could be collected and used to compare the effectiveness of the program at both levels.

For this purpose, children's mathematical progress was determined using the ENRP assessment interview and framework of growth points (Clarke, 2001). The children were assessed by their classroom teachers in both March and November, and the growth points reached by the children determined. Table 1 shows the mean growth in each of the Number Domains for the children who participated in an EMU program and the comparison groups.

Table 1
Mean Growth Points Gains for Children in the EMU Program and Comparison Groups.

Mathematics Domains	Mean gains in Growth Points			
	Grade 1		Grade 2	
	EMU *n*=42	Compar. Group *n*=469	EMU *n*=63	Compar. Group *n*=364
Counting	1.64	1.01	1.24	1.16
Place Value	0.86	0.68	1.25	1.07
Addition & Subtraction	1.00	0.99	1.24	1.05
Mult. & Division	1.21	0.98	0.94	0.51

The comparison groups were comprised of ENRP trial school children who did not participate in an EMU program, but had the same growth point profiles as children who did participate.

The data presented in Table 1 shows that the mean gains were greater for children who participated in an EMU program, with one exception. Mean growth point gains for Addition and Subtraction for the Grade 1 groups were almost identical. The comparisons suggest also that the EMU program was effective for both Grade 1 and Grade 2 children.

The data in Table 1 can also be used to compare the mean gains for Grade 1 and Grade 2 children participating in the EMU Program. The gains are greater for Grade 1 children in Counting and Multiplication and Division, and for Grade 2 children in Place Value and Addition and Subtraction. However, statistical analysis of the effect of grade level on the effectiveness of the EMU program demonstrated that grade level was significant for only two of the four number domains. Grade 2 was significantly more effective for Addition and Subtraction ($F(1, 101) = 7.249$, $p = 0.008$), and for Place Value ($F(1, 101) = 7.991$, $p = 0.006$). There were no significant differences attributable to grade level for the Counting or Multiplication and Division domains.

These findings suggest that in determining whether the EMU Program is more effective at one grade level than another, then the mathematical domain must be considered. However, even though children made significantly more progress in Addition and Subtraction and Place Value in Grade 2, children make important progress in Grade 1 also. It may be that given the assistance in Grade 1, then children might be better able to benefit from the regular classroom program, and might not require an intervention program in Grade 2.

A further examination of the growth point data for each of the number domains shows that there are several growth points that act as barriers or prolonged transition points for some children. Another measure of the effectiveness of the EMU program at each grade level may be the program's effectiveness in assisting children to move beyond these barriers. These prolonged transitions or barriers can be identified using an area graph. Figure 1 shows the Counting area graph for the low-attaining trial school children in 2000 who had the same ENRP profiles as the Grade 1 children participating in an EMU program.

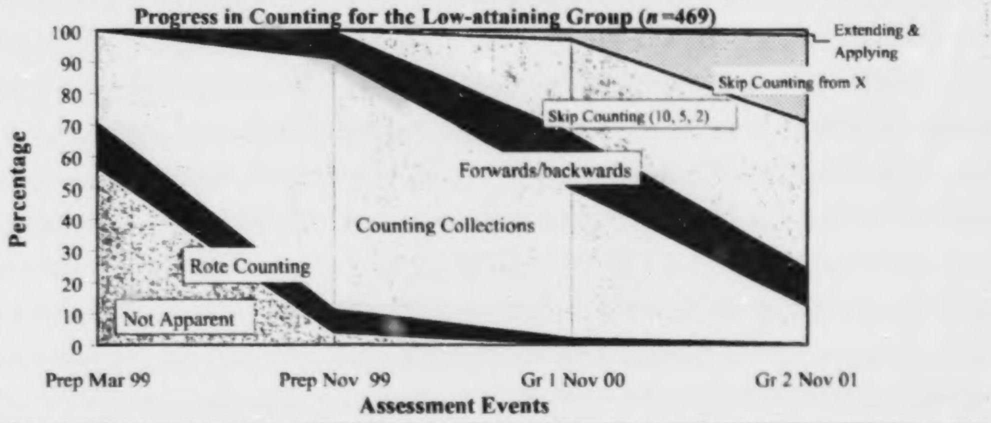

Figure 1. Low-attaining students (%) achieving Counting growth points over time.

The different shadings in the graph represent different growth points for Counting. The four data sets are points on the horizontal axis, and the vertical axis represents the percentage of the group achieving each growth point. The areas bounded by the lines give an indication of the difficulty students have in achieving particular growth points. For example, the large area in the middle of the figure indicates that many Prep students and Grade 1 students can count collections but are not yet counting forwards and backwards by ones from varying starting points. Indeed, many children remain on growth point 2 for extended periods. The figure suggests that counting forwards and backwards by ones from various starting points presents a barrier but, once overcome, students move readily to the subsequent point, as is indicated by the narrowness of the band associated with the next growth point. By the end of Grade 1 and throughout Grade 2, growth point 5 (counting by 10, 5 and 2 from X) becomes another significant transition point for many children.

Similar analysis for the domains of Place Value, Addition and Subtraction, and Multiplication and Division suggest further growth points transitions that are prolonged for many children. These are summarised in Table 2.

In comparing the effectiveness of an additional assistance program in Grade 1 or Grade 2, it is useful to compare how many children made these important transitions in their mathematical learning. Table 2 shows the percentage of Grade 1 and Grade 2 children participating in an EMU program who made these transitions.

Table 2.
The Percentage of Children Who Made the Difficult Growth Point Transitions in 2000.

Difficult Growth Point Transitions or Barriers	Gr 1 (%) *n*=42	Gr 2 (%) *n*=63
Counting (3) – counting forwards & backwards by ones from variable starting points	50	65
Counting (5) – counting by tens, fives and twos from variable starting points	10	14
Place Value (2)– reading, writing, interpreting and ordering 2 digit numbers	45	73
Place Value (3) – reading, writing, interpreting and ordering 3 digit numbers	10	33
Addition and Subtraction (2) - count-on	57	67
Addition and Subtraction (3) - count-back, count-down-to, count-up-from	12	37
Mult. & Division (2) – Uses models to solve multiplicative & sharing situations	81	63
Mult. & Division (3) – Uses abstract strategies without modelling	0	13

Table 2 shows that, with the exception of the modelling growth point in Multiplication and Division, a higher percentage of Grade 2 children made each transition than did Grade 1 children. Although it might be expected that these transitions would be more achievable for the older students, it is interesting to note that, for several growth point transitions, the difference in the percentage of Grade 1 and Grade 2 children making the transition is no more than 15 percent. Therefore, even though more Grade 2 children make these transitions, this may not be significant when the factor of intervening a year earlier is considered and the difference this makes to children's subsequent learning in Grade 2.

These findings indicate that there is no clear answer as to whether the EMU Program is more effective at one grade level than another. The answer varies according to the mathematical domain being considered. Indeed, there may be other factors to consider. This possibility was explored by interviewing teachers, parents, principals to gain there perspectives on whether the intervention may be more effective in Grade 1 or Grade 2.

Perspectives of Parents and Teachers

The 21 Specialist teachers were interviewed and asked whether the EMU program was most effective in Grade 1 or Grade 2. The responses are summarised in Table 3.

Table 3

Specialist Teachers' Perceptions of the Most Effective Grade Level for the EMU Program

Perception of most effective year level	Number of responses
Grade 1	7
Grade 2	2
Both Grade 1 and Grade 2	10

The summary indicates that about half of the teachers thought the program was effective in either grade. One teacher qualified this response by explaining that she thought that both grades were effective, but that the program was more beneficial in Grade 1. Of the teachers who thought one grade level was more effective, most thought that Grade 1 was more effective. Reasons for this opinion included the following, "in Year 2 it is a stigma; they are more self-conscious. In Grade 1 [the children perceive] it as a privilege to participate. You are intervening to stop incorrect learning. It is a safety net. "

One specialist teacher of Grade 2 children believed that his students would have benefited from having the additional assistance program in Grade 1. Another teacher commented, "sorting out common problems in Grade 1 is really good, so [children] don't see themselves as not good at maths." Another teacher explained, "if you want to make quick progress, then run the program in Year 2. The program is more beneficial in Year 1, but the progress is slower."

Two teachers believed the program was more effective in Grade 2. One teacher commented, "it is more effective with Grade 2. It could be maturity. The boys [Grade 1] wanted to play *Snap* all the time."

Three teachers were unable to answer this question because their schools made the decision to run the program with Grade 2 children only because of literacy being a focus in Grade 1. They were therefore not able to make a comparison between the effectiveness of the program the different grade levels. One teacher explained,

> It was a lower primary unit decision to focus on Year 2. The Year 1 children are taken out so much for Reading Recovery. We have a lot of candidates for Reading Recovery at our school and children for top-up [literacy] programs, so lots of children are taken out. Therefore, we didn't want to be taking children from the Year 1 level again. Many of the kids who need additional assistance for maths also need Reading Recovery.

The classroom teachers, parents and principals of two Grade 2 children participating in the EMU program were asked whether the EMU program would be more effective in Grade 1 or Grade 2. The children are referred to in this paper as Sarah and David. Pseudonyms are used also for the teachers. David's teacher thought that Grade 1 was the preferred year for the EMU program.

> If there is an obvious difficulty with a child in their developing of their mathematical understanding I think the earlier you can work with them the better…But then that doesn't also eliminate that there may be children for whatever circumstances in Grade 2 who need the help. I don't think they should be barred from it, but a child around the age of a Grade 1 child is [at] a premium age to start working on difficulties.

This teacher saw no difficulty with children doing Reading Recovery and EMU in the same year. She explained, "it's all for the benefit of the child's understanding and development."

Sarah's classroom teacher also thought Grade 1 would be the preferable year for participation in the program. However, she would like children to have the opportunity to participate in Grade 2 also:

> When it's gone to…Grade 2, you've really left your run very, very late. There's things that people have moved on with. …When are they going to catch that up? I would aim to have them return to …the normal classroom as quickly as possible. I'm not sure if it all can be achieved in the one year though-if it's one of those things that might be beyond one year. It might be more like a two year target. …I think we could make an error if we say "this is only for grade ones" ah la Reading Recovery. "Only people in Grade 1 are eligible." If it has to be that we can only offer it in one year, it would probably be the Year 1 [that] I would look at.

Both parents interviewed would have liked their children to have participated in the EMU program in Grade 1. Sarah's mother commented:

> I think it would have been great if Sarah could have done it last year in Grade 1. But as I said … she was falling behind in her reading so you couldn't really … do two things at once-I think she needs to concentrate on one thing or another. …Grade 2 is great but I think the earlier the better with the maths. I think if they find it difficult after Grade 2 and into Grade 3 I think they'll just keep suffering and getting further and further behind … because they just don't have the basics there and then going back to the real basics … probably would make them feel a bit stupid because they don't know… the really, really fundamentals of maths. So I mean Grade 2's been great, but she probably needed the help last year. But then again it's really hard to tell with her because she probably did lack knowledge in maths even ..... Yeah, with Sarah even in Grade 1 it would have been great but you know Grade 2's been great too. She really sort of enjoys it thoroughly so I don't have a problem with it but, yeah Grade 1 would probably be a good time.

Sarah's mother also pointed out that participating in Reading Recovery in Grade 1 may have had a detrimental effect on Sarah's learning of mathematics because she missed out on mathematics classes to do Reading Recovery:

> Mrs Jones did say, I think at the first … parent teacher interview, that Sarah was a bit slow in her maths. But we did know that because she was taken out from a lot of maths… because she was in Reading Recovery. Mrs Southby said, that was her teacher in Grade 1, the only thing is it's [Reading Recovery] been wonderful with her reading but she is missing out on some of the maths which we have started to notice. So that was probably one of the downfalls of her being pulled out of half an hour every day of that class. But it … [Reading Recovery] was excellent, but then again… everything does have it's little downfalls somewhere along the line. But [Mrs Southby] did mention that …even last year, …her maths were a bit behind.

This comment highlights the importance of planning appropriate times for withdrawal programs. It appears that running *Reading Recovery* during mathematics times may be detrimental for some children.

David's mother also preferred Grade 1 for the EMU Program. She explained, "if he starts at an earlier age, then when he gets older, if he's still got the difficulty, he's got the help at an earlier age… I'd prefer the earlier age, Grade 1.

David's principal also had an opinion about the most effective grade level for the EMU program. He believed that the EMU program was most effective for the Grade 1 children.

> The testing [in our school] has shown that, especially the Grade 1s, [have] made rapid improvement and rapid progress. Their case has sort of caught up with the mainstream in the classes. Grade two- …they've had definite improvement but nothing like the Grade 1s. Whether that's related to age and whether that sort of endorsed the Reading Recovery sort of philosophy, that children at that age, Grade 1, are the best age group to intervene, I don't know, but from results here, that seems to be the

> case…Grade 1s tend to be making the better progress and excellent progress.

The data suggest that the most effective level for the EMU program depends on several factors. Indeed, the effectiveness of the program depends partly on the mathematical domain being considered. The views expressed by the parents and teachers interviewed suggest that Grade 1 is preferable to Grade 2 because children are provided with assistance at an earlier age. Providing children with assistance in Grade 1 is important. However the teachers suggest that Grade 2 should be considered also.

Conclusion

Providing specialised assistance at critical points in their education is an important way to enhance the learning of children who experience difficulty learning mathematics. However, it is not clear whether intervention programs are more effective in Grade 1 or Grade 2. Analyses of growth point data presented in this paper suggest that the EMU Program was effective in both Grade 1 and Grade 2, but that Grade 2 children made significantly more progress in Place Value and Addition and Subtraction. Grade level had no significant effect on children's learning in the Counting or Multiplication and Division domains. This suggests that the effectiveness of the EMU program depends, in part, on the area of mathematics being considered. However, if a choice needed to be made between running the EMU Program in Grade 1 or Grade 2, the views expressed by the parents and teachers interviewed suggest that Grade 1 is more appropriate. Indeed, it seems preferable to provide children with assistance as early as possible, before they experience failure.

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Beginning Teachers and Technology: Developing Identities as Teachers

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This paper reports on a study that investigated the pedagogical practices and beliefs of pre-service and beginning teachers in integrating technology into the teaching of secondary school mathematics. A case study documents how one teacher's modes of working with technology changed over time and across different school contexts, and identifies relationships between a range of personal and contextual factors that influenced the development of his identity as a teacher. This analysis views teachers' learning as increasing participation in sociocultural practices.

Much of the existing research on the role of technology in mathematics education has been concerned with effects on curriculum content or student learning (Penglase & Arnold, 1996). However, less attention has been given to the relationship between technology use and issues of pedagogy, in particular the impact on teachers' professional learning in the context of specific classroom and school environments. This paper reports on the preliminary stages of a study that aims to address this gap in current knowledge. The study is situated at the interface between pre-service education and initial professional experience of secondary mathematics teachers, in order to investigate the pedagogical practices and beliefs of beginning teachers who have graduated from a technology enriched teacher education program.

From a theoretical perspective, this research extends earlier work which applied sociocultural perspectives on learning to develop models of technology enriched mathematics learning (Goos, Galbraith, Renshaw, & Geiger, 2000), and models of mentoring in pre-service teacher education (Goos, 1999a). The aims of this preliminary study are to identify modes of working with technology adopted by pre-service and beginning teachers, and to explore the influence of a range of personal and contextual factors on the pedagogical identities they construct.

Sociocultural Perspectives on Teacher Learning and Development

Learning to Teach

Although research on mathematics teacher education has grown rapidly in the past 10 to 20 years, influential voices have argued that teacher education is an under-theorised field of inquiry, lacking coherent conceptual frameworks that address the complexity of individuals acting in social situations (e.g., Cooney, 1994; Lerman, 2001). Particularly relevant to the present study is research on teacher socialisation, the goal of which is to explain how teachers acquire the beliefs, values and attitudes of their professional culture. Studies of teacher socialisation from a functionalist perspective typically identify influences such as the beliefs that students bring to the pre-service course from their own schooling, and the classroom practices they observe and experience as novice teachers (Brown & Borko, 1992). Such approaches view teachers as being passively moulded by

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external forces to fit the existing culture of schools – thus producing the common explanation for why beginning teachers are unable to implement innovative approaches (e.g. those involving use of educational technologies) that they may have experienced during their pre-service courses (e.g., Loughran, Mitchell, Neale, & Toussaint, 2001). However, Lerman (2001) claims that the study of teacher beliefs, and of apparent mismatches between espoused and enacted beliefs, is often too static and decontextualised to adequately describe these (dis)connections between theory and practice. As an alternative, he points to the work of Vygotsky (1978) and followers in proposing that teachers' learning is better understood as increasing participation in sociocultural practices that develop their identities as teachers.

There are yet few studies that have applied sociocultural theories to teacher learning, particularly in pre-service teacher education. Some of these have adopted a neo-Vygotskian approach, extending the well known concept of the Zone of Proximal Development to additionally incorporate the social setting and the goals and actions of the participants (e.g., Blanton, Westbrook, & Carter, 2001; Goos, Evans & Galbraith, 1994). In a teacher education context, the ZPD can be thought of as the novice teacher's emerging skills that have not yet developed fully, but which are taking shape under the guidance of other people. However, this gap between present and potential ability is not the only factor influencing future development. Valsiner (1987) has described two additional zones of interaction, the Zone of Free Movement (ZFM) and Zone of Promoted Action (ZPA), that can account for the contextual constraints and affordances of the teaching environment and the teaching actions specifically promoted.

According to Valsiner (1987), the ZFM represents environmental constraints that limit freedom of action and thought. For pre-service or beginning teachers, elements of the ZFM might include:

1. their students, whose (perceived) abilities and behaviours may constrain teaching actions;
2. curriculum and assessment requirements, which influence choice of topics, teaching methods, and the time available to teach required content;
3. resources, in the form of teaching materials, computers or calculators, or specially equipped rooms, whose availability has a bearing on teachers' planning decisions;
4. the relationship between pre-service or beginning teacher and supervising teacher or more experienced colleagues, and the extent to which their beliefs about teaching and learning coincide.

Although these elements clearly have an external existence, teachers may also construct personal ZFMs within which constraints – or affordances – exist as a result of their interpretation of the external environment.

While the ZFM suggests which teaching actions are *possible*, the Zone of Promoted Action (ZPA) represents the efforts of a teacher educator, supervising teacher, or more experienced teaching colleague to *promote* particular teaching skills or approaches. It is important that the ZPA be within the novice teacher's ZFM, and is also consistent with their ZPD – that is, the actions promoted must be within the novice's reach if development of their identity as a teacher is to occur. Additionally, pre-service teachers develop under the influence of two ZPAs – one provided by their university program, the other by their supervising teacher(s) during the practicum – which do not necessarily coincide. Unlike functionalist approaches, this sociocultural model facilitates an analysis of teacher learning and socialisation that considers the person-in-practice, and examines how identities develop as involvement in practice increases (Lerman, 2001).

Teaching with Technology

Research on technology use by mathematics teachers has identified a range of factors influencing uptake and implementation, including: previous experience in using technology; time and opportunities to learn (pre-service education, professional development); access to hardware (computers and calculators), software, and computer laboratories; availability of appropriate teaching materials; technical support; support from colleagues and school administration; knowledge of how to integrate technology into mathematics teaching; and beliefs about mathematics and how it is learned (Fine & Fleener, 1994; Manoucherhri, 1999; Simmt, 1997; Simonsen & Dick, 1997). In terms of the concepts introduced in the previous section, studies such as these have catalogued elements of a teacher's Zone of Free Movement without necessarily considering possible relationships between the setting, actions, and beliefs, and how these relationships might change over time or across different classroom or school contexts.

From a sociocultural perspective, technologies such as computers and graphics calculators can be viewed as cultural tools that not only re-organise cognitive processes but also transform classroom social practices. In an earlier study (Goos, Galbraith, Renshaw, & Geiger, 2000), metaphors were developed to describe how such technologies can provide a vehicle for incorporating new teaching roles. Teachers can see technology as a *master* if their knowledge and competence are limited to a narrow range of operations, especially in situations where external pressures force implementation. Technology is a *servant* if used by students or teachers as a fast, reliable replacement for pen and paper calculations – that is, in reproducing rather than transforming existing teaching and learning arrangements. However, when teachers develop an affinity for technology as a *partner*, there is potential for students to achieve more power over their own learning, for example, by using a screen projection unit interactively to construct a mathematical argument in whole class discussion. Technology becomes an *extension of self* when seamlessly incorporated into a teacher's pedagogical and mathematical repertoire, such as through the integration of a variety of technology resources into course planning and the everyday practices of the mathematics classroom. These categories elaborate different ways in which teachers may appropriate technology as a cultural tool, and also provide a means of tracing trajectories of professional growth as teachers construct new pedagogical identities.

The Study

Background

This study represents the preliminary stages of a three year project that spans the transition from pre-service to beginning teaching of secondary school mathematics. In the pilot phase reported here, participants comprised all Bachelor of Education students enrolled in the mathematics curriculum studies class of 2000 (n = 18) at The University of Queensland. The BEd program is available to undergraduates as a four year dual degree or to graduates as a single degree taken in four semesters over eighteen months. Students undertake mathematics curriculum studies as a single class group in a one year course that corresponds to the fourth year of the Dual Degree and the first two semesters of the

Graduate Entry program. All students complete two, seven week blocks of practice teaching during this year.

BEd mathematics curriculum students experience a technology-enriched pre-service course (taught by the author) that features regular and intensive use of graphics calculators, computer software, and internet applications (see Goos, 1999b). Integration of technology into mathematics education is emphasised through continuous personal access to a graphics calculator for the duration of the course (including the practicum), assignment tasks that ask students to prepare technology related classroom activities, and encouragement for students to contribute to the professional community through publication of resources they have produced and presentation of student workshops at professional association seminars and conferences. The course thus offers a teaching repertoire, or Zone of Promoted Action, that privileges technology as a pedagogical resource.

Research Design

The research design for this pilot study had two components: (a) a cohort study of pre-service students' practicum experiences in technology integration; and (b) individual case studies of selected students that allowed snapshots of experience to be captured at two developmental stages, during the second block of practice teaching (August 2000) and towards the end of the first year of full-time teaching (November 2001). For the cohort study all students completed a Technology Survey of their practicum schools to record information on the availability and accessibility of technology resources (computers and graphics calculators), and the frequency and mode of technology use during lessons they observed or taught, and in assessment tasks. Survey findings were discussed and compared in an audiotaped whole class interview when students returned from the first block of practice teaching.

Four students were also selected for individual case studies to represent a range of initial school placements (government vs independent schools, capital city vs regional location) and program formats (Dual Degree vs Graduate Entry). These students were visited in their schools during the second practicum session and again after graduation as described above. The school visits involved lesson observations, collection of teaching materials, and audiotaped interviews. Observations focused on teachers' modes of working with technology, using categories developed in an earlier study and described in the previous section of this paper. A Post-lesson Interview assisted participants to reflect on pedagogical beliefs that influenced lesson goals and methods (as in Goos, 1999a), while a more general Technology Interview sought perceptions of constraints and opportunities affecting technology integration, and the influence of technology on mathematics curricula, learning, teaching, and assessment. The purpose of these interviews was to gain insights into factors shaping the formation of beginning teachers' professional identities.

This paper draws on the Technology Survey, lesson observations, teaching materials, and interviews from one case study participant (Geoff) to compare his modes of working with technology over time and in different school settings, and to examine how changing relationships between his Zones of Proximal Development, Free Movement, and Promoted Action generated an environment for development of his identity as a teacher.

Geoff's Practice as a Pre-Service Teacher

Geoff's practicum placement was in a co-educational government school in a low SES Brisbane suburb. The student population of 430 was ethnically diverse, with many students having recently arrived in Australia on refugee visas. Geoff was assigned to teach a Year 10 Mathematics class and a Year 11 Numeracy class, the latter being the focus for observation and interview for the purposes of this study. Geoff explained that Numeracy was a low status, school based subject, regarded by students as "a repository for misbehaving students", or for "druggos and dropouts". Yet he was determined to challenge this perception by designing interesting activities that demonstrated how mathematics is used in real life situations. He commented that students' negative attitudes towards mathematics often come from "ingrained mathematical practices from Years 1 to Years 10 and Years 12 that maths is some kind of remote thing. It's some kind of island that you visit and then you go back to your other lessons". Geoff's goals for these students were concerned as much with building their self-respect and encouraging their engagement with their futures beyond school as with teaching mathematical content.

Geoff was an experienced computer user and spoke enthusiastically of his desire to integrate technology into his teaching, mentioning in particular the potential for technology to speed up calculations and "make things easier to understand because ... it's dynamic and not static". Nevertheless, after almost fourteen weeks of practice teaching, he had only had one opportunity to use technology in a mathematics lesson. This involved creating an internet research activity for the Numeracy class that required them to use the Australian Bureau of Statistics website to produce a fact sheet on a health issue of their choice, such as alcohol or drug use. The fact sheet was to include a graph (e.g. Excel chart) that compared how this health issue affected different age groups, genders, or countries.

Constraints and Affordances in Teaching with Technology

At the time of this study, mathematics syllabuses in Queensland only encouraged, rather than mandated, the use of technologies such as computers and graphics calculators. Geoff's school was poorly resourced in this area, with no graphics calculators and only two computer laboratories that were almost continually booked out to Information Technology or Business Studies classes. Geoff felt that teachers in this school did not regard mathematics as a subject worthy of computer use. In the Technology Survey of his school Geoff noted that no technology was used in any of the mathematics classes he had observed. He commented that the mathematics staff seemed to be generally uninterested in learning about technology and unconvinced of its benefits for mathematics learning, especially for low status mathematics subjects such as Numeracy. Each of these elements of Geoff's Zone of Free Movement could be interpreted as constraints that might limit his teaching actions. Furthermore, while Geoff's supervising teacher allowed him to take the initiative in planning and delivering Numeracy lessons, the Zone of Promoted Action offered by this supervision excluded technology, and thus was not well matched with the ZPD that defined the direction in which Geoff hoped his teaching would develop. In this respect, neither did the supervisory ZPA coincide with that offered by the pre-service course.

Despite these many hindrances, Geoff did design and implement a technology-based activity with his Numeracy class. In theoretical terms, he achieved this by re-interpreting aspects of his teaching environment, or ZFM, so as to *afford* at least some use of technology in ways that were consistent with the actions promoted by the university course. First, he found there was little direct opposition towards introducing technology activities into the Numeracy class "because basically it's a class that nobody cares about". Also, he was aware of the general expectation within the practicum environment that student teachers would be adventurous in trying out new ideas, including those involving technology. Thus he was able to construct his practice as a pre-service teacher of low status mathematics students to further develop his emerging identity as a teacher for whom technology was an important pedagogical resource. However, at this stage his teaching experience had included technology only in the role of a *servant*, to facilitate his students' information searching or production of accurate graphs.

Geoff's Practice as a Beginning Teacher

After graduation Geoff found employment at an academically oriented, independent girls' school with an enrolment of over 1000 students. Geoff taught two senior secondary Mathematics classes, and also a Year 8 class which was observed during a visit to the school near the end of his first year of teaching. In the previous lesson with this class, Geoff had presented a graphing task that introduced students to use of a motion detector in conjunction with a graphics calculator and viewscreen. The aim of the activity was for individuals to walk towards, or away from, the motion detector so as to match a pre-selected distance-time graph displayed on the calculator screen. Students conducted "walking contests" within their working groups, followed by a hotly contested "walk off" to determine the most accurate walker in the class. In discussing his rationale for this task, Geoff referred to the motivational benefits of having students physically involved in creating a graph of their own movement, the capacity for the technology to provide instant feedback to walkers so they could adjust their movement to better match the target graph, and the thoughtful interaction this facilitated between walkers and observers. He also pointed out the mathematical skills required to make an accurate match, such as scale reading, estimation, and knowledge of the meaning of slope. At the beginning of the following (observed) lesson, Geoff reinforced these skills through a simulated graph matching activity, where a student volunteer "walked" the graph he had drawn on the whiteboard as he moved his pen along the x-axis to represent the passage of time. In these lessons, he worked with technology as a *partner* that entered into the mathematical practices and collective memory of the classroom.

Constraints and Affordances in Teaching with Technology

The teaching environment within this school contrasts significantly with that experienced by Geoff during his practicum. As a newly graduated teacher, Geoff came to an apparently well resourced school with an explicit policy of emphasising technology use across all subject areas. All students from Year 9 upwards were required to buy their own graphics calculator, and peripherals such as data loggers and viewscreens were readily available, as was school based professional development on the use of this equipment. Thus Geoff's Zone of Free Movement appeared to afford teaching actions consistent with his beliefs about mathematics learning and teaching, as expressed in his justification for the graph matching activity. Furthermore, the Zone of Promoted Action offered by his teaching colleagues seemed to be consistent with both his development as a teacher (i.e. his ZPD) and the ZPA offered by his pre-service course, in that new graduates teaching at the school were actively supported in integrating technology into their practice. For example, as part of a move to include at least one technology-based assessment task per semester in every mathematics subject, at each year level, Geoff had designed a Year 8 assignment on tessellations that involved students in web-based research, and using Microsoft Paint to create their own tessellations. As Geoff commented, "I have basically been given a brief to go ahead and ... try whatever I like, and do anything I like with technology".

However, other, less obvious, elements of the school context constrained Geoff's practice in more subtle ways. For example, some classrooms were designated as technology centres and specially fitted with equipment such as computers, internet and intranet connections, and data projectors. However, the timetabling of classes into these special rooms was neither transparent nor flexible, with the result that some teachers and classes regularly allocated these rooms rarely used the available technology while others who wished to use these resources were unable to gain access. Apart from these classrooms the school had only three dedicated computer laboratories which, according to Geoff, were almost always fully booked to non-mathematics classes. These components of Geoff's ZFM tended to undermine his goal of infusing technology as a *partner* in assessment tasks as well as learning activities. In particular, his original plan to include oral presentations and computer demonstrations of the tessellation assignment mentioned above had to be modified when it proved impossible to arrange access to the specially equipped technology centres for all Year 8 classes.

At the end of his first year of teaching, Geoff was looking for further challenges in exploring what technology could do:

> I know what things the graphics calculator can do, and I have a pretty good knowledge of Excel, but really now that teachers know how to include this in their pedagogy, I suppose the emphasis would be now on getting the most out of it. Instead of just knowing what to do, how to really take this technology and explore it to its fullest extent and use all of the resources that [it] has to offer instead of taking bits and pieces that might be good. I suppose unlocking the potential ... of what this technology has to offer. [...] I have been sort of nominated by the Maths faculty to go out and delve into the world of Microsoft PowerPoint because I believe there is a lot more to PowerPoint than what meets the eye in terms of the teaching tool ... how you can use animations to explain mathematical concepts, how you can integrate the whole thing into [your teaching] and have it available on line for every teacher. PowerPoint is not just something you can put bullet points up on the screen with ... There is [no inservice] for teaching with PowerPoint because everyone just takes it for granted as a display [tool]. I might get myself a book or two on it and maybe try to do an inservice of my own and try to get my head around how useful it can be.

Here he maps out the landscape of his ZPD in a way that suggests he is moving towards using technology as an *extension of self*, and anticipates seeking out – or perhaps even generating – a ZPA that will further develop his identity as a teacher.

Discussion

This study addresses one of the major themes in current debates on educational reform – the need for teachers to become more effective, confident, and creative users of technology in their teaching – by examining how beginning teachers of secondary school mathematics integrate technologies such as computers, graphics calculators, and the internet into their practice. Rather than analysing teacher beliefs about technology and its role in mathematics education, and possible connections (or conflicts) between beliefs, the teaching repertoire offered by the pre-service course, and practicum experiences, the study applied sociocultural perspectives on learning to focus on beginning teachers' identities and the settings in which those can change (as proposed by Lerman, 2001). A case study of one teacher's early professional experiences demonstrated how he developed that part of his pedagogical identity concerned with technology use, by negotiating changing relationships between his teaching environment, actions, and beliefs.

Clearly, it is not possible to explain Geoff's appropriation of technology as being determined solely by the material and human resources available to him in technology-poor and technology-rich school settings. Nor is it meaningful to describe his initial socialisation into teaching practice in functionalist terms that separate theory from practice. Instead, the sociocultural analysis presented here revealed how Geoff was an active agent in his own development as a teacher, not simply reproducing the practices he observed nor yielding to environmental constraints, but instead re-interpreting these social conditions in the light of his own professional goals and beliefs. Reading this as interactions between Zones of Proximal Development, Free Movement, and Promoted Action provides a dynamic way of theorising teacher learning as identity formation.

In contrast with research that suggests beginning teachers are unlikely to implement innovative approaches promoted by their pre-service courses, this study documented different ways in which Geoff was able to work with technology in quite different school settings. In addition, it appears that this aspect of his teaching identity developed to the extent that his modes of working became more varied and sophisticated over time, as indicated by the metaphors of technology being used first as *servant*, then later as a *partner* and *extension of self*. Thus these categories appear useful not only for describing different models of teaching and learning with technology, but also for anticipating a trajectory of development. In this regard, there is potential for technologically knowledgeable beginning teachers to act as catalysts for technology integration in schools, as Geoff's experience in his first year of teaching demonstrates. This observation raises interesting questions for further research on how novice teachers might develop their pedagogical identities by sharing their technology-related expertise with more experienced colleagues.

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Affective Development in Mathematics: A Case Study of Two Preservice Primary School Teachers

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Preservice primary school teachers come to their initial teacher education programs with beliefs, attitudes, values and feelings about mathematics and mathematics education. These have been formed through their experiences of mathematics prior to tertiary study and they can powerfully impact on their development as a teacher of mathematics. Then, during their teacher education they have another set of experiences of mathematics, which further impact on their views and emotions about the discipline. This study focuses on two preservice primary teachers in the first year of their initial teacher education and through a series of interviews and tasks it explores their affective views towards mathematics. It seems from the study that tertiary mathematics education courses need to explicitly and continually revisit the preservice teachers' affective positions if positive development is to occur.

The beliefs, attitudes, values and feelings of preservice and inservice teachers towards mathematics have become a topic of increasing interest for mathematics educators as is evidenced by the growing number of articles and papers published in recent times. A growing awareness of the impact of affective factors combined with the realisation that many teachers have debilitating views of mathematics, mathematics teaching and learning is concerning. The literature includes a number of studies attempting to address the negative cycle of beliefs, attitudes and feelings preservice teachers bring to mathematics. Overall the results are inconclusive with some reporting limited positive change (eg., Aldridge & Bobis, 2001), some lamented little change (eg., Artzt, 1999) and few studies report success in reforming negative or debilitating views. McLeod (1992) notes in his comprehensive review of research in the area that "although affect is a central concern of students and teachers, research on affect in mathematics education continues to reside on the periphery of the field"(p. 575).

This paper reports an investigation of the affective positions of two preservice primary school teachers, particularly as they relate to their experiences of mathematics and mathematics education. Initially the study focussed on their experiences prior to commencing their tertiary teacher education. The second phase explored their experiences during their first semester in teacher education and in particular their development through their first course on the teaching of mathematics. Finally, the study explored their beliefs, attitudes and values after their practicum experience.

Literature Review

Teaching includes an affective dimension and a teacher's beliefs, attitudes, values and feelings impact on their classroom work (Hargreaves, 1998). A number of studies have researched facets of the affective domain in mathematics education, including: (1) beliefs about mathematics, mathematics teaching and learning (eg., Ball, 1990; Mayers, 1994; Schuck, 1997b); (2) values (eg., Clarkson, Bishop, FitzSimons & Seah, 2000); (3) attitudes (eg., Sullivan, 1987; Way & Relich, 1993); and (4) feelings (eg., Carroll, 1994; Kelly &

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Tomhave, 1985). Perhaps the saddest aspect of these studies is the overwhelming sense that mathematics is perceived by many in negative and destructive ways. Carroll (1994, p. 131) noted that "many people seemed to be constrained by negative attitudes, beliefs and feelings about mathematics and mathematics learning".

Teacher Beliefs and Attitudes

It is of concern that primary school teachers, not immune to these negative views of mathematics, teach mathematics and their views influence their mathematical pedagogy (Ball, 1990; Bobis & Cusworth, 1994; Schuck, 1997a, b; Sullivan, 1987). Thompson (1992) suggested that most teachers view mathematics in an absolutist or instrumental fashion. Further to this, Mayers (1994) found that teachers who held an absolutist or instrumental view tended to teach through exposition, practice and rote learning. This confirmation of the connection between a teacher's affective views and practice was noted by Schuck (1996) when she stated; "… the work of primary school teachers in mathematics is generally seen to differ from their work in other subject areas; mathematics is transmitted…" (p. 119).

A number of studies have found that the absolutist and instrumentalist views of mathematics predominate amongst preservice primary school teachers and that these views block the development of mathematics as a human activity (Cooney, Shealy & Arvold, 1998; Mayers, 1994; Schuck, 1997a, b). Many preservice teachers have beliefs about mathematics and mathematics teaching that are narrow (Schuck, 1997b) enacting these through a transmission-model of teaching (Biddulph, 1992).

Often these narrow views about mathematics are accompanied by negative attitudes towards the subject. In a study with preservice primary teachers Grootenboer (2000) reported that almost 70 percent of the participants had a negative attitude to mathematics, and Biddulph (1999, p. 66) found that "… between one-half and two-thirds [of the primary teacher education students] had distinctly negative feelings about the subject". Biddulph elaborated on the attitudes of this group: "The students in the negative category expressed feelings that ranged from lacking enthusiasm, to being nervous, scared, feeling a total failure and terrified, being totally intimidated and hating or loathing it" (p. 67). In a case study of one "mathsphobic" preservice teacher, Carroll (1994) observed that a lack of confidence and destructive feelings about mathematics made it difficult for the participant to learn basic mathematical ideas, let alone teach them in a meaningful way.

Experiences of Mathematics

Most preservice teachers' experiences of mathematics have been in a classroom. Grootenboer (2001) reported that in his study many participants could recall in great detail their mathematics teachers even though they could not recall a great deal of the content knowledge they studied. Around two-thirds of the respondents described, with some emotion, disliked or unappreciated teachers. Furthermore, Carroll (1998, p. 1) contended that "often students' attitudes became negative during secondary school". It would seem then, that for many their experiences of mathematics have been unpleasant and negative, and through these experiences they have developed their attitudes and feelings towards the subject (Carroll, 1994; Grootenboer & Lowrie, 2002; McLeod, 1992).

Lortie (1975) developed the concept of an "apprenticeship of observation" to explain the learning of preservice teachers during their years of experiencing teaching first-hand as a school student. For mathematics education, Ball (1990) commented:

> Prospective teachers are … unlikely to acquire an appropriate view of the discipline as a result of precollege mathematics experience. The view they hold is likely to shape not only the way they teach mathematics once they begin teaching but also the way in which they approach *learning to teach* mathematics: A vicious cycle emerges. (p. 463)

If indeed there is a cycle linking school experience to affective development and teaching practice then the most likely point for an intervention to break it is within initial teacher education.

The Study

The research presented in this paper reports on the affective development of two preservice primary school teachers through the first year of their three-year teacher education programme. The first participant, Brad (pseudonym), came straight to teacher education from secondary school where he had been a successful sportsman and school leader. Marina (pseudonym), the second participant, was in her late twenties and she started her tertiary teacher education with a range of life experiences. The study focuses on their affective views in reference to their experiences: (1) as a school student; (2) during their first course in the teaching of mathematics; and (3) on practicum.

Theoretical Framework

In essence, the study was an exploration of the participants' personal interface with mathematics and their perceptions of those experiences. The desire for a holistic, human methodology with an emphasis on experience and perception meant that a phenomenological framework was most appropriate (Cohen, Manion & Morrison, 2000). Such research is descriptive, emphasising experience as the object of inquiry and incorporating sense impressions, perception, feelings, beliefs and values as being significant (Creswell, 1998).

Methods

The qualitative study employed two modes of data collection. The first was interviewing and occurred in February, July and November 2001. The 60-minute interviews were semi-structured and at the time they were audiotaped and later transcribed by the researcher. Each interview focussed on a different phase of their experience with mathematics and mathematics education. The first interview was directed at school experience, the second toward tertiary experience within their course on the teaching of mathematics and the third focussed on related experiences during practicum. The second mode of data collection was a journal which the participants maintained throughout the study. At times certain questions and prompts from the lecturers in their introductory course on the teaching of mathematics guided the participants' journal entries.

Data analysis utilized phenomenological techniques where initially the researcher must note and record any preconceptions about the topic of interest (Moustakas, 1994). The data were then read and re-read in order to gain a feel for the phenomenon. Coding was used to identify key themes across the descriptions using the *NUD*IST* computer program (Qualitative Solutions & Research, 1997), and then the overall descriptions of the phenomenon were crafted (Creswell, 1998).

Findings

The experiences and associated affective responses of the participants were explored in relation to three particular contexts outlined previously. Although these particular experiences are unique, they do impact the experiences that they precede, therefore they will be discussed in chronological order. Where possible, the participants' own words are used to explicate and exemplify the findings.

Experiences as a School Student

Generally, the participants could remember little of their mathematics at primary school other than having quizzes and tests on their "times-tables". Marina recalled feeling "quite dumb in standard one" because she couldn't recite all her tables quickly, and her teacher had made her write them out several times during the lunch-break. In her first interview she said:

> I know that knowing your times-tables is really important – probably one of the most important things you need to learn at primary school, but I'm still not too confident about my sevens and twelves. I think this is something I'm going to have to brush-up on if I am going to be a teacher!

Brad remembered playing with some "coloured sticks" (Cuisenaire rods) and his times-table tests every Friday. He suggested that the key to passing these tests was not to necessarily know the answers, but to sit next to someone who did and, when you could get away with it, mark your own test.

Both Brad and Marina could recall greater detail of their secondary school mathematics classes, and indeed both described in great detail some of their teachers. However, neither of them could recall a mathematics teacher they liked or respected, in fact they described their teachers as being sarcastic, disinterested and incompetent. Brad described his year ten mathematics teacher:

> Mr Howard [pseudonym] was good at maths himself, but he couldn't relate to me. I was into sports and he wasn't, and he thought I just wasted all my time playing rugby and cricket when I should have been studying. When I failed a test or didn't get something he'd just say that I needed to do more work and spend less time chasing balls. … I really hated him in the end and I just gave up.

He said that after his year with this teacher, he basically gave up on mathematics and didn't try to understand it until in his last year when he thought he needed to pass mathematics to get accepted into a teacher education programme.

In describing their mathematics classes, the experiences of the two participants sounded remarkably similar, regardless of year level, teacher or school. Below is part of one of Marina's descriptions she penned in her journal:

> Class began with the teacher explaining stuff from the blackboard with heaps of notes for us to copy down. Then he would do two or three examples which we also copied down. At this point most of us were confused! We would be set some pages of work to do from the textbook, and after we ruled up our books properly we would do the exercises. Most of the time we'd copy the answers from the back of the book, although sometimes the teacher wanted to see some working. At the end of the lesson we'd get told to complete the exercises for homework.

Brad's experiences were almost identical, as were Marina's when she outlined her classes at different levels of her secondary schooling.

After their experiences of mathematics at school, both participants thought that mathematics was important, but neither of them knew why. Furthermore, they described mathematics as being mostly about numbers and "the basics", irrelevant to life, mysterious and difficult for all but geniuses to understand. Marina added that she was also very anxious about having to do "mathematics courses" as part of her teacher education as she was a "maths dummy". Brad also said that he was "mathematically challenged", and he'd do it because "it was important for kids to learn maths" even though it was "largely irrelevant".

Experiences in Their Tertiary Teaching of Mathematics Course

The students undertook a course in the teaching of mathematics during the first semester of their teacher education programme. This course involved two 90-minute sessions a week for twelve weeks and its rationale espoused a constructivist approach to teaching and learning mathematics. Their classes incorporated a range of activities including lectures, workshops, readings, role-plays, discussions and field trips. After the course, Marina commented:

> I loved the course! I never thought I would have said that about a maths class! Maybe it is just because I am older and more focussed now, but I do enjoy maths and I can see it all around me everyday. ... Matt [pseudonym – one of the course lecturers] was really interesting and passionate about his stuff – maybe I have caught this love of maths off him! He spent time with me, explained things and challenged me to look at maths again. He even helped me with my times-tables!

Brad also wrote:

> Yep, it was primo! I had a lot of fun and enjoyed the classes, which is a change. I thought the field trip was cool – it showed me that maths is useful and all around us. I can see that maths is more than just working from a textbook and blah, blah, blah from the blackboard. ...

They still thought that mathematics was important and that "all kids need to know the basics", the basics being arithmetic, times-tables and fractions. However, both participants also emphasised the need to "understand mathematics rather than just being able to do it", and that mathematics was relevant to life outside the classroom.

Experiences During Their School Practicum

Marina's practicum was in a year three class in an innovative school which was involved in the Early Numeracy Project (ENP). Her associate teacher allowed Marina to have a small group of students for mathematics each day and she encouraged her to try out some of the ENP material. Marina said that "her teacher was a treasure who really loved the kids", and in her mathematics teaching "Beryl [pseudonym] always tried to make learning fun and relevant – she was setting them up for success". In reconciling her practicum experiences with her prior views Marina commented:

> That stuff that we learned in lectures [in the mathematics education course] was spot-on. On the prac I could really see how kids can be switched onto maths – about teaching to their hearts as well as their heads! Kids can actually enjoy maths – amazing – I never thought it would be possible!

Clearly there was a sense of astonishment for Marina in seeing mathematics differently, although she later acknowledged:

> It is still a struggle. If I don't consciously think about it then I go back to my old ways and I become anxious and rigid. ... It's like a bad habit of the brain, something I need to consciously and consistently work on or else I go back to thinking [about mathematics] how I did before.

For Brad, his practicum experience was quite different. His time was spent in a year seven class where they were streamed and they followed a "traditional mathematics program". Brad described the mathematics classes he experienced:

> Every lesson was the same. The teacher had a blue folder with all the worksheets for the unit, so after their pre-test the kids would chug their way through the worksheets one at a time until the end. After three weeks, the teacher gave them the test. The teacher had a few notes on the board at the start of each lesson which some of the kids copied down, but she never really taught them stuff from the front. … It was boring man and I nearly fell asleep sometimes!

When asked to reflect upon his experiences he commented:

> I think that stuff we learned with [the mathematics education course lecturer] was fun, but when it comes down to it, it doesn't work in real teaching. … Kids can't do step 42 if they don't get step 41, and so you need to teach everything step by step. This is hard with 30 kids in the class but streaming definitely helps. … The students have to know the basics.

Interestingly, although Brad expressed little enthusiasm for the mathematical pedagogy he experienced on his practicum, his views about mathematics and mathematics teaching seemed to be consistent with those of his associate teacher.

Discussion

While this study has only focussed on the experiences of two preservice primary school teachers, there appears to be some important features that have emerged. Firstly, the participants perceived their school mathematics as a negative experience, which caused them to develop feelings of anxiety and inadequacy. This finding is indeed sad and disappointing, particularly when it appears to be consistent with the majority of the literature (eg., Carroll, 1994; Grootenboer & Lowrie, 2002). Furthermore, their experiences were consistent with absolutist and instrumental views of mathematics as is evidenced by the emphasis they placed on times-tables and "the basics". Even though Brad had been at school immediately prior to starting his initial teacher education and Marina had not been to school for at least twelve years, their school experiences were remarkably similar, and this was despite a major curriculum reform in between. Both of their experiences were characterised by teaching that was narrow and based on a transmission-model, which again is consistent with the findings of others (eg., Biddulph, 1992). These experiences had facilitated the development of Brad and Marina's affective views of mathematics, which they brought to their tertiary teacher education program.

It was heartening to note the changes in Marina and Brad's affective dispositions after their participation in their tertiary mathematics education course. This was probably a predictable outcome, and indeed one would be worried if such a course facilitated no growth in beliefs about mathematics, or change in attitude towards the subject. However, it was fascinating to note the continuing strong conviction and commitment to times-tables as being important. A number of other studies have reported reform in preservice teachers affective views through tertiary mathematics education courses, but perhaps what is more important is if these changes are significant enough to last into the teachers' inservice careers. The strength of the beliefs, attitudes and feelings that preservice teachers develop as school students during their "apprenticeship of observation" render any long-term effect from their tertiary courses problematic (Richardson, 1996). Indeed, there is scope for longitudinal research to explore the long-term impact of tertiary mathematics education courses. The current study did however, follow the preservice teachers through one post-course experience, namely the school practicum.

The participants' experiences had been similar in many respects up to this point, but during their school placements they encountered mathematics education in different ways. For Brad and Marina, their experiences were significant events in their affective development and they indicate the power of the practicum placement to confirm or deny the learning of the tertiary course. This has been noted by a number of authors who have expressed concern over the possible miseducative nature of school practicums (eg., Calderhead, 1988). Clearly for Brad this was the case as his experiences were confirming of his views prior to entering his initial teacher education program. Perhaps it was because his practicum experiences matched up with his own school experiences, that this reality was too strong for him to conceive of mathematics, mathematics teaching and learning in other ways. For Marina however, her practicum experiences continued to support her new perspective of mathematics and she was able to continue her affective growth in more positive and healthy ways. Interestingly, she was still able to note the struggle she had as she fought against her previous views that she described as her "default mode".

Although this study was only small, the weight of the literature in general means that mathematics educators need to accept, at least at present, that many preservice teachers are going to start their initial teacher education with poor and/or negative views of mathematics. Given that, a fundamental concern of their mathematics education courses needs to be affective reform, otherwise teachers will begin their teaching careers with beliefs, values, attitudes and feelings towards mathematics that hinder positive and effective mathematics teaching and learning. Furthermore, there needs to be some thought given to how preservice teachers' views about mathematics and mathematics teaching develop through their practicum experience, and how these can be reviewed if their mathematics education coursework is over. Finally, there is a great need for some longitudinal research into the long-term changes and developments in teachers' affective views towards mathematics as they progress through their teaching careers.

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One Student's Understanding of the Concept of Function

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This study aimed to investigate the ways in which one student developed his understanding of concept of function and the use of his concept image of function when solving problems. This student's responses from a function concept questionnaire, a function test and interviews were classified into one of three categories: action, process or object concept of function.

It is widely accepted by mathematics educators and teachers of mathematics at all levels that the concept of a function plays an important role in the algebra curriculum. The improvement of instructional strategies to promote higher levels of student understanding of the function concept is a key goal of the secondary school mathematics curriculum. Researchers including Breidenbach, Dubinsky, Hawks and Nichols (1992) and Sfard (1991) have attempted to find better ways of building students' understanding of the function concept. It has been suggested that using a graphing calculator and a computer can assist students to construct a richer understanding of the function concept (Breidenbach et al., 1992; Dubinsky & Harel, 1992; Hollar, 1996; O'Callaghan, 1998; Slavit, 1994).

This study is part of a wider study of the introduction of graphing calculators to Thai secondary mathematics students. The study investigates the ways in which the students' develop the concept of function and describes their learning in terms of the "action-process-object" framework (Asiala, Brown, De Vries, Dubinsky, Matthews & Thomas, 1996).

The Concept of Function

Vinner and Dreyfus (1989) classified the mathematical concept of function into two distinct aspects, concept definition and concept image. A concept definition is a formal mathematical definition of the concept. It is directly introduced to the students in words of phases. On the other hand, a concept image is a set of all the mental pictures, and a set of all the properties associated in the student's mind with the concept name. The concept image includes the visual representations, experiences and impression evoked by the concept name (Thompson, 1994). Vinner and Dreyfus (1989) stated that the student's concept image results from experiences with the concept, and is not necessarily the same as the concept described by the definition. In order to handle the concept, according to Vinner (1983), a student needs a concept image rather than a concept definition. Therefore, it is important to investigate the student's images of mathematical concepts. This study supports this claim by exploring a student's concept image of the concept of function.

Dubinsky and his colleagues (Asiala et al., 1996; Dubinsky, 2000) proposed the APOS theory developed from Piaget's idea of reflective abstraction. In the APOS theory, the construction of a mathematical concept begins with manipulating previously constructed mental or physical objects to form *actions*. An action is a transformation of mental or physical objects and the individual can carry out the transformation only by reacting to

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external cues that give precise details of the steps to take. A student holding an action concept is able to calculate values of a function for specific values from the domain, or is able to find specific points on a graph representing the function.

For an interiorized *process*, an individual reflects on the action and constructs an internal operation that the individual is able to describe, or reflect on, as the steps in the transformation without necessarily performing them. At this stage, a student is able to think of function as receiving one or more inputs, or values of the independent variable, perform one or more operations on the inputs and return the results as outputs or values of the dependent variable. The student also thinks of the function as a formula, graph, table of values, mapping diagram, or function machine. The constructed process can be reversed or it can be coordinated with other processes to form a new process.

An *object* is constructed through the encapsulation of a process. A student becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such a transformation. As an object, function is thought of as the correspondence between two variables. The student can de-encapsulate the object back to the process concept in order to manipulate it to form a new process. In this stage, the student also thinks of a function as something to which actions or processes may be applied. Finally, the actions, processes and objects can be organized in *schema*. According to Vinner and Dreyfus (1989), a concept image comprises the set of all mental pictures associated in the student's mind with the concept name, and all of the properties related to them. Moschkovich, Schoenfeld and Arcavi (1993), and Slavit (1994) stated that a student who understands the function concept as an object can use a function and its properties in solving a novel problem. Slavit (1994) also stated that a student who perceives the function concept as an object has the ability to realize the equivalence of procedures that exist in different representations and to generalize procedures across function classes.

It is possible that a student may think about a function in one manner, but uses a different concept of function to solve problems (Slavit, 1994). Slavit stated that a "student may talk in general about function as a relation between two sets, but that same student may think of only specific functional properties when problem solving" (p. 50). This implies that investigating students' thinking related to the concept of function should incorporate a range of situations and activities to allow for the display of the full range of thinking.

Methodology

The study reported in this paper investigated one student from a wider study of 13 year ten (approximate age of 15-16 years) students who studied in the first semester of academic year 2001 in a Thai secondary school. The study was not part of the regular teaching program and was conducted after school time. The researcher taught the class. The teaching experiment consisted of a series of teaching episodes over 7 weeks, with 2 one and a half-hour episodes per week. The following topics were addressed in the course: introduction to a graphing calculator, relation, function, linear function and linear equation. A constructivist approach incorporating peer collaboration and the use of a graphing calculator was applied. Within each teaching episode, the students worked on the learning tasks in groups of two or three. They performed the tasks by discussing and sharing information with their group-mate. In each session, some groups were asked to present their answers to the class using an overhead projector. At the end of the study, a function concept questionnaire and a function test were administered, and three students were interviewed. The interviews were videotaped. The investigation reported here focuses on one of the three students interviewed, namely "Piya". Piya complete year 9 with a grade point average 3.43 (4 point-scale) and received "A" in the mathematics course. He had no experience in using a graphing calculator either in and out of school and he had not been introduced to the function concept in his earlier mathematics studies.

Two written instruments were used to gather data. The function concept questionnaire consisted of two parts. The first focused on students' concept definition of function as well as asking for one example of a function in a practical situation. The second part focused on students' concept image of function, having them classify various relations given as either algebraic equations, or tables of values, or graphs or sets of ordered pairs. Students' responses were elaborated on in the interviews.

The function test was developed from tests used by O'Callaghan (1998), Hollar (1996), Eisenberg and Dreyfus (1994), and Slavit (1994). The test was designed to assess each of the following aspects of the conceptual knowledge of function: interpreting functions (5 questions); modelling functional problem situations (4 questions); and transforming between different representations and within the same representation of a function (4 questions). Students' responses were also elaborated on in the interviews. The interviews were conducted on the next two days following the test.

Based on the APOS theoretical framework, all student's responses from the function concept questionnaire, the function test and the interviews were classified into one of three categories: action concept, process concept or object concept. The responses of the one student who was the focus of the study reported here were classified and then an overview of the student's concept of function was constructed.

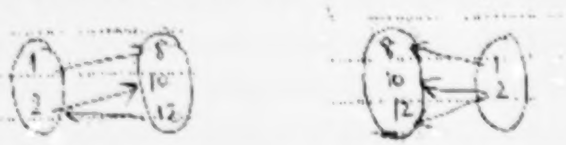
Results

Concept Definition and Concept Image of Function

The student's responses from the function concept questionnaire were used to explore the student's concept definition and concept image of function during the interview.

The student, named Piya, gave the definition of a function as:

> Two sets of numbers map with each other, such that for every element in the first set maps only one time with an element of the second set, but each element in the second set can be mapped with elements in the first set more than one time.



Not a function A function

Figure 1. Piya's examples of a function.

Piya's response indicated that he held an *object* concept definition of function. He thought of a function as a relationship between two sets of numbers, a mapping, with a univalent property. His concept definition of function was limited to the mapping between two sets of real numbers as he was unable to expand on this definition in the interview. He gave two examples (Figure 1) using mapping diagrams to support his definition, but the directions of the arrow lines were mixed. In the first example, he used an arrow line directed from 12 to 2. It appears that he intended to show that the number 2 mapped with 10, and if 12 was also mapped with 2, then the mapping diagram did not represent a function. In his explanation he stated: "…2 with 10 and it cannot map with 12, one number of the first set cannot map with two numbers of the second set". As in the first diagram, Piya used arrow lines in the wrong direction for the second mapping diagram. He explained: "1 maps 8, but one element of the second set, 2, can map with elements of the first set more than once, because from the definition each element of the first set can map with only one element of the second set". He appears to be using a learned definition but is confused about the meaning of a "many to one" relationship.

Piya's answers to the second part of the function concept questionnaire involving the identification of functions showed that he believed that parabolas, represented by either a graph or an equation were not functions because they are not one-to-one. When talking about the graph of a parabola, Piya checked first with a vertical line and then checked again with a horizontal line.

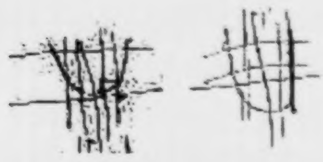


Figure 2. Piya's working on parabolas.

TR (Teacher researcher): Why do you answer that $y = x^2-2x+5$ is not a function?

P: *Its graph is a parabola. The parabola makes two points.* These points are the points that the horizontal line crosses the graph as the picture. (Later) I think all parabolas are not functions. Although, a vertical line passes the graph at only one point, a horizontal line passes the graph at more than one point.

The one-to-one relationships belief was also apparent in Piya's explanation of whether a relationship expressed in a table was a function, many-to-one relationships being classified as not functions. However, Piya classified a discontinuous graph with a split domain (Figure 3) as a function using the vertical line test even though it was a many-to-one relationship. For other non-linear graphs, Piya applied the vertical line test first. If vertical lines met the graph in only one point, he then applied a horizontal line test as he had for a parabola.

$$y = \begin{cases} 0; & x < 1 \\ x; & 1 \le x \le 3 \\ 1 - x; & x > 3 \end{cases}$$

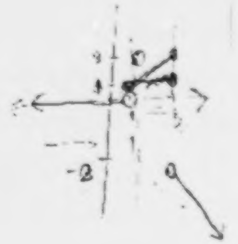


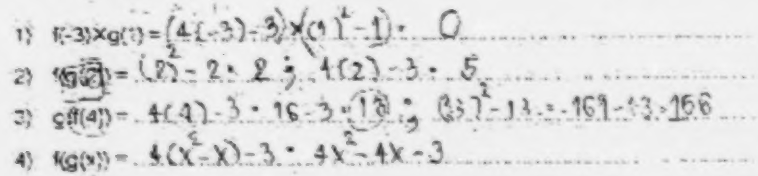
Figure 3. Piya's use of the vertical line test

Concept Image of Function Used in Solving a Problem

Interpreting a problem situation. Piya had not been exposed to the composition of functions during the instructional period. He demonstrated his ability to master a problem situation involving the composition of functions. His work and transcript of interview were as follows.

Problem: Let $f(x) = 4x-3$ and $g(x) = x^2-x$, find the followings:

1) $f(-3) \times g(1)$ 2) $f(g(2))$ 3) $g(f(4))$ 4) $f(g(x))$



1) $f(-3) \times g(1) = (4(-3)-3) \times (1^2-1) = 0$

2) $f(g(2)) = (2)^2-2 = 2 ; 4(2)-3 = 5$

3) $g(f(4)) = 4(4)-3 = 16-3 = 13 ; (13)^2-13 = 169-13 = 156$

4) $f(g(x)) = 4(x^2-x)-3 = 4x^2-4x-3$

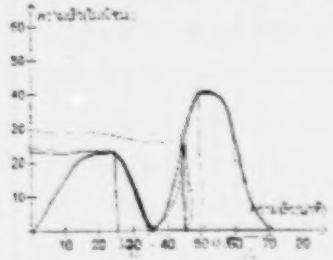
Figure 4. Piya's work on composition of functions.

P: To find $f(-3) \times g(1)$, firstly find $f(-3)$ by substituting $x = -3$ in f and secondly, find $g(1)$ by substituting $x = 1$ in g. Then $f(-3) \times g(1) = [4(-3)-3] \times [(-1)^2-1] = 0$. Because the second term equals zero (0). To find $f(g(2))$, find $g(2)$ first, we get 2, and substitute $x = 2$ in the equation of f and get 5. To find $g(f(4))$, we find $f(4)$ first and get 13. Then find a value of g by substituting $x = 13$ in the equation of g and we get $g(f(4)) = 156$.

Piya exhibited his ability to objectify his understanding of the function concept. He thought of the function as leading to actions or process, for example calculating $f(4) = 13$ and $g(f(4)) = g(13) = 156$. Later in the test and interview, Piya manipulated the perceived object to find a new object when he substituted $g(x) = x^2 - x$ in $f(g(x))$ and simplified. That is, he created a new process (composition of functions) which required an object view of function.

The following problem involved interpreting a function represented by a graph.

Problem: The graph below gives the speed of a cyclist on his daily training ride. During his ride, he must climb a hill where he pauses for a drink of water before descending. Use this graph to answer the following questions as accurately as possible.



1. Find the time when speed equals 30 mph.
2. During what time intervals was the speed increasing?
3. During which 10-minute interval did the speed decrease most?
4. When was the cyclist at the top of the hill?

Figure 5. Piya's work on a *v-t* graph.

Piya demonstrated his process and object conceptions of function when solving question 3, 4 and 5. A misconception was shown in interpreting the graph for question 5.

TR: Great. For question 3 your answer is the time interval 0 to 20 minutes and 35 to 50 minutes. Why?

P: Within the interval 20 to 25 minutes, the speed is almost constant.

TR: For the question 4, why do you think that the three intervals--20 to 30, 30 to 40 and 60 to 70 are the intervals that the speed decreased most?

P: Can I change my answer? The interval that the speed decreased most is the interval 55 to 65. When compared with the interval 25 to 35 minutes that the speed is increasing.

TR: When is a cyclist at the top of the hill?
P: From the graph, in the interval 50 to 60 minutes the cyclist is at the top of the hill.

Working with question 3, Piya demonstrated a process concept of function. He was able to interpret the graph in terms of interval reading. He could find the time interval that the speed increased. While doing question 4, Piya coordinated two processes, speed during the times 25 to 35 and 55 to 65 minutes in which the speed decreased, and by comparing a changing rate of the speed between the two intervals, chose the interval that the speed increased most, the interval 55 to 65 minutes. In question 5, Piya displayed his object understanding of function. He considered all points or all features of the graph (whole graph reading), and answered that in the interval 50 to 60 minutes the cyclist is at the top of the hill. It was an incorrect answer. His misconception was that the graph represented the picture of the hill and that the maximum point of the graph represented the top of the hill. He did not link the given information from the situation that the cyclist stopped riding for a drink of water at the top of the hill. That was, at the top of the hill the speed of the cyclist was zero miles per hour.

Modelling the problem situation. The following problem required students to model the situation with equations.

Problem: On your new job, you can be paid in one of the two ways
Plan A: A salary of 50,000 baht.
Plan B: A salary of 35,000 baht plus a commission of 5% on gross sales.
For what gross sales is plan B better than plan A?

From his work, Piya displayed his understanding of function as an object. He knew that the problem situation could be represented with linear equations. He assigned a variable x as a monthly salary, and y as gross sales and constructed equations representing the relationships between the two variables for plan A and plan B. In order to find the amount of gross sales making plan B better than plan A, he then created an equation representing the situation by setting gross sales for plan A equal to gross sales for plan B and solved for x. It indicated that Piya viewed functions in the form of equations as objects on which he could operate. Unfortunately, he obtained the wrong answer. Mistakes were made in the process of formulating equations representing the problem situation. Instead of setting the amount of gross sales for x and the monthly salary for y, he set the monthly salary as x and the gross sales as y ($y = 50000x$ for Plan A and $y = 35000x - 20$ for Plan B). When asked for reasons, Piya could not explain why. He said, "I have no ideas about these".

Translating between different functional representations. Problems involving translating between different functional representations are usually solved in one of two ways, using a local strategy or a global strategy. The local translation strategy, or point-wise translation strategy, involves the student in checking several values of the independent variable in each representation and determining if they produce the same values of the dependent variable. This approach corresponds to a process view of function. The global translation strategy, an object view of function, involved the student using function properties and behaviours in the processes of translating. For example, when translating between two representations of a linear function, the student realizes that each has the same slope, and then checks the y-intercept or any other point. In the interview, Piya demonstrated a view of function as a process concept. He used only the local point-checking strategy when translating between different representations. The following are some of Piya's comments during the interview.

Problem: Determine which of the following functions are the same (Figure 7).

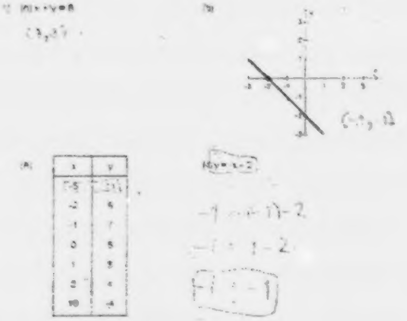


Figure 7. Piya's work during the interview.

TR: Do you think functions in (a) and (c) are the same?

P: Compare the table in (c) with the expression in (a), you see, sum of each pair of x and y in the table is equal to 6. That is, data in the table satisfy the equation $x+y=6$.

TR: Why are the graph in (b) and equation $y = -x - 2$ in (d) equal?

P: From (d), a graph of the function $y = -x - 2$ has the same point that the graph crosses the x-axis--for $y = 0$ we get $x = -2$, as the graph in (b). Let's check a point (-1, -1) on the graph in (b). The point (-1, -1) satisfies the equation $y = -x - 2$ in (d). Then the function in (b) is equal to the function in (d).

In comparing the function representations, Piya called on his process view of function. However, when asked for an explanation of the general linear function $y = mx + b$, he displayed his object concept understanding of function by stating:

> The constant m represents the slope of a straight line. If m is positive, the line makes an acute angle against the x-axis. If the constant m is negative, the line makes an obtuse angle against the x-axis....[The constant b] tells the y-intercept of the graph. That is a point (0,b). For example, an equation $y = 5x+6$ passes a point (0,6).

Discussion and Conclusion

This study has provided an insight into one student's concept definition and the concept image of function. Vinner and Dreyfus (1989) stated that a student's concept image used in problem situations is not necessary the same as the concept determined by the definition. Students may discuss and think of functions in one manner, and use different conceptions of function to solve problems. Piya's concept definition of function was interpreted as an object concept. He defined the function as a relationship (mapping) between members of two sets of numbers under a univalent property. On the other hand, Piya used a variety of concept images to solve the given problem situations. In an identification of function task, Piya's concept image of function included it being a one-to-one relationship in some cases and a relation with a univalent property in other cases. Discontinuous graphs, graphs for which equations could not be specified, and equations with a split domain could be considered as function if they satisfied one of his concept images of function. In the interpretation of problem situations, he thought of a function as something to which actions and processes could be applied, and the properties and behaviours of the function were considered. In the modelling task, Piya displayed an object concept of function. He viewed functions as objects, which could be compared and operated on. Within the translation tasks, Piya used a local translation strategy, the point-checking strategy, to translate between different functional representations using a process concept. While Piya displayed a wide range of concept images and properties involving actions, processes and objects related to the function concept, his schema was at this stage immature. He was confused about a number of key ideas but his responses in the interview indicated that he was able to resolve those confusions with a little help from the teacher/researcher.

The study also indicated that perceiving a process or object concept of function did not guarantee the student's success in a problem solving situations. Success required the student to link perceived concept images of function with related processes needed in the problem solving situations. As Hiebert and Carpenter (1992) stated "understanding involves recognising relationships between pieces of information" (p. 67). Further research could investigate students' ways of using the concept of function in a wider range of problem solving situations.

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Inaccurate Mental Addition and Subtraction: Causes and Compensation

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This paper reports on a study of seven Year 3 students' diminished performance in addition and subtraction mental computation. Although all students were identified as being inaccurate, three students used some variety of mental strategies, while the other students used only one strategy that reflected the written procedure for each of the addition and subtraction algorithms taught in the classroom. Interviews were used to identify students' knowledge and ability with respect to number sense (including numeration, number and operations, basic facts, estimation), metacognition, affects, and memory. Two conceptual frameworks were developed, one representing the flexible mental computers, and the other representing the inflexible mental computers. These frameworks identified factors and relationships between factors that influence flexibility in these inaccurate mental computers. The frameworks were compared with a framework of an ideal mental computer. These frameworks showed that inaccuracy resulted from disconnected and deficient cognitive, metacognitive, and affective factors; and in some cases might have been affected by deficient short-term memory. It appeared that students' choices of mental strategies resulted from different forms of compensation for varying levels of deficiencies.

Researchers and educators have stressed the importance of including mental computation in number strands of mathematics curricula (e.g., McIntosh, 1996; Reys & Barger, 1994; Sowder, 1990; Treffers & de Moor, 1990; Willis, 1990). Reasons for its inclusion are that mental computation: (1) enables children to learn how numbers work, make decisions about procedures, and create strategies (e.g., Reys, 1985; Sowder, 1990); (2) promotes greater understanding of the structure of number and its properties (Reys, 1984); and (3) can be used as a "vehicle for promoting thinking, conjecturing, and generalizing based on conceptual understanding" (Reys & Barger, 1994, p. 31). In effect, mental computation promotes number sense (National Council of Teachers of Mathematics, 1989; Sowder, 1990). In fact, Willis (1992) suggested that mental computation should be the main form of computation, with written computation to serve as memory support. However in the existing Queensland curriculum document, *Years 1 to 10 mathematics teaching, curriculum and assessment guidelines* (Department of Education, Queensland, 1987), addition and subtraction mental computation is not mentioned.

Proficiency in mental computation has been the focus of several research projects (e.g., Beishuizen, 1993; Heirdsfield, 1996; McIntosh & Dole, 2000; Reys, Reys, Nohda, & Emori, 1995). In The Netherlands, where mental computation is taught before written computation, mathematics programs emphasise the use of aggregation (N10) as a more efficient mental strategy. However, weaker students tended to use less efficient separation strategies (Beishuizen, 1993). Reys, Reys, Nohda, and Emori (1995) found that accuracy in mental computation was associated with strategies other than mental image of pen and paper algorithm. In contrast to these findings, McIntosh and Dole (2000) reported higher accuracy when students employed mental image of pen and paper algorithm than when they employed alternative mental strategies (although these alternative strategies revealed number sense). Heirdsfield (1996) also found that accuracy in mental computation did not need to be accompanied by employment of a variety of efficient mental strategies.

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Therefore, while some research appears to indicate that accuracy in mental computation is a result of efficient mental strategies, other research has reported accuracy as a result of employment of strategies that reflect pen and paper algorithms.

Research undertaken by this author investigated mental computers and the factors that supported accuracy and flexibility (Heirdsfield, 1998, 2001a, 2001b). This study investigated the part played by number sense knowledge (e.g., numeration, number facts, estimation and effects of operations on number), metacognition (metacognitive knowledge, strategies and beliefs), affects (e.g., beliefs, attitudes), and memory (working memory – Baddeley, 1986 – and long term memory) in mental computation. It showed that students proficient in mental computation (accurate and flexible) possessed integrated understandings of number facts (speed, accuracy, and efficient number facts strategies), numeration, and number and operation. These proficient students also exhibited some metacognitive strategies and metacognitive beliefs, and affects (e.g., beliefs about self and teaching) that supported their mental computation. Further, proficient mental computers had good short-term recall to hold interim calculations and recall number facts, and well developed central executive (in working memory) to attend to the demanding task of mental computation and retrieve strategies and facts from a well-connected knowledge base in long term memory. Proficient mental computers chose alternative and efficient strategies, as they possessed extensive and connected knowledge bases to support these strategies. Thus, there was evidence of the importance of connected knowledge, including domain specific knowledge, and metacognitive strategies, affects and memory for proficient mental computation. As a result of this study, a conceptual framework identifying associated factors involved in proficient mental computation was developed (Heirdsfield, 2001b).

This leads to the question as to what are the effects on mental computation of less knowledge and fewer connections? It would be expected that one effect would be less accuracy. The purpose of this paper is to report on seven students who were inaccurate in addition and subtraction mental computation, but they exhibited differing levels of flexibility of mental strategies. These students were participants in a study that investigated addition and subtraction mental computation in Year 3 students (Heirdsfield, 2001c). Conceptual frameworks for these inaccurate students are developed and compared with a framework for the "ideal" mental computer (flexible and accurate) that was developed in the large study from which these students were drawn.

Method

Participants

The participants were seven students, selected from a population of sixty Year 3 children from three classrooms, representing two Independent schools in Brisbane. Both schools served students from high and middle socio-economic areas. The students were selected on the basis of an interview that identified accuracy and flexibility in mental computation. Although all students were considered inaccurate, three students employed some variety of strategies; therefore they were identified as flexible. The other four students employed a single strategy consistently. Therefore, these four students were identified as inflexible.

Instruments

The students were presented with a series of tests and indepth interviews. These were number fact knowledge, mental computation (one-, two-, and three-digit addition and subtraction), computational estimation, numeration, effects of operation on number, and memory. In order to choose neuropsychological tests relevant to these aspects, Lezak (1995) was consulted. The neuropsychological tests, which were used with the Year 3 students, were aimed at investigating short-term recall and executive functioning. The tests were modifications of a Digit Span Test (short-term recall) and a maze test (central executive, e.g., planning and attention).

Further, questions were asked addressing attribution, self-efficacy, beliefs, and metacognition. The students were also required to complete the *Student Preference Survey* (*SPS*) (McIntosh, 1996), to identify whether they would and could solve computational tasks mentally.

Procedure

The students were withdrawn from their classroom on a one to one basis, and interviewed in a quiet room. The interviews were videotaped, and each interview session lasted for no more than 30 minutes at a time. Because of the variety of aspects covered, each child received four interview sessions (three sessions in the pilot study).

Analysis

For the purposes of identifying flexibility in mental computation, mental computation strategies were identified using the categorisation scheme (based on Beishuizen, 1993; Cooper, Heirdsfield, & Irons, 1996; Reys, Reys, Nohda, & Emori, 1995) that divided the strategies into the following categories: (1) *separated* (e.g., 38+17: 30+10=40, 8+7 = 15 = 10+5, 40+10+5 = 55 – *separation left to right*; 38+17: 30+10=40, 40+8=48, 48+9=57 – *cumulative sum*); (2) *aggregation* (e.g., 38+17: 38+10=48, 48+7 = 55); (3) *wholistic* (e.g., 38+17 = 40+17-2 = 57-2 = 55); and (4) *mental image of pen and paper algorithm* – following an image of the formal setting out of the written algorithm (taught to almost automaticity in the schools the students attended).

The analysis of the interviews incorporated three stages. First, each interview for each student was analysed separately. Second, relationships across interviews for each student were considered (e.g., whether understanding of the effect of operations on number was used for mental computation, whether the same number facts strategies were employed in both the number facts test and in the mental computation interview). Third, analysis compared commonalities and differences across students.

Mental computation responses were analysed for strategy choice, flexibility, accuracy, understanding of number facts, computational estimation, numeration, and the effects of operations on number. It was also noted whether the students could access alternative mental strategies, when encouraged to do so. Number facts were analysed for accuracy, speed and strategy choice. Estimation strategies were identified and proficiency and flexibility were noted. Analysis of students' responses to numeration tasks was based on Ross's five levels (1986), which included canonical and noncanonical understanding of number. Also, evidence of multiplicative understanding (e.g., ten tens are the same as one hundred) was investigated. The tasks addressing the effect of operations on number were analysed for understanding of arithmetic properties (e.g., associativity, inverse, the effect of changing the addend and subtrahend) as they apply to computational relationships (e.g., 70-43=27, ∴70-44=26).

For the analysis of the memory tests, Lezak (1995) was consulted. As so few students were interviewed, it was decided to compare individuals' raw scores for the Digit Span Test, and note any trends with memory problems evident in mental computation tasks. The maze completion times and the number of errors, such as retracing lines or entering blind alleys, were recorded. Although there might be a tenuous link between executive functioning in completing mazes and executive functioning in completing mental computation tasks, the fact that a student could attend to a task and plan would affect mental functioning in any domain. Evidence for planning and decision-making was compared with the same in the mental computation interviews.

Results

Comparison of Flexible and Inflexible Inaccurate Mental Computers

Although all students were inaccurate, there were differences in the mental computation strategies they employed. These strategies appeared to be used to compensate for limited and disconnected knowledge. A comparison between the flexible and inflexible students is made in Figure 1.

	Not accurate and flexible (n=3)	Not accurate and inflexible (n=4)
Mental computation	Strategies: Separation strategies only (left to right, right to left, cumulative). Alternative strategies: yes (including wholistic), but not always successful.	Strategies: Used mental image of pen and paper algorithm. Alternative strategies: yes, but rarely successful
Number facts	Accuracy: accurate for addition, generally slow and inaccurate for subtraction Strategies: Derived Facts Strategies (DFS), recall, and count. Used count mostly in mental calculations.	Accuracy: inaccurate and slow Strategies: DFS (very few though), recall, and count. Used count mostly in mental calculations.
Computational estimation	Varied	Mostly poor.
Numeration	Varied, canonical, noncanonical, proximity of numbers.	Generally poor, mostly only canonical understanding with material.
Number and operation	Mostly poor.	Mostly poor.
Metacognition	Some strategies, mostly inaccurate beliefs.	Mostly no strategies, inaccurate beliefs.
Affects	Varied beliefs and no strong beliefs evident.	Varied beliefs.
Working Memory:	Evidence of central executive (planning and attention). ST recall score: ≥ 6	Evidence of diminished central executive. ST recall score: ≤ 5

Figure 1. Comparison of flexible and inflexible inaccurate mental computers.

In general, both groups of students lacked sufficient understanding of number facts, estimation, numeration, and number and operation to support advanced mental computation strategies. Although the students were unsuccessful with the taught procedures, the flexible students attempted to compensate by inventing strategies, although most (but not all) of these strategies were not high order strategies. Some number facts strategies, numeration understanding, and metacognitive strategies assisted the employment of alternative mental calculation strategies. Thus, the flexible students attempted to compensate for their lack of procedural understanding, but their knowledge was diminished and disconnected. On the other hand, the inflexible students attempted to compensate by employing the teacher taught strategy, which required little conceptual understanding (and also provided a mental image to support memory). However, because of a lack of procedural understanding, errors still resulted. None of the inflexible students held accurate perceptions of their mental computation abilities. Also, poor short-term recall and diminished executive functioning compounded these deficiencies. In other words, their knowledge was so deficient and disconnected, even strategies that had been taught (but not learnt) could not be followed.

Results of Flexible Students

The three flexible students used some variety of strategies, but few high order strategies. Most strategies were *separation* (*left to right*, *right to left*, and *cumulative sum/difference*). Although, one student also attempted to employ *aggregation left to right* and *wholistic* strategies for addition in the indepth interview.

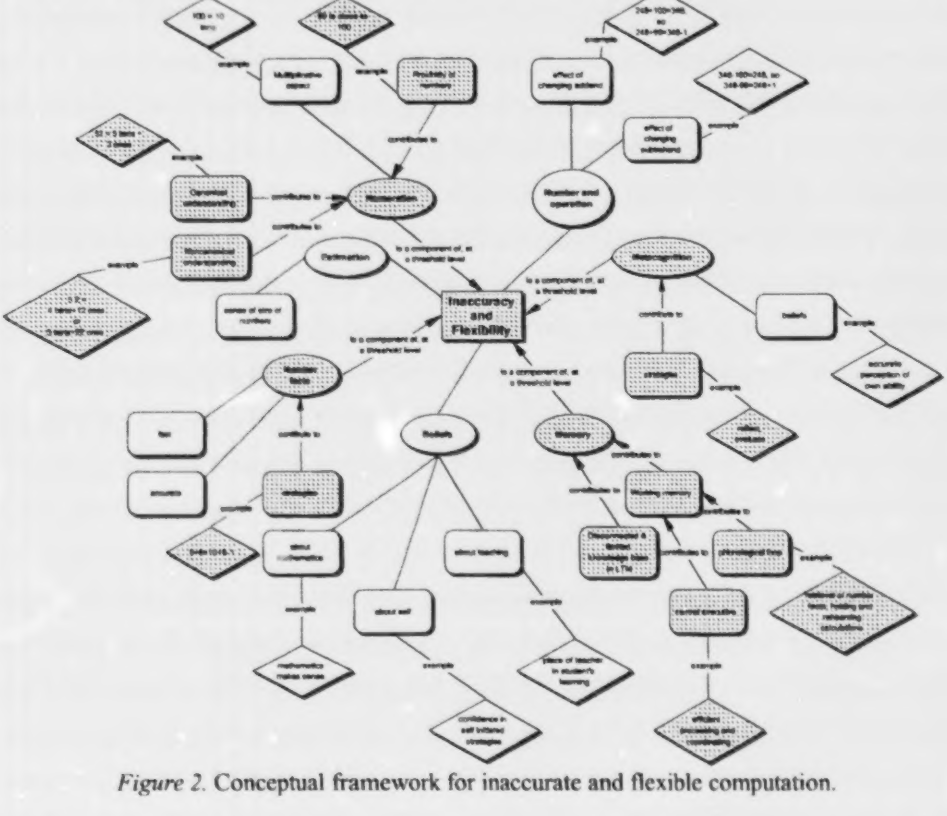
Inaccuracy and Flexibility
Numeration
Estimation
Number and operation
Metacognition
Number facts
Beliefs
Memory
Multiplicative aspect
effect of changing addend
effect of changing subtrahend
sense of size of numbers
beliefs
strategies
accurate perception of own ability
about mathematics
about teaching
about self
mathematics makes sense
place of teacher in student's learning
confidence in self initiated strategies
Working memory
phonological loop
central executive
efficient processing and coordinating
example
contributes to
is a component of, at a threshold level

Figure 2. Conceptual framework for inaccurate and flexible computation.

In general, the students lacked sufficient understanding of number facts (although some efficient strategies were employed), numeration, and number and operation to support advanced mental computation strategies. Estimation was poor and, thus, did not support mental computation. Further, these students were unsuccessful with the taught computational procedures, so they compensated by inventing strategies, although most (but not all) of these strategies were not high order strategies. Some numeration understanding

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and metacognitive strategies assisted mental calculation using alternative strategies. The students were unable to use teacher taught procedures to compensate for their deficiencies.

Thus, lack of procedural understanding of the pen and paper procedures resulted in the students' inventing mental strategies. However, as they did not possess sufficient understanding of number facts and number and operation, they were rarely successful. Some numeration understanding and metacognitive strategies assisted the invention of mental strategies, however, there was insufficient understanding to support high-level mental strategies. Thus, they attempted to *compensate* for their lack of procedural understanding, but their knowledge was *disconnected*. A conceptual framework for the inaccurate and flexible students is presented in Figure 2. This framework was developed in comparison to that of a proficient mental computer (Heirdsfield, 2001b). To show missing factors, cells are left unshaded. Dappled effect in cells indicates that knowledge was at a threshold level (compared with proficient mental computers). Arrowheads are eliminated to show disconnected knowledge.

Results of Inaccurate and Inflexible Students

The four inflexible students predominantly employed *mental image of pen and paper algorithm*. They all reported "seeing" or imagining the vertical format of the pen and paper algorithms.

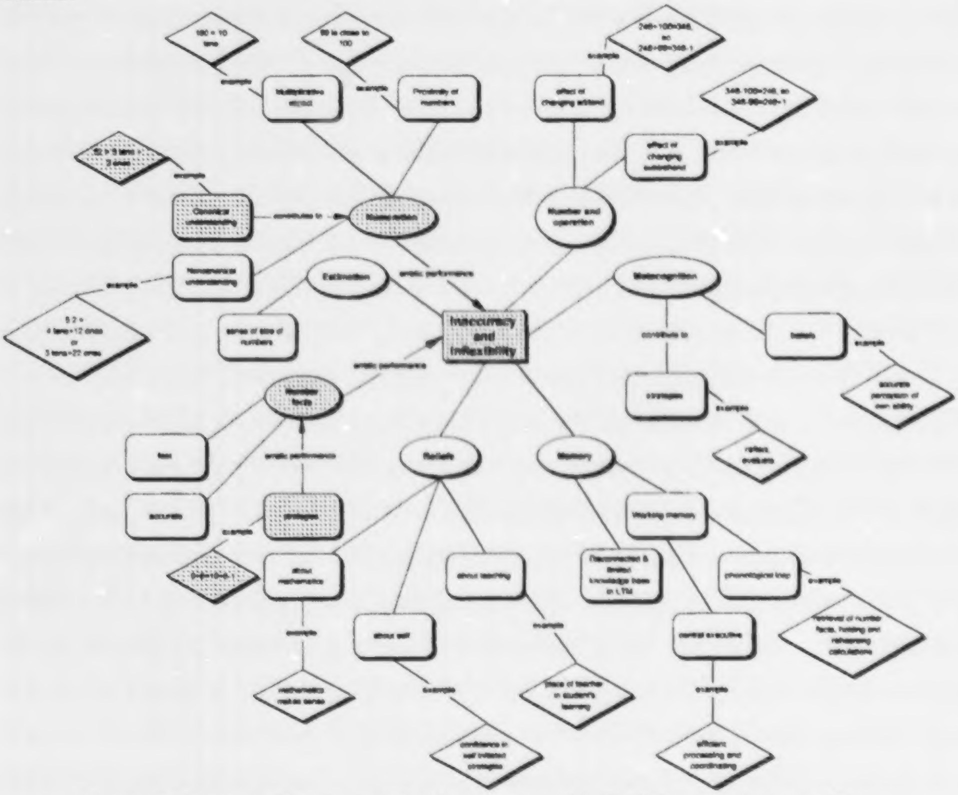
Inaccuracy and Inflexibility
Numeration
Estimation
Number and operation
Metacognition
Number facts
Beliefs
Memory
erratic performance
contribute to
example
100 = 10 tens
99 is close to 100
52 = 5 tens + 2 ones
52 = 4 tens+12 ones or 3 tens+22 ones
246+100=346, so 246+99=346-1
346-100=246, so 346-99=246+1
effect of changing addend
effect of changing subtrahend
sense of size of numbers
fast
accurate
strategies
beliefs
accurate perception of own ability
reflect, evaluate
about mathematics
about self
about teaching
mathematics makes sense
confidence in self initiated strategies
Role of teacher in student's learning
Working memory
Disconnected & limited knowledge base in LTM
phonological loop
central executive
Retrieval of number facts, holding and rehearsing calculations
efficient processing and coordinating

Figure 3. Conceptual framework for inaccurate and inflexible mental computation.

In general, poor number facts knowledge (i.e., inaccurate), and poor understanding of estimation, numeration, and number and operation contributed to inaccuracies in mental computation, and the inability to access alternative strategies. Also, poor short-term recall and diminished executive functioning compounded these deficiencies. To compensate for a poor knowledge base and memory overload, they attempted to employ an automatic strategy, which required little conceptual understanding (and also provided a mental image to support memory). However, because of a lack of procedural understanding, errors still resulted. None of these inflexible students held accurate perceptions of their mental computation abilities. Finally, the inflexible students exhibited little or no understanding of any of the factors investigated in relation to mental computation, resulting in *deficient and disconnected* knowledge.

In summary, the students exhibited *deficient and disconnected* knowledge. To compensate, the students resorted to an automatic strategy, but lack of procedural understanding and other deficiencies resulted in inaccurate application of this strategy. A conceptual framework for the inaccurate and inflexible students is presented in Figure 3. This framework was developed from that of the proficient mental computer. To show missing factors, cells are left unshaded. Dappled effect in cells indicates that knowledge was at a threshold level (compared with proficient mental computers). Arrowheads are eliminated to show disconnected knowledge.

Discussion

Some key differences between the two groups of inaccurate computers were number fact strategies, numeration understanding, metacognitive strategies, and working memory. It was interesting how different forms of compensation resulted from different knowledge bases.

It would seem obvious that knowing number facts by immediate recall would aid in mental computation, as there would be less memory load. Many of the flexible students (and to less extent the inflexible students) used efficient *Derived Facts Strategies* (c.f., *count*) when they could not recall number facts in the number facts test. However, they resorted to *count* for calculating interim calculations during mental computation. The reason might lie in the extra load placed on working memory when *DFS* are employed. *Count* seems to be a more primitive strategy that they and the inflexible students resorted to for interim calculations. Therefore, permitting students to use pen and paper would alleviate working memory load due to lack of number fact knowledge. Further, students should be encouraged to develop efficient *DFS*, where understanding rather than speed is a focus.

Some numeration understanding supported alternative low-level mental computation strategies. However, overall, numeration understanding was lacking in both groups of students. Whether teaching numeration as prerequisite knowledge is necessary or whether numeration understanding can be improved in conjunction with computational understanding is open to question. The success of teaching experiments (e.g., Buzeika, 1999; Kamii, 1989) where students are encouraged to formulate and discuss self-developed computational strategies seems to indicate that the development of efficient computational strategies has a positive effect on the development of numeration.

Most importantly, students need to realise that mathematics should make sense. More emphasis should be placed on students making meaning, rather than students learning procedures that are not/cannot be successfully employed.

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Single or Combination Grades in Mathematics in the Early Years

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As part of the Early Numeracy Research Project, children's mathematical growth in 295 single grade and 275 combination classes in grades P – 2 was compared. Significant differences were found with negative effects for students in the upper grades of combination classes, though overall differences were not significant. Data also indicates that deliberate assignment of weaker students to the upper grade of a combination class as an exercise in homogeneity can have a negative effect on mathematical growth.

One decision many primary schools must make is how to organise their classes. This decision is often made taking a mixture of philosophy and expedience into account. In the early 1980's and 1990's multi-age groups were popular in the early years and were even mandated in some US states (Lodish, 1992). The rationale was one that aimed to optimize each child's opportunity to grow intellectually, socially and personally through interaction with both mixed and same age peers and involved approaches designed to meet individual needs rather than grade level needs (Surbeck, 1992). Sometimes though, the decision to have multigrade classes is to do with numbers and staffing and one "combination" class is created with more than one grade level, often with other grades remaining as a single grade level. At other times the decision is made for all classes at particular levels to be multigrade – for example grade 1and 2 multigrade classes or sometimes grades prep- 2 multigrade classes though the teachers involved may or may not embrace the philosophy of the planned multi-age classes. Mason and Burns (1997) refer to the multigrade classes created for expediency as combination classes to distinguish them from the multi-age or non-graded classes where there is a distinctive philosophy in both design and practice, and I will use this terminology in this paper.

A comparison of combination classes and single grade classes in a meta-analysis of 45 studies (Veenman, 1995) found no achievement differences and very small affective effects favouring combination classes. Further analysis on 51 studies, considering aspects such as country of origin as well showed no differences (Veenman, 1996). Nearly all studies of achievement differences included a component of mathematical achievement. Burns and Mason (1995) claimed a tendency to assign less problem children and more independent students, as well as more experienced teachers, to combination classes in schools where there are also single grades, at least in California where the study was based. This raises questions about the interpretation of research findings due to selection bias.

Teachers in combination grades tended to group the students according to grade and teach them as separate groups, while teachers in the single grades were more flexible in their approaches (Pratt & Treacey, 1986 in a study in WA at first and second grade level, and Mason & Good, 1997, in a study of mathematical instruction in the US with grades 1-6, reported in Mason & Burns, 1997). Teaching approaches within combination classes were to teach two separate curricula, especially in mathematics, but one group for most other curriculum areas.

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Analyses in most studies of combination versus single grade classes have shown either no significant difference in achievement in mathematics (as well as none in other areas), or slight effect sizes favouring the combination classes, but, as previously mentioned, selection bias was not considered in these studies. It is of interest though that the particular grade of the children has not also been a focus, particularly looking at the higher and lower grade levels in the combination classes.

Attitude studies have shown significant differences, with the higher grade level students showing the more negative attitudes towards the multiple groupings (Smith, 1993, grades 3-6). Few of the analyses of achievement differences between combination and single grade classes, though, seem to have considered the differences between the upper grade students and the lower grade students in the combination classes. Burns (1996, reported in Mason & Burns, 1997) found that the lower grade combination students, after controlling for ability and independence, scored significantly higher on standardised achievement tests (including mathematics) than the lower grade or single grade students (200 grades 3 through 6 classes). This is one of the few studies to consider this factor carefully.

Other research has focussed more on the setting or tracking of classes and has produced mixed findings with set or tracked classes sometimes being beneficial to the better students but usually detrimental to the weaker students (Gamoran & Berends, 1987). Even among the better students there were often detrimental effects (Boaler, Wiliam, & Brown, 2000). Teachers changed their practice when they believed their class was homogeneous and taught as if the students were identical, disadvantaging students at both ends of the spectrum. Sometimes with combination classes there has been a tendency to put the better children from the lower grade in with weaker children from the upper grade. This could lead to the teachers again thinking more of their class as being homogeneous. There have been very few studies that have specifically looked at mathematics during the first three years of school and considered the effect on children's mathematical learning of the schools' class assignment practices.

Method

As part of the Early Numeracy Research Project (ENRP) (Clarke, 2001) data was collected on the mathematical learning of over 10,000 students in grades Prep-2 over a three year period. The main student data base of the ENRP included information on the year level of the student and the teacher to whom the class had been assigned for mathematics as well as data on children's mathematical development in a number of different domains from the areas of Number, Measurement and Space. Within the project there were trial schools where there was full participation in all aspects of the project, and a set of reference schools where the project just collected data on student progress and classes.

Some of the classes were single grade classes but there were also many combination or multi-grade classes. The principal and coordinator in each school were asked what factors they considered when assigning students to classes and their assignment policy, particularly in reference to combination grades. To enable comparisons of student growth in different types of classes, a variable 'grade type' was added to the data base with the values single-grade, lower grade combination, middle grade combination and upper grade combination.

Since the scales for each child's mathematical development had previously been converted to interval scales (Horne & Rowley, 2001), ANOVAs were used for each of the domains making a direct comparison between students in single and combination grades. This was followed by ANOVAs comparing the mathematical growth of students with different grade types both overall and in separate grade levels.

In a few classes children had been assigned to the upper grade level of a combination class because of previous lack of achievement. In one school, although they worked in P-2 classes for other subjects, for mathematics they effectively set the children with one teacher taking the preps and a few weak grade 1 students and the rest of the children being divided according to achievement. Where it was possible to identify such childrn, in any school, assigned due to lack of achievement, their growth in those classes was compared to the growth of children with a similar mathematical learning profile in other classes in the project schools.

Results

Within the trial schools, a variety of approaches was used for assigning students to classes. In most schools the classes were planned to be of mixed ability. There were some schools that had multigrade classes and others with single level classes. In some cases the multigrade classes were because of numbers, while in other schools the decision had been made to have all grade 1 and 2 classes as multigrade, or in two cases all grades as Prep-2 multigrade. While the original idea may have been multi-age classes or non-graded classes with a philosophical rationale, this evidence was not collected and the current practice tended to mean most of these classes could be regarded as combination classes. For this reason all will be considered as combination classes in this analysis. The main factors taken into account in the placing of children in class had been social interaction and behavioural factors. There were also a small number of schools who adjusted their classes for mathematics, so that the classes were closer to homogeneous, for example by placing "weaker" Grade 1 students with classes that consisted predominantly of Prep children. These children are discussed, following a consideration of combination and single grades. In both the 35 trial schools participating in the project and the 35 matched reference schools there were some classes which had a single year level (single) and some which had either two or three grade levels present in the one class (combination). There were 97, 104 and 94 single grade classes and 84, 96 and 95 combination classes in each of 1999, 2000 and 2001 respectively.

Table 1

ANOVA of Performance of Children According to Grade Type, 1999-2001.

	Number domains			All domains		
	df	*F*	*p*	*df*	*F*	*p*
1999	3, 5191	8.789	0.000	3, 3372	10.158	0.000
2000	3, 5315	24.169	0.000	3, 5279	31.502	0.000
2001	3, 5450	9.599	0.000	3, 4476	22.343	0.000

A comparison of the growth of children's mathematical understanding between single and combination grades shows no significant difference (using $p < 0.01$ because of the number of comparisons), either in the trial or reference schools or for any grade level. However when the *grade type* of particular children in the combination grade was considered, differences emerged. When the position in the class was taken into account, there were significant differences in the growth in both the trial and reference schools. Table 1 shows the ANOVA results for the whole cohort using average growth across the number domains and average growth across all domains as the dependent variables, with grade type as the independent variable.

Table 2
Average Mean Growth According to Grade type (Single and Combination).

	Change number domains (growth points)*			Change all domains (growth points)*		
	1999	2000	2001	1999	2000	2001
single grade	0.86 (*n* = 2623)	0.96 (*n* = 2794)	0.97 (*n* = 2861)	0.83 (*n* = 2129)	0.80 (*n* = 2772)	0.84 (*n* = 2817)
lower grade combination	0.87 (*n* = 1260)	0.89 (*n* = 1217)	1.03 (*n* = 1313)	0.81 (*n* = 1132)	0.70 (*n* = 1210)	0.82 (*n* = 1289)
middle grade combination	0.81 (*n* = 138)	0.79 (*n* = 145)	0.93 (*n* = 65)	0.70 (*n* = 108)	0.66 (*n* = 145)	0.72 (*n* = 65)
upper grade combination	0.77 (*n* = 1174)	0.79 (*n* = 1163)	0.80 (*n* = 1222)	0.75 (*n* = 1038)	0.67 (*n* = 1156)	0.66 (*n* = 1205)

* the change figures here are expressed as an average growth within a domain in ENRP Growth Points

Whether growth in all domains together is considered, or just the number domains, the message is the same: There is a significant difference in performance. Table 2 provides a means of considering the direction and extent of these differences. It shows the mean growth for each of the years, 1999 to 2001, for the number domains and all domains, for the different positions within the grades.

Table 3
ANOVA Across Grade Levels Comparing Single Grades and Position in Combination Grades.

Grade		Change number domains			Change all domains		
		1999	2000	2001	1999	2000	2001
Prep	*F*	2.84	4.12	0.05	8.19	3.16	2.02
	df	1, 1801	1, 1811	1, 1813	1, 1150	1, 1799	1, 1388
	p	0.092	0.043	0.690	0.014	0.076	0.239
1	*F*	4.58	5.23	2.40	6.38	7.96	3.64
	df	3, 1761	3, 1753	3, 1837	3, 1129	3, 1743	3, 1387
	p	0.003	0.001	0.053	0.000	0.000	0.085
2	*F*	7.78	5.99	0.31	10.12	6.68	1.69
	df	2, 1624	2, 1746	2, 1795	2,1088	2, 1732	2, 1696
	p	0.000	0.003	0.504	0.002	0.001	0.465

From the analysis of variance, it is clear that the position of the year level within a combination class had a significant effect on growth of mathematical understanding. The table of means shows that it makes very little difference to the growth in understanding if the children are in a single grade or in the *lower* part of a combination grade but that the growth is significantly less if the children are in the *upper* part of a combination grade. Similar, but not quite as extensive differences are evident between single grades and the *middle* part of a combination grade.

Table 4
Mean Growth by Grade Level Comparing Grade Types

Grade		Change number domains (n)			Change all domains (n)		
		1999	2000	2001	1999	2000	2001
P	single grade	0.89	1.00	1.07	0.86	0.84	0.95
		(1454)	(1435)	(1425)	(1251)	(1424)	(1399)
	lower grade combination	0.84 (349)	0.93 (378)	1.08 (393)	0.78 (297)	0.80 (377)	0.90 (387)
1	single grade	0.81	0.97	0.91	0.74	0.76	0.73
		(562)	(630)	(638)	(433)	(625)	(629)
	lower grade combination	0.91 (854)	0.88 (784)	1.02 (853)	0.83 (792)	0.66 (779)	0.80 (836)
	middle grade combination	0.81 (138)	0.79 (145)	0.93 (65)	0.70 (108)	0.66 (145)	0.72 (65)
	upper grade combination	0.79 (211)	0.86 (198)	0.82 (287)	0.72 (166)	0.71 (198)	0.64 (278)
2	single grade	0.85	0.88	0.83	0.85	0.74	0.71
		(607)	(729)	(798)	(445)	(723)	(789)
	lower grade combination	0.59 (57)	0.85 (55)	0.78 (67)	0.57 (43)	0.71 (54)	0.75 (66)
	upper grade combination	0.77 (963)	0.78 (965)	0.79 (935)	0.75 (872)	0.66 (958)	0.67 (927)

Of course, such statistics simplify the situation and while they do indicate a cause for concern that is discussed below, they do not mean that being in the upper part of a combination grade always leads to lesser growth. Tables 3 and 4 show the ANOVA statistics and the mean growth respectively with the data split into grade levels. As expected, there is a very small difference in growth at the Prep level (though significant in 2000 for the number domains and in 1999 across all domains at the $p < 0.05$ level) since the comparison was between single grades and the lower part of combination grades. However in Grade 1 and 2 there was a significant difference in both of the first two years with any disadvantage residing in being part of the upper grade level of a combination. In 2001 the difference in means was in the same direction but was not statistically significant.

The analysis assumed that no account of student understanding and performance is considered when assigning students to classes. However, there were a small number of classes where these were among the criteria in placing students in the combination class. This tended to happen particularly when a small number of high-attaining Grade 2 children were placed in a Grade 2/3 combination or when a small group of low-attaining Grade 1 or 2 students were placed with a lower grade level. In these latter cases, the teacher knew these children were considered to be weak, and the expectations of the teacher were more likely to be that these children would operate at a level commensurate with other children in the class. While these students could be identified, the sample size of only one or two classes is too small for any differences to be really considered statistically as the differences may be a teacher effect or class effect rather than the effect of the allocation. While little importance can be placed on the figures, a small group of students in Grade 2, who were taught with the Grade 1 students in 2000 because of their low achievement, were compared with other students in classes where achievement was not considered in class assignment. The comparison group was selected as all students with the same low profile of scores in the number domains at the start of the year.

Table 5
Mean Growth Comparing Low-Attaining Grade 2 Students Placed in a Lower Class to Others with the Same Profile.

	Mean growth (in growth points)	
	Students with same profiles	Students in combination class
Number domains	1.02 ($n = 311$)	0.58 ($n = 11$)
All domains	0.78 ($n = 306$)	0.43 ($n = 11$)

The data is shown in tables 5 and 6. The difference is significant ($p < .05$), but the number of students placed in classes because of their low achievement was small. This evidence indicates that such placements could be detrimental to children and should be avoided but it must be remembered that this is a small sample.

Table 6
ANOVA Comparing Grade 2 Students Placed in a Lower Class Because of Low Attainment to Others with the Same Profile.

	df	*F*	*p*
Number domains	2, 322	5.19	0.023
All domains	2, 317	5.84	0.016

A similar comparison of the 12 Grade 1 students who were placed in with Prep classes did not give the same result, but there was a difficulty in matching the profiles of the children. Within one school, they were clearly the lower-attaining students at the year level, but they had higher profiles than many other Grade 1 children in other schools, so the comparison sample was very large. The ANOVA showed no significance at the .05 level. The mean growth figures are shown in Table 7.

Table 7
Mean Growth Comparing Grade 1 Students Placed in a Lower Class Because of Low Ability to Others with the same Profile.

	Students with same profiles	Selected students
Number domains	0.94 ($n = 666$)	0.54 ($n = 12$)
All domains	0.73 ($n = 666$)	0.72 ($n = 12$)

It can be seen that these students improved less in the number domains but overall there was little difference, and the differences were not significant.

Discussion

These results differ from Veenman's (1996) meta-analysis but it should be noted that the early years were not a focus of the analysis and standardized achievement tests were used rather than the more strategies focussed interview assessment. While the implications of this data as presented are that combination classes are not as good for mathematical development of the students in the upper grade of a combination class as a single grade class, and have little or no advantage for those in the lower grade, there are other factors which need to be considered.

As mentioned earlier all of these grades were considered as combination classes but some may have had teachers who were committed to the idea of multi-age grades. If claims from multi-age advocates are true then the inclusion of possible multi-age philosophy classes as combination classes should have reduced the differences in favour of the combination classes. When classes were combination, although often in other subjects the students were taught as a group, there was a tendency for the teachers to separate the students by grade level or perceived ability level for mathematics. This was also indicated by Mason and Burns (1997). In many cases the multi-age philosophy does not appear to operate in the mathematics components of the program, though in this study there was no clear indication of whether the teachers were committed generally to a multi-age philosophy. This within class grouping is of interest and raises questions about homogeneity versus heterogeneity of within class groupings. Research on tracking and setting has shown possible detrimental effects and few positive gains (Boaler, Wiliam, & Brown, 2000; Gamoran & Berends, 1987; Linchevski & Kutscher, 1998). Homogeneous grouping within a class is a form of setting though with more flexibility possible.

While upper grade students overall showed lower growth, there were classrooms where this was not the case. For instance if grade 1 and 2 combination classes are considered there were some where the growth of grade 2 students was greater than that of grade 1 students while in other classes this situation was reversed. In 44 of the 86 combination classrooms in 2000, the growth of the students in the lower grade was greater than that of the upper grade students by at lest 0.1 growth point on average (in a few cases as high as 0.5 of a growth point). There were 14 classes, however, where this was reversed with differences greater than 0.1 growth points on average in favour of the upper grade level (and as high as 0.4).

The statistics indicate some aspects that need further consideration. Studies are needed of the characteristics of the teacher attitudes and beliefs, which enable strong growth across both grades of the combination class and of effective teaching approaches in combination classes for both groups. We must find how to alter the current situation with combination classes tending to show weaker growth, particularly, when for reasons of expediency, combination classes will be with us for a while to come.

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A Model for Teaching Numeracy Strategies

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The New Zealand Numeracy Project is based on two kinds of number models. One kind using the Forward Number Word Sequence, the Backward Number Word Sequence, Numeral Identification, and number facts that need to automatically recalled by children are defined as knowledge models. The other using Steffe's counting types and extensions is called a strategy model. Material written to support teachers assume two different teaching models are appropriate corresponding to whether the teaching involves knowledge or strategy. This paper maps the construction of the strategy teaching model. In particular it considers the influence of Pirie-Kieran Theory and Mathematics Recovery on its development.

In New Zealand pilot numeracy projects in 2001 and 2002 (Thomas & Ward, 2001, 2002; Higgins, 2001, 2002) have led to a numeracy project for teaching children from years 1 to 8 which, as resources allow, will be available over the next few years to all New Zealand primary teachers. Teachers will be provided with assessment and teaching materials. These are extensions and modifications of the Count Me in Too materials used in 2001 (NSW Department of Education and Training, 1999a, 1999b), the work of Wright (Wright 1991a, 1991b, 1991c, 1994, 1998, Wright, Martland, & Stafford, 2000) and the work of Young-Loveridge. (Anderson, Lindsey, Shulz, Monseur & Meiers, 2002; Young-Loveridge, 1991). A major element in these projects are the forward and backward number word sequences models derived from Fuson's work (Fuson, 1988) by Wright (Wright, 1991c) and the numeral identification model created by Wright (Wright, 1991c). These are defined as *knowledge* models in the New Zealand Numeracy Project. The other kind of model was formulated from Wright's (Wright, 1991a, 1991c, 1992b) tabular overview of the development of Steffe's theory of counting types (Steffe, von Glasersfeld, Righards & Cobb; 1983; Steffe & Cobb, 1988). In this theory there is a progression of five counting types in which children solve number problems using strategies ranging from counting objects to part-whole reasoning. This is defined as a *strategy* model for the New Zealand Numeracy Project. This distinction between knowledge and strategy is similar to that used by Wright (Wright *et al.*, 2002). As the New Zealand Numeracy Project is designed to be used in the first eight years of school rather than the first three years (NSW Department of Education and Training, 1999a, 1999b) three extra stages have been added to the counting types. This set of new types represent an increasingly sophisticated sequence of part-whole stages. There has also been some slight modifications of the other counting types.

Critics who argue that splitting number into *knowledge* and *strategy* is arbitrary have a point. What may start as strategy for a child will become knowledge and to separate them is somewhat artificial. For example, suppose a child is asked the value of 8 + 8. A child says she adds 2 to 8 to make 10 and then adds 6 to give 16. She is using a part-whole strategy. But as the child improves her speed with this kind of reasoning, this method moves from using a strategy towards becoming knowledge thus blurring the distinction. However, the

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dichotomy between knowledge and strategy is maintained in the teaching materials for *pedagogical* reasons. It is assumed that teaching for knowledge and teaching for the initial development of strategy development warrant very different teaching models. This paper is concerned with the construction of a teaching model that is suitable for strategy teaching.

Assessing Number Knowledge and Strategy

All teachers involved in the New Zealand Numeracy Project are provided with enough release time to interview individually all children in their class. The interview tool is designed to determine children's number knowledge and strategies in the sense already defined.

For questions designed to determine children's ability with the forward number word sequence, backward number word sequence and numeral identification the method of recording results is typically straight forward for teachers; the questions get progressively harder in that the size of the numbers used increases and the teacher assigns a level corresponding to the highest numbers the child can reliably work with. It was a basic assumption in the writing of teacher material that teachers do find the teaching of knowledge informed by the diagnosis straight forward and well within their current models of teaching. Thus the material supplied makes no explicit reference to a model for teaching knowledge. For example, the teacher material has activities like clapping and saying every second number in order to practice the forward number sequence.

In marked contrast to assessing for knowledge teachers often find assessment of strategy using Steffe's five counting types and variations far more difficult. This because answers to questions can not used to assign a counting type or stage by simply looking at the number size. Questions can be answered correctly using a range of counting types. For example a year three child might solve 8 + 5 by a counting from one with fingers, by counting on, using a part-whole method or recalling a basic fact. The teacher must infer which counting type each child uses from a range of clues. These include whether the child uses materials, sub-vocalises when counting, speed of response, head nodding, direction of the eyes and the child's verbal explanation of the methods used. Use of recall of a known fact to solve the problem further complicates the assessment for the teacher as recall tells the teacher nothing about the child's ability to use counting types. This often represents a shift in teacher's thinking about assessment as well as an awareness raising about the different counting types children use (Hughes, 1995). Adding to the complexity a child may respond with different counting types to similar questions. For example Sharon, a year 3 student, says 8 + 7 is 15 very quickly; she explains she doubled 7 and added 1. She would be rated in New Zealand (Ministry of Education, 2002) as *Stage 5 Early Additive Part-Whole*. However, Sharon works out 9 + 6 by counting on from 9. Now Sharon could be described as *Stage 4 Advanced Counting*. In a case like Sharon's many teachers who are struggling to come to terms with assessing for strategy rather than right/wrong answers are somewhat confused by Sharon's use of two different counting types. Is she an *advanced counter* or a *part-wholer* or a mixture of both? Teachers are advised by their visiting facilitators to rate such a child at the highest counting type displayed. The reason for this advice is so the teacher can group children by the highest counting type for purposes of strategy teaching. Teachers in the project are expected to teach knowledge lessons to whole classes but strategy lessons in groups defined by the highest counting type available to the children in that group. This is in sharp contrast to teaching in England's National Numeracy Strategy. While there are a large number of similarities between the English and

New Zealand numeracy projects there are at least two core differences. In England the teacher materials do not distinguish knowledge and strategy, and materials are written for year levels with a strong emphasis on regular whole class teaching (Department for Education and Employment, 1999). In New Zealand the teacher material distinguishes between knowledge activities for whole class and strategy activities; teachers group their children by similarity of counting type and try to encourage the construction of harder and more sophisticated counting types. The merits of the English approach versus the New Zealand approach at this time is a matter for further research.

The origins and nature of the New Zealand Numeracy Project number strategy teaching model is now presented.

P-K Theory

The strategy teaching model is influenced Pirie & Kieren Theory (Pirie & Kieren, 1989, 1992, 1994; Pirie & Martin, 2000). Figure 1 shows some of the key features of Pirie & Kieren's Dynamical Theory for the Growth of Mathematical Understanding.

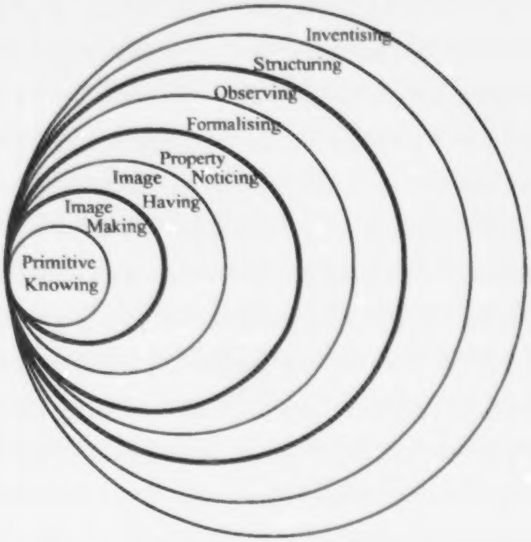

Figure 1. A dynamical theory for the growth of mathematical understanding.

Some of the stages in this model and the P-K idea of *folding back* influenced the development strategy teaching model. These influences are discussed as the various stages of the strategy teaching model is explained.

A Model for Teaching Steffe's Counting Types and their Extensions

Figure 2 shows the main features of the strategy teaching model used in the writing of teacher materials for broadening and extending children's counting types.

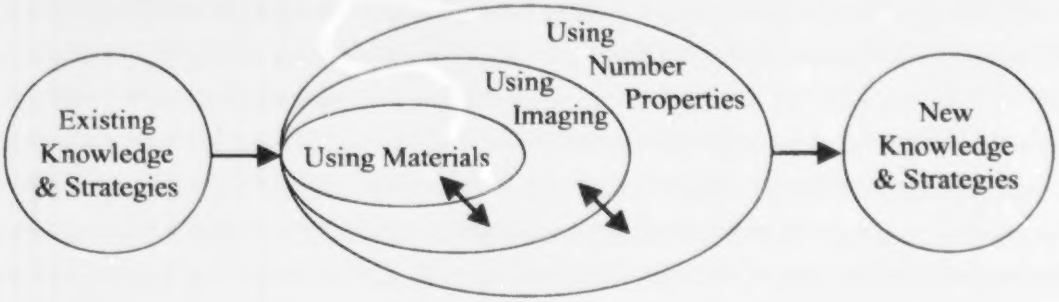

Figure 2. Strategy teaching model.

Existing Knowledge and Strategies

Existing Knowledge and Strategies is similar to *Primitive Knowing* in P-K Theory. *Primitive Knowing* is defined as "...all previously constructed knowledge, outside of the topic, that students bring to the learning of a topic" (Pirie &Martin, 2000). In the P-K theory *Primitive Knowing* is the foundation on which the *Image Making* level of understanding and later levels are built. Since *Primitive Knowing* is not a change in understanding *Existing Knowledge & Strategies* is placed outside the rings in Figure 2 to highlight its difference from these change in levels of understanding.

Using Materials

New Zealand teachers, especially those teaching children in the early years of school, have a long tradition of using materials in the teaching of mathematics. So most activities in the teacher material begin explicitly with a familiar section named *Using Materials*.

Using Imaging

The *Using Imaging* phase is a response to the real problem that many children fail to make the desired abstractions out of the *Using Materials* phase. Von Glasersfeld (1992) explains the difficulty.

> Mathematics is the result of abstraction from operations on a level on which the sensory or motor material that provided the occasion for operating is disregarded. In arithmetic this begins with the abstraction of the concept of number from acts of counting. Such abstractions cannot be *given* to students, they have to be made by the students themselves. The teacher, of course, can help by generating situations that allow or even suggest the abstraction. This is where [materials] can play an important role, but it would be naive to believe that the move from handling or perceiving objects to a mathematical abstraction is automatic. The sensory objects, no matter how ingenious they might be, merely offer an opportunity for actions from which the desired operative concepts may be abstracted; and one should never forget that the desired abstractions, no matter how trivial and obvious they might seem to the teacher, are never [obvious] to the novice. (p. 6)

Extensive use of materials without some way of promoting abstraction can lead to problems for children's learning. Ross (1989) reports for students "... even extensive experience with embodiments like base-ten blocks, and other place–value manipulatives does not appear to facilitate an understanding of place value...". In a similar vein Hart (Hart, 1989) notes the typical child who says "bricks is bricks and sums is sums" and does not connect materials with the desired abstractions. Hart notes the need for a bridge between "bricks" and "sums". The *Using Imaging* phase in the teaching model is an attempt at providing such a bridge.

Bobis (1996), arguing for the power and usefulness of visual imagery, provides a clue to a phase to follow the *Using Material* stage in the model.

> The use of concrete materials is important, but rather than moving directly from physical representations to the representations to the manipulation of abstract symbols to explain the traditional abstract procedures of algorithms, it is suggested that the emphasis be shifted to using visual imagery prior to the introduction of more formal procedures. (p. 21)

The *Using Imaging* stage where physical material is removed from the children's view is a deliberate attempt to introduce the use of visual imagery of absent objects into teaching and learning.

In P-K theory the meaning of *Image* in the *Image Making* and *Image Having* phases is more complex than the creation of picture images. Thus the *Using Imaging* phase in the teaching model is only tenuously linked to P-K Theory.

In the *Using Materials* phase the manipulatives have normally selected so that at the *Using Imaging* phase children can readily imagine the materials in their absence. For example, some activities start using counters on tens frames at the *Using Materials* phase rather than unstructured collections of counters phase because the tens frames are easier to image in their absence than an unstructured collection of counters.

Within the *Using Imaging* phase there are frequently two sub-phases. The first involves shielding or screening. This idea is derived from Mathematics Recovery (Wright, 1991a; Wright, *et al.* 2000). Shielding occurs when the teacher hides the objects that they have previously used in the *Using Material* phase so they are out of sight of the children. Then the teacher invites the children to imagine the rearrangements she is making on the material to solve the problem. For example children working in the part-whole counting type group may be asked to add 8 and 6. Children are asked to imagine what the hidden 8 counters on a tens frame look like, then to describe the groups of counters they moved around in their heads in order to get 14.

Failure to solve problems correctly at the *Using Imaging* phase may occur for a host of reasons. But whatever the reason the lack of success should make teachers aware of the fact that successful manipulation of material has *not* led to successful learning. The strategy teaching model suggests that when children do not make connections at the *Using Imaging* stage in the model that they fold back to *Using Materials*. In Figure 2 the return arrows indicating this *folding back* process. The idea of *folding back* is derived from P-K Theory (Pirie & Martin, 2000).

> When faced with a problem that is not immediately solvable at any level, an individual needs to return to an inner layer of understanding. The result of this folding back is that the individual is able to extend their current and inadequate and incomplete understanding by reflecting on and then reorganising their earlier constructs for the concept… (p. 131)

The *folding back* idea in the strategy teaching model has been altered to be an action provoked by explicit behaviour of the teacher whereas in P-K theory this is an action that i children autonomously take.

In the second sub-phase in *Using Imaging* the teachers' book provides activities in which children image without the support of shielding or folding back. Success at this stage cues the teacher to attempt to move on to the final phase of *Using Number Properties*.

Using Number Properties

In P-K theory *Property Noticing* "involves noting distinctions, combinations or connections between images, predicting how they might be achieved and recording such relationships" (Pirie & Kieren, 1992). And *Formalising* "entails consciously thinking about the noted properties, and abstracting commonalities. The person now has a class-like mental object not dependent on meaningful images" (Pirie & Kieren, 1992). In the strategy teaching model the P-K ideas in *Property Noticing* and *Formalising* led to the construction of the *Using Number Properties* phase. In this phase the objective is that children will abandon the use materials or imaging, if they indeed have not already done so, and proceed to reason directly with the numbers and their properties. The device used consistently in *Using Number Properties* phase of each learning activity is to push the number size up to the point where imaging the numbers is a burden, and solutions are best found by reasoning with abstract number properties. For example consider a group of children who have successfully used imaging and part-whole reasoning to solve problems like 9 + 16 and 25 - 7. For a problem like 7 + 89 it is difficult to *Use Imaging* but the problem is solvable simply by part-whole reasoning *when* the children are able to connect to number properties without needing images. Should children not make connections the double arrow in Figure 2 indicates *folding back* to the *Using Imaging* stage.

New Current Knowledge and Strategies

If the process has been successful children now have *New Current Knowledge and Strategies* have been created. The model may now be reused from the beginning.

Restrictions on the Use of the Strategy Teaching Model

In the New Zealand numeracy project Steffe's five counting types have been modified and extended to make a nine stage model; the strategy teaching stages in Table 1 shows these counting stages. In particular Steffe's last counting type *Explicitly Nested Number sequence- Part/whole Operations* which includes procedures other than counting-by-ones such as compensation, using addition to work out subtraction, and using known facts such as doubles and sums which equal ten is expanded. One stage becomes four part-whole stages to emphasise the importance of part-whole thinking and reflect the increasing sophistication in thinking required.

The shaded boxes (Table 1) indicate where *Using Imaging* or *Using Number Properties* is inappropriate because the counting type is too elementary to use these stages.

Table 1
Counting Stages and Strategy Teaching Stages

Counting Stages	Strategy Teaching Stages		
	Using Materials	Using Imaging	Using Number Properties
Emergent		▨	▨
One-to-one Counting		▨	▨
Counting from One on Materials		▨	▨
Counting from One by Imaging			▨
Advanced Counting			
Early Additive Part/Whole			
Advanced Additive Part/Whole			
Advanced Multiplicative Part/Whole			
Advanced Proportional Part/Whole			

Discussion

A large scale experiment in the use of the strategy teaching model involving literally thousands of New Zealand teachers is under way this year. 75% of the teaching material supplied to teachers in the Numeracy Project in 2002 has been written to include this model. Evidence about the model's effectiveness will be gathered and reported at the end of the year. Small scale anecdotal evidence from developing the model over two years by a small group of the Numeracy Project's facilitators give cause for some optimism about the results. In particular teachers seem to readily accept the *Using Images* phase as being an obvious and natural stage that has frequently been missing in their teaching.

The teaching model is presented to teachers by their school's facilitator. For many of the facilitators the model was quite knew at the beginning of 2002 when they undertook training for facilitating the project. What remains to be seen is how these facilitators interpreted the model, and how effective they are at communicating the use of the model to their teachers.

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Visualising and the Move from Informal to Formal Linear Measurement

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Forty-three children, aged 8 and 9, were interviewed about their understanding of informal and formal measurement. Results showed that they visualised units accurately if the number of units was small. For many this was their preferred method of measuring. When asked to iterate an informal unit or use rulers in unconventional ways, they made errors that indicated ignorance of many of the unwritten rules of measurement. Successful children used visualization and the width of a finger to measure.

Linear measurement entails comparison, either of two quantities or by using intermediary non-standard or standard units for this comparison. As several authors have summarised, this requires appreciating the need for conservation of length, an understanding of units and when they can be subdivided for measuring continuous quantity, why and how units can be iterated, and the appropriate use of measuring tools (e.g., Nunes & Bryant, 1996; Outhred & McPhail, 2000; Piaget, Inhelder & Szeminska, 1960).

Age-related differences have been shown in understanding of measurement. Nunes and associates carried out a series of studies with children aged 5 and 6 years many of whom are quite naïve about measurement (summarised in Nunes & Bryant, 1996). Bragg & Outhred (2000) interviewed children from Grades 1 to 5 and showed interesting changes in this age range, especially between Grades 2 and 3. The study reported here was concerned with students at the age, in New Zealand, when they usually move from informal to formal measurement (*Mathematics in the New Zealand Curriculum*, Ministry of Education, 1992).

Children may appear to make the transition to formal linear measurement easily, as they are already familiar with the most common measuring tool, the ruler. This familiarity may disguise the need to understand the principles given above. To measure correctly they must understand that the length of units is conserved in that all units are the same length, whether informal or formal. They need to understand that when they put one unit down after another in iteration, the two units must just touch and not overlap. Gaps are not allowed. Some of these concepts are built into rulers but need to be observed when the object to be measured is longer than the ruler. Formal measuring instruments add other important and difficult concepts, namely that in the use of standard measuring instruments measurement starts at 0 and scales can be used to give fractional parts of continuous length.

One reason why formal measurement is difficult may lie in the limited opportunities that our children have outside school in which precision in measurement is required. Out of school, children may help measure ingredients for baking or may measure how tall they are (an excellent task for comparison) but they rarely engage in activities that require precision. In the Philippines, people use a phrase "puwede na yan" which can be translated as "that's good enough". This concept of "good enough" is not limited to the Philippines, but is common in most areas of measurement and especially those that children have experience with. In asking the 43 students in this study where they used measurement outside of

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school, only one mentioned measurement in which precision was needed. This child spoke of working with his father, who was always building things around the house. That child might be expected to know how to use a measuring tape and the importance of starting with 0. The other 42 children either could think of no examples of when they measured outside of school or gave examples in which precision was not necessary. As adults do less tailoring or carpentry at home, children see fewer models out of school that require precision.

Method

The authors interviewed 43 children, from four schools, individually using five tasks and standard initial questions but with follow-up questions chosen to cover predetermined topics. Twenty-two children were between age 8 years and 8 years 11 months (called age 8) and 21 were between 9 years 0 months and 9 years 9 months (called age 9). Interviews lasted 15 to 20 minutes. Careful notes were kept on students' activities during the interview on recording sheets that prompted interviewers to explore and record children's answers, their use of visualization, their measuring processes, their use of units and fractional units, and their experience with measuring. For items involving a ruler, specific notes were made on appreciation of the need for 0 at the start of a ruler or in measuring, where the child started measuring from, and uniformity of units where theses were not given. Children also wrote on their sheets. Teachers were also interviewed.

The first two items required children to visually judge the number of ribbons or tiles would fit in a given length and then use informal methods to measure. A third item required them to finish drawing a ruler than someone else had started drawing. A fourth item asked them to measure four dimensions with a "broken ruler". Items 3 and 4 were adapted from Nunes and Bryant (1996). The final task asked them to give the length of string with an s-shaped bend in it that was pictured next to a ruler (task from the TIMMS-R).

Students came from four schools of different character, as we expected different results from children who attended these schools. One was a small Montessori school, one was an expensive private girls' school, one was a small lower middle class primary school, and one was a large primary school from an upper middle class area. We found no major differences in the accuracy of children between schools, which in itself was surprising. Therefore data from all schools are given together. All teachers followed a national curriculum for measurement, which for these ages covers the transfer from informal measurement to formal measurement. Where there were small differences that might be the result of teaching these are referred to in the results and discussion.

Results and Discussion

Children's responses were analysed in relation to concepts needed for measurement, especially the unwritten rules for units and iteration.

Analysis of Responses to Tasks 1 and 2, Use of Informal Measurement.

The first item required students to estimate how many ribbons could be cut from a given length when both were pictured on their sheets. Then were then asked to check their estimate but given no tools for this.

Table 1

Number of Children, Age 8 and Age 9, giving the Listed Responses when asked How Many Ribbons could be cut from a Strip

Response	Age 8		Age 9	
	For visual estimate	By informal measuring	For visual estimate	By informal measuring
3	8	2	17	5
3 with a bit leftover	4	17	3	15
4	8	1	1	1
Other responses	2 (5,1½)	2 (18, 1½)	0	0

Note: The strip given was 3.4 times the length of the ribbon.

A higher percentage of 9-year-old children (19 or 95%) than 8-year-old children (12 or 55%) were accurate for this visual estimate. It could be argued that the children who estimated 4 rather than 3 were not far off. This shows very good accuracy in visualization of length in this age group. Asked how they did this, one girl said *"I looked at it from that bit and then I tried to get the same amount"*. The unusual responses like 18 and 1 ½ were the result of children losing sight of the interviewers' goal for this task.

Most children measured with their fingers or with their pencil. Some did not measure with an instrument but continued to trust their visualization. Thus given a measuring task with no tools, and asked both to visualise and then to measure, these young children showed marked accuracy, in a comparative measurement of this size.

On the next task they were again asked to estimate visually and then were given two tiles and asked to measure the length, including parts of tiles that would be needed to just fill the length.

Table 2

Number of Children, Age 8 and Age 9, giving the Listed Responses when asked to Visualise and Measure How Many Tiles Would be Needed to Fill a Space and Leave no Gaps.

Response	Age 8		Age 9	
	Visual estimate	Measure	Visual estimate	Measure
4 (and parts)	10	3	8	1
5 (and parts)	8	18 (all with parts)	9	15 (13 with parts)
6	3	1	2	5
Other	1 (7)		2 (7, 3½)	

Note: The strip was 5.11 times the length of the tiles

This was a more difficult measure to visualise, possibly because more units fit into the space. About the same percentage of 8 and 9-year-old children were accurate in their estimate, 36% and 43% respectively. It was interesting that 45% of the 8-year-olds and 38% of the 9-year-olds underestimated, but few children overestimated when asked to visualise.

Measuring was to be done with two tiles that could be alternated to measure the space. Children's manual dexterity may have influenced their accuracy. Despite this, 82% of the 8-year-old children and 71% of the 9-year-old children measured reasonably accuracy.

However, even reasonably accurate answers demonstrated some misconceptions. Errors in measurement using the two given tiles were related to failing to respect the unspoken rules of iteration. They included:

- The need for each unit to be right next to the previous one. This unspoken rule was broken by children who overlapped the tiles so that six tiles fit in without any fractional bits left over. It was also broken by a child who laid his pencil down to mark the end of one tile, and laid the next tile on the other side of the pencil.
- The need to count only completely iterated units, and give the remaining portion a fractional name. Placing a final tile over the boundary so that a space slightly greater than 5 tiles was called 6 broke this rule.
- The need to count all iterations. For example, one child counted 4 iterations, counting only those put down after the first tile.
- The need to count each unit or partial unit only once. This rule was broken by children who laid a final tile over the right-hand edge, and counted it both as six and as ¼. For example, one child said that the answer could be 5 ¼ or 6 ¼. She measured 5 ¼ but unsure whether or not to count the partial one as both 6 and ¼.

Other methods were not wrong but failed to take advantage of having a measuring unit. Of children who did this, one continued to visualise with considerable accuracy, deciding that 5 ½ tiles would be needed to fit the space. Another used both tiles to iterate as one unit, pushing them both along rather than alternating. One measured with her fingers spread to the width of a tile. One put both tiles down, but then used only the second one to iterate.

There were both careless measurers who had to be reminded to be careful, and some very careful children. For example, one boy who had initially estimated 4¾ measured once and got 5, measured a second time and got 4 7/8 and measured a third time, getting 5 1/8. For an actual answer of 5.11 or 5 1/9 this was very accurate indeed.

In an attempt to get at children's understanding of fractional parts of units, we asked what they would call the leftover bit if there were one. The children either gave the leftover portion an informal name, like a bit or a slice, or used a common fraction. A quarter, a half and a third were all used sensibly. Others called the bit either a cm or – surprisingly in a country that abandoned imperial measurement 35 years ago - an inch. Sometimes these terms were used together, suggesting that the child had an idea that there were words for bits but was uncertain what they were. For example, when asked what was left over, one girl said *"½ - a cm? 2 cm?"*

Analysis of Children's Responses to Item 3, Finishing a Drawing of a Ruler

The third task explored children's understanding of the most common formal or standard linear measuring instrument in their environment, a ruler. This task explored whether or not they could show an understanding that units on a ruler needed to be of the same length and whether or not they understood the importance of starting with 0 for a complete initial unit. The drawing that they were given had lines for 3, 4, 5, and 6 with marks halfway between these. Only the numbers 4, 5, and 6 were given. If completed correctly this ruler would have had 0 on the left margin, without a space before it, if their units were of the same size.

The rulers used in these New Zealand schools are identical. They all have centimetres on one side and millimetres on the other starting from the other end. Either "cm" or "mm" is printed at the left of the ruler and there is no 0. The marks do not start at the left end of the ruler but start about 5 mm in. When children completed the drawing of a ruler on this task they did not have their school rulers present, but they were very likely to have a mental image of this tool as they use it daily, for underlining if not for measuring. For measuring they must learn which of the aspects of their rulers are essential to measurement and which can be ignored. With these factors, some errors in their use are not surprising. For example, in a previous study (Irwin, 2001) one child, when asked what came between 0 and 1, said that "cm" came between them and an older child said that there was no relation between centimetres and millimetres because they were on opposite sides of the rule and went in opposite directions.

Drawings were analysed for evidence that they appreciated that the first unit ended with 1, and that all units were similar is size.

Table 3

Starting Point Marked on Rulers Drawn by Children Age 8 and Age 9.

Starting point of drawn ruler	Age 8	Age 9
Acceptable Drawings		
0	8	7
cm where 0 should have been	1	2
Unacceptable Drawings		
Notable space before 0 or cm	3	7
No mark before 1	3	2
Starting at 1	1	2
0 placed at the 0.5 mark	5	0
Nothing before 2	0	1
Units very uneven	1	0

Thus less than half of the students in either age group started their rulers in a way that showed understanding that the first unit was the same size as other units and started with 0 (41% and 43% respectively).

A separate analysis was made of the evenness of the units that children drew. This was done in analysing only the portions before 3 and not those at the right hand end. Some leeway was allowed if the child's intent was clear. Half of the children in each age group drew adequately regular units (50% of Age 8 and 52% of Age 9), suggesting that these children had an understanding of this essential aspect of measurement units. Those who made units of irregular size usually did so because of confusions about where the numbers should start.

Children who were unsuccessful on this task ignored unwritten rules of measurement that

- All units must be the same size.
- "1" marks the end of the first unit.

Teaching appears to have made a difference for some of these children. Seven of the 15 students who started their rulers with 0 at the very start of the space were from one class. There were also several children from the same class who made a variety of incorrect drawings (starting at 1, putting the 0 at the 0.5 position, or putting no marks before 1), so if teaching did aid these seven students, it did not prevent other errors.

Responses to Task 4, Measuring Four Lengths with a Broken Ruler

For this task children were given a portion of a ruler that ran from 3 to 7.5 cm on one edge and from 236 to 287 mm on the other edge. This portion of a ruler was made from a photocopy of their school rulers. This item was included to allow us to observe children's understanding of units when given a portion of a ruler that did not start at or near 0. They were asked to measure the height and width of a pictured house and then the height and width of its door. The first two dimensions were longer than the ruler and the second two were shorter than the ruler.

We wanted to see if they counted the spaces for units rather than the lines or numbers, and what technique they used for iterating the broken ruler when it was not long enough. Photocopying warp made the measurements more difficult than intended, but not beyond the capabilities of careful children. The correct answers to measuring four dimensions were 5.6 cm, 8 cm, 3.3 cm and 2.8 cm. One child of 8 measured these with a broken ruler and got the answers: 5 cm and 7 mm, 7 cm and 9 mm, 3 cm and 2 mm and 2 cm and 8 mm.

Table 4
Students Measuring each Segment with Enough Accuracy to Show that they were Using the Units on the Broken Ruler.

Response	Item a [5.6]		Item b [8]		Item c [3.3]		Item d [2.8]	
	Age 8	Age 9	Age 8	Age 9	Age 8	Age 9	Age 8	Age 9
Accurate	12	9	9	7	7	7	8	10
1 cm up – counting lines	6	7	6	6	13	10	7	7
Inaccurate	4	5	7	8	2	4	7	4

Note: Accuracy for Item a [5.6] was considered 5 to 6, For Item b [8] 7.5-8.5, for Item c [3.3] 2.8 – 3.8, and for Item d [2.8] 2.3 – 3.3.

The differences between age groups were not marked in most cases. More 8-year-olds than 9-year-olds were accurate on Item a. The majority of these children did not iterate the ruler but imagined the units that would come before the 3 on the ruler, an effective strategy. However, visualisation was not always used successfully. Some students reported visualising all the lines before the start of the broken bit and counting them, ignoring the unit that came before the first line.

Visualising on Item b was more difficult as both the beginning and end of the ruler would have to be pictured. Children were more likely to iterate the ruler, which caused more confusion in counting lines or spaces. These two techniques were the source of some of the quite inaccurate measurements.

On the third and fourth measurements there was no need to either visualise spaces or to iterate. Many more students referred only to the numbers on the ruler without reflecting on what was missing. Some students made the same measuring technique on these two items, and others changed their method of measuring, suggesting that they were not certain of what to do. We were reminded by their activity that this was a school task, without the payoffs for incorrect measurement that occur when precision is needed in everyday life.

Some of the answers coded as 'quite inaccurate' were the result of reading from the mm side of the ruler, without understanding the units or iterating. For example one boy gave the fourth reading as "263 mm". Others were reasonably accurate in their measurement but did not know what to call the units. For example "*7 and 9 inches – no millimetres – I forget the names*".

Two successful children used the width of a finger to measure one cm when this was needed before or after the broken ruler. When this successful method was described to their teacher, the teacher reported that they had spent time measuring various body parts, and the width of a child's finger at this age was usually 1 cm. This seemed an excellent way of transferring a skill gained in informal measurement to formal measurement.

Responses To Task 5, Measuring a Picture of a Folded String Placed Next to a Ruler.

This task required students to use the units of a pictured ruler to measure a string with an s-shaped bend in it that was drawn above the ruler. The total to be measured was 7 cm, made up of portions of 3 cm, 1 cm, and 3 cm. Again, students were required to know that the space of the unit, not the lines, were what was to be measured.

Table 5

Methods Used by Children, Age 8 and Age 9, in Measuring Segments of a Picture of a Folded String above a Picture of a Ruler.

Technique used	Age 8		Age 9	
	Technique	Accurate	Technique	Accurate
Visualising	10	1	8	1
Measuring	11	1	12	3
Visualising and measuring	1	0	1	0

Some children who visualised on this task reported that they stretched out the string in their mind, sometime to the beginning of the ruler and sometimes to 1. For example "*I brought the end* [mentally] *to cm and thought it would probably be that long*"

Very few children measured accurately. As shown above, only two of the 8-year-old and four of the 9-year-old children were accurate in their measurement. Most children could see that the middle piece was 1 cm, looking at the space between the lines. For the longer segments they counted lines or stretched back visually only to 1. Children who measured matched a spread of their fingers to the ruler, used their finger to measure 1 cm, or lined up the segments with the ruler in the picture.

Implications

Children demonstrated a hierarchy of skills on these increasingly difficult tasks. Their first preference appears to have been for visualising. We were surprised how frequently and how accurately these young children visualised to get linear measurements. Even when not asked to estimate they often reverted to picturing units when measurement with iterated tiles or a ruler was difficult. When the number of units to be visualised were small they did well, both on estimating that three ribbons could be cut from a longer strip and on the task with the broken ruler where they had to estimate the length of 3½ cm. However, while visualising worked for picturing units it was not effective for picturing how long a string would be if pulled straight, a much longer distance.

Next to this preference for visualising came using their fingers to measure, either a finger's width, which was very close to 1 cm, or fingers spread to match a given length. The first of these measurements was very accurate, but the latter often did not conserve the distance when moved to a different place.

Iterating informal units to measure was moderately successful, but in doing this several students showed that they were uncertain of the unwritten rules of measuring units. Some of the same unwritten rules were broken in drawing a ruler as in measuring with two tiles. Errors in drawing and using a ruler may be exacerbated by the rulers that the children use daily. However, confusions about whether the spaces or the lines were to be counted could not be blamed on the rulers that they used.

Accurate understanding of the use of rulers in the unusual situations in these interviews proved the most difficult skill to use, especially in the last task in which neither the ruler nor the object to be measured could be moved. These children were starting to use rulers for standard measurement, but many were not transferring the rules for units and iteration from informal measurement.

Visualizing called upon their experience of what would fit in a space. This may be a task that children have much more experience with than they do with precise measurement, as Owens and Outhred (1998) found for measurement of area. It depends on a spatial skill similar to that used in putting puzzles together. As this group reported almost no situations in which they had to use precise measurement outside school, this must be seen as a school task rarely reinforced out of school.

When the children's teachers were interviewed, they were not surprised by the children's weaknesses. Some thought that formal measurement was introduced too early, reflecting their understanding of these children's difficulties. However, they were pleased to hear of the successful use of a finger width as an intermediary between informal and formal measuring. Similarly, the teacher who had emphasised the role of 0 on a ruler had taught a useful piece of knowledge.

These interviews suggest that children's skills in visualisation should be fostered. They also suggest that the use of body lengths can help a successful transfer from informal to formal measurement.

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Variation in a Chance Sampling Setting: The Lollies Task

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Responses of 73 students to an interview protocol based on selecting 10 lollies from a container with 50 red, 20 yellow, and 30 green are categorised with respect to centre and spread of numerical answers and to reasoning expressed in justification of the answers. Results are compared to earlier survey research and small-scale interview studies.

The motivation for the sequence of studies of which this one is a part arose from a survey item used in the National Assessment of Educational Progress (NAEP) in the United States, known as the "Gumball Task". Although it was set in the context of drawing a number of gumballs in a chance setting, the wording of the question encouraged an exact answer (the expected value), instead of a likely range (Zawojewski & Shaughnessy, 2000). The task was rewritten in three alternative forms by Shaughnessy, Watson, Moritz, and Reading (1999) in an attempt to explore more deeply students' understanding of variation and to move away from the strong emphasis on centres. In association with trials of the rewritten tasks in surveys with 324 students in the United States and Australia, Shaughnessy et al. developed a two-dimensional coding system based on a "centring scale" and a "range scale". The "centring scale" for a set of six predictions, for example, classified responses using the mean to determine whether the response was "low", "five" (mean), or "high". The "range (r) scale", informed by a simulation of 1000 trials, resulted in a classification scheme of "narrow" ($r \leq 1$), "reasonable" ($2 < r < 7$), or "wide" ($r \geq 7$). Each response was then classified based on centre and spread, with the optimum classification being a "five-reasonable" prediction.

Two small studies followed the surveys, where students were interviewed to gain more appreciation of their understanding of the "Lollies Task", as it was renamed for Australia. Torok and Watson (2000) used a similar protocol to that used here as well as several others involving variation and conducted 16 interviews with students in grades 4, 6, 8, and 10. The responses were categorised and clustered into similar groups that formed a four-tiered hierarchy demonstrating an increasing sophistication in the understanding of the proportional ideas and the variation involved in the tasks. The categories they found were those displaying (i) "weak appreciation of variation", (ii) "isolated appreciation of aspects of variation and clustering", (iii) "inconsistent appreciation of variation and clustering", and (iv) "good consistent appreciation of variation and clustering" (p. 155). Reading and Shaughnessy (2000) also interviewed 12 students using a similar protocol to this study and reported on four case studies from one student in each of grades 4, 6, 9, and 12. These students reflected many of the characteristics of the four levels observed by Torok and Watson, with the grade 12 student expressing conflict in choosing between multiple choice responses representing strict probability and sampling variation.

Following these four studies, this report focuses on two research questions. First, following the survey work of Shaughnessy et al. (1999) what are the distributions of students' responses given in an interview setting with respect to centres, spreads, repeated values in predictions, and change in predictions after experimentation? Second, following

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the work of Torok and Watson (2000) and Reading and Shaughnessy (2000), are four levels of understanding confirmed for the overall context of the Lollies Task? Does the structural complexity fit that observed earlier?

Method

Interview protocol. The Lollies Task was the first and longest protocol of an interview focusing on aspects of variation in chance and data in a 45-minute session. The task was based on a container with 50 red, 20 yellow, and 30 green lollies in it. The first part of the protocol was a series of questions asking for predictions of how many red lollies are likely in handfuls of 10 (see Figure 1). Students were later given the opportunity to perform the experiment for themselves and go back and change any answers already given. The second part of the protocol (not shown here) was a graphing exercise that asked the students to imagine that 40 students performed this experiment and to "draw" what this might look like. Students who had trouble producing graphs or understanding the question were presented with labelled axes to help with the task. All interviews were video taped and transcribed for later analysis.

1. Suppose you have a container with 100 lollies in it. 50 are red, 20 are yellow, and 30 are green. The lollies are all mixed up in the container. You pull out 10 lollies.

a) How many reds do you expect to get?

b) Suppose you did this several times. Do you think this many would come out every time? Why do you think this?

c) How many reds would surprise you? Why do you think this?

2. Suppose six of you do this experiment.

a) What do you think is likely to occur for the numbers of red lollies that are written down?

______, ______, ______, ______, ______, ______ Why do you think this?

3. Look at these possibilities that some students have written down for the numbers they thought likely.

(a) 5,9,7,6,8,7 (b) 3,7,5,8,5,4 (c) 5,5,5,5,5,5 (d) 2,3,4,3,4,4

(e) 7,7,7,7,7,7 (f) 3,0,9,2,8,5 (g) 10,10,10,10,10,10

Which one of these lists do you think best describes what might happen? Why do you think this?

4. Suppose that 6 students did the experiment. What do you think the numbers will most likely go from and to?

a) From __________ (lowest) to __________ (highest) number of reds. Why do you think this?

Now try it for yourself: ______, ______, ______, ______, ______, ______

Figure 1. Part of the Lollies Task interview protocol.

Sample. The sample for this study consisted of 73 students from public schools in the Australian state of Tasmania in a preparatory class (prep) ($n = 7$), grade 3 ($n = 18$), grade 5 ($n = 18$), grade 7 ($n = 15$), and grade 9 ($n = 15$). Students in grade 3 and above were chosen based on interesting or unusual responses to survey items. The prep students were considered to be bright and articulate by their teacher who had created an innovative program in mathematics throughout the school year. The students in grades 3, 5, and 7 were considered to cover a range of average to higher ability levels, whereas the grade 9 students were selected mainly from classes considered to be of average ability.

Analysis. The criteria for categorisation of the numerical responses to questions 1 to 4 in Figure 1were the same as those used by Shaughnessy et al. (1999). Although the analysis of reasoning took place with background knowledge of the Torok and Watson (2000) study, all student explanations to the lollies task were analysed by the authors using a clustering procedure (Miles & Huberman, 1994) based on similarities in the types of reasoning shown. For this study, only the responses to the Lollies Task were considered whereas Torok and Watson had clustered responses to the complete interview, including other protocols, for example, focusing on variation in weather.

Results

First Research Question: Initial Responses

Initial analysis of the interviews focused on the choices of centres and ranges. Outcomes for responses to the parts of the protocol where students were asked for point estimates (Q1a)), lists of 6 possible outcomes (Q2a)), a choice of seven outcomes (Q3), and a range (Q4a)), are reported by grade in Tables 1 to 3. For point estimates many students said "about" or "probably" while writing a single number. Over half of these focused on 5 as an expected outcome (see Table 1). For the listing of the 6 outcomes, approximately three-quarters of the students used "five" as the centre, with slightly more students favouring a lower centre (< 4) than a higher centre (> 5). An oscillation effect was evident across grades with a peak at grade 3, a decrease in grade 5, an increase in grade 7, and another decrease in grade 9. The results for spread show an increase in the "reasonable" category for the higher grades, with a levelling out across grades 7 and 9. Over half of the students in each grade level except prep (see Table 1), combined a "five" centre and a "reasonable" spread in their choice of 6 outcomes. The tendency to predict repeated values increased to grade 7 and declined in grade 9. Only 10% of students overall changed some of their predicted values after doing 6 trials and there was no trend over grades.

Table 2 shows that for the multiple-choice question (3), half of the students chose the "five-reasonable" response (b) across grades (slightly less for grade 9). This is consistent with the results of Q2a) for grades 3 and 5, an increase in performance for the prep students, and a decrease in performance for the students in grade 7 and 9 for the "five-reasonable" combination. Very few students changed their multiple-choice responses after completing their 6 trials.

Categorisation of responses to Q4a) was determined using the classification scheme of Shaughnessy et al. (1999). The "wide" classification always included a spread of at least 7 and numbers greater or less than 5 and hence the centre did not seem relevant. The "reasonable" grouping, however, could be separated with "low", "five", and "high". Table 3 shows the results with an increase in "five-reasonable" responses across grade levels up to grade 7 and a decline for grade 9, paralleled by a decline in "wide" responses up to grade 7, then an increase at grade 9. This is inconsistent with the results for the spread of the 6 student-generated outcomes in Q2a), and also with Q3, which offered less opportunity for a wide choice to be made. For ranges with "reasonable" variation but not a centred response, grades 3 and 5 preferred "low-reasonable" estimates, whereas grades 7 and 9 preferred "high-reasonable" estimates. Again few students changed their ranges after the 6 trials.

Table 1

Percent Distribution of Q1a) and Q2a) Responses for Each Grade and Overall

	Prep (*n* = 7)	Grade 3 (*n* = 18)	Grade 5 (*n* = 18)	Grade 7 (*n* = 15)	Grade 9 (*n* = 15)	Total (*n* = 73)
Q1a) – Single outcome						
≤ 4	29	39	17	13	33	26
5	57	39	55	67	53	53
≥ 6	14	22	28	20	13	21
Q1a) – Change after trials						
	0	0	11	0	7	4
Q2a) – 6 outcomes (centre)						
Low	14	11	22	13	27	18
Five	57	83	67	80	73	74
High	29	6	11	7	0	8
Q2a) – 6 outcomes (spread)						
Narrow	0	0	11	7	7	5
Reasonable	43	61	78	87	87	74
Wide	57	39	11	7	7	21
Q2a) – Five & Reasonable						
	14	50	55	73	67	56
Q2a) – Repeated values in predictions						
	14	44	72	80	67	60
Q2a) – Changed predictions after trials						
	14	6	0	20	13	10

Table 2

Percent Distribution of Q3 Responses for Each Grade and Overall

Choice (Q3)	Prep (*n* = 7)	Grade 3 (*n* = 18)	Grade 5 (*n* = 18)	Grade 7 (*n* = 15)	Grade 9 (*n* = 15)	Total (*n* = 73)
(a) High, reasonable	0	0	0	7	13	4
(b) Five, reasonable	57	50	50	47	40	48
(d) Low, reasonable	14	28	22	20	27	23
(c) Five, narrow	14	0	11	20	7	10
(f) Five, wide	0	17	17	7	13	12
(e) (g) Other	14	6	0	0	0	3
Change after trials	0	6	11	0	0	4

Table 3

Percent Distribution of Q4a) Responses for Each Grade and Overall

Range (Q4a)	Prep (*n* = 7)	Grade 3 (*n* = 18)	Grade 5 (*n* = 18)	Grade 7 (*n* = 15)	Grade 9 (*n* = 15)	Total (*n* = 73)
Low, reasonable	0	17	17	7	0	10
Five, reasonable	14	22	44	67	47	41
High, reasonable	14	6	11	13	13	11
Wide	71	56	28	13	40	38
Change after trials	14	6	17	0	0	7

Second Research Question: Reasoning Expressed

Students were asked to give reasons for all of their numerical responses. Often students relied on similar reasoning throughout the protocol, such as "more red", "50% red", or "anything can happen", when asked for predictions. It was hence possible to combine an assessment of reasoning with observation of consistency or inconsistency in numerical answers to describe a developmental progression based on the 73 responses.

Level 1 – Intuitive ikonic reasoning. Of the 22 students observed to respond at Level 1, 12 students produced incompatible values for their choices of 6 possible outcomes and the range of 6 outcomes. One grade 5, for example, chose 1s and 2s, then a range of 3 to 8, and appeared confused at times. Another 8 students who gave similar explanations at this level gave ranges indicating all possible outcomes (e.g., 0 to 10 or 1 to 10), so there could be no contradiction with their chosen values. Only two students, both preps, gave compatible values for their choice of 6 possible outcomes and the range of 6 outcomes.

The reasoning expressed at Level 1 was intuitive, mentioning guessing, favourite numbers, location in the container, lollies being mixed up, numbers of lollies that would fit in the hand, "you could get any", and "wouldn't get all in one go". All of the prep students were classified in Level 1, displaying only ikonic reasoning. One prep student, however, seemed to have an intuitive idea of proportionality; to Q1a) he responded "because there's 50 and 5 like … 10." He could not clarify this answer further, however, and when asked to justify his choices for the 6 predictions (4, 6, 5, 3, 8, 2) he said he chose them because "4 + 6 = 10, 5 + 3 = 8 … and 2 is my second best number." Graphs generated by the students tended to be imaginative (see Figure 2), drawings of the lollies, or lists of numbers.

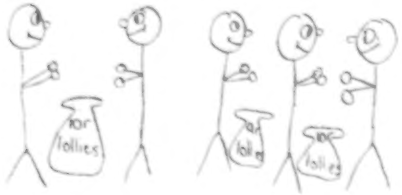

Figure 2. Level 1 representation of 40 trials.

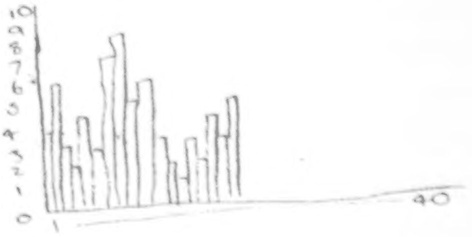

Figure 3. Level 2 representation of 40 trials.

Level 2 – "More red" but inconsistent reasoning. Of these 25 students, 7 students made low initial estimates (2 to 4) and 5 chose the Q3 response (d) that was also low (two chose (f)). Overall their six choices were low as well. Of the ranges, two were inconsistent with the six choices, one included all possible, and four were reasonable (2-6, 2-8 or 1-8). When faced with higher outcomes from their own trials, four changed some of their initial six choices to higher values. The rest of the students in this group (18) gave initial responses associated with middle or higher outcomes. Inconsistencies between the responses for the six suggested outcomes and the range occurred for two students. Five students included all possible numbers in their ranges and five others did not amend their ranges in the light of discrepant experimental outcomes.

Students at Level 2 justified their numerical responses based on "more red" without explicit mention of proportion in relation to the other two colours. One student, for example, explained his "five-reasonable" list of 6 outcomes with "because there are more reds than any other colour", and another student in response to Q3 justified her "five-reasonable" multiple-choice answer with "because there are more red than yellow and green." In the graphing task, most students generated their own representations as lollies or numbers, with a few drawing series graphs for the trials demonstrating wide variation (see Figure 3). With the help of a grid, some appeared to appreciate frequency but not centre.

Level 3 – "More" or "half" red with centred reasoning. At this level, 18 students discussed variation around the centre with non-extreme ranges consistent with their predictions. Four students in this level expressed a strong preference for the middle value and chose (c) in the multiple-choice question, with strong probabilistic reasoning, for example "half the number is red, so there is 50/50 chance of getting red." Only one student missed changing a range after the experiments were performed and four students had difficulty providing a graph consistent with their number choices and discussion.

At this level students used reasoning based on "more" or "half" red. Although, some relied on reasoning similar to that found in Level 2, an intuitive acknowledgement of "centre" was almost always present, for example, "mostly around 5 and mostly reds." Other students were more definite in their preference for "half" when explaining their numerical choices. In response to Q2a), one student reasoned that "some might go up higher, some might be lower, but half of them is red." Similarly, another student, after choosing the "five-reasonable" choice in Q3 explained "(b) – All in the middle, [other alternatives] mostly all high or low". Although many students continued to have difficulty creating their own graphs, with the help of labelled axes, ten provided idiosyncratic graphs indicating variation about the middle. One is shown in Figure 4.

Level 4 – Distributional reasoning. Eight students displayed a strong appreciation for the proportion of reds in the container as well as variation, giving reasons to Q1a) like "about 5, because there's 50 red, so divide 100 by 10 and then divide 50 by 10 to get 5." Seven could represent this in a distribution of outcomes, one of which, however, was an idiosyncratic graph by a grade 3. Another grade 3 responded appropriately to Q1a) by stating "5, because 50 is half of 100, so 5 is half of 10" and backed up his "five-reasonable" prediction of 6 outcomes by saying "they're all around the 5 mark", but could not understand the graphing task. One grade 9 chose c) in the multiple choice, because "the average number would be 5, so the most likely", but drew an appropriate distribution with explanation with variation. Most of the distributions drawn were wider than statistically appropriate, with the exception of one student who provided the tally in Figure 5. Given the lack of large-scale experimentation by the students, they showed an adequate understanding of the relationship of centre and variation in the task.

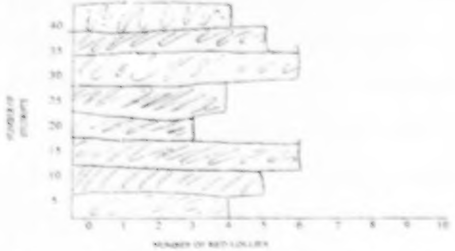

Figure 4. Level 3 representation of 40 trials.

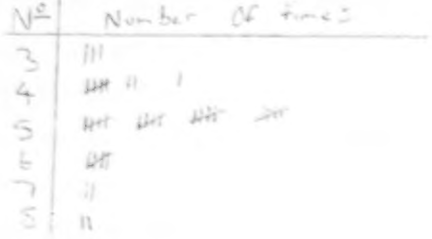

Figure 5. Level 4 representation of 40 trials.

Overall, the distinguishing features among the levels were related to appreciation of the relationship between the centre and spread of predicted values. All students understood the task and most had intuitions about both middle and spread. At Level 1 these intuitions could not be further supported with appropriate reasoning, and at Level 2 there was an attempt at justifying choices but this was inconsistent over different questions. Level 3 responses were more consistent in terms of both centre and spread but lacked strong proportional and/or distributional thinking. This was demonstrated at Level 4. Table 4 shows the grade levels of the students at each developmental level. There was a trend for increasing level of performance up to grade 7 but a drop in grade 9.

Table 4

Percent Distribution of Reasoning Levels for Each Grade and Overall

	Prep (n = 7)	3 (n = 18)	5 (n = 18)	7 (n = 15)	9 (n = 15)	Total (n = 73)
Level 1	100	44	39	0	0	30
Level 2	0	28	56	20	47	34
Level 3	0	17	6	60	33	25
Level 4	0	11	0	20	20	11

Discussion

Initial responses. Although this study used questions similar to Shaughnessy et al. (1999), outcomes were slightly different. Shaughnessy et al. had an "unclear" category for each question, for responses that were ambiguous or not appropriately answered. In the present study, a higher percentage of students were classified in the "five" and "reasonable" categories, probably a result of using an interview protocol where students read and talked through the question carefully, resulting in no missing data. The interviewer was also free to probe students' thinking to clarify nebulous comments. This was likely to contribute to a higher percent of optimal responses per grade level when compared to the Shaughnessy et al. study. The interview setting also provided the opportunity for students to express intuitive ideas of variation, for example "about 5". The fact that they only recorded the number "5" on the answer sheet probably suggests what occurred for many students in the earlier surveys. Compared to Shaughnessy et al., who found that students completing a new survey after watching the 6 trials performed in front of the class showed greatly improved responses, students in this study, performing the experiments themselves, generally were not influenced to change their predictions. For many responses, more than in the survey study, no change was required; however in other cases, change would have been appropriate. With their original work was in front of them, not starting over, perhaps a sense of "ownership" precluded some students from changing their minds.

Because this study interviewed grade 9s chosen from classes of average ability, it is not surprising that there was a decrease in "five-reasonable" responses for grade 9 in Q2a) and Q4a). For Q3 in the current study, there was a uniform performance across the grades, with a slight decrease for grade 9. The Shaughnessy et al. (1999) study, however, using combined data from all versions of the task, found fluctuations in performance across all grades, with an increase for grade 9, and a decrease for grade 12.

Both Shaughnessy et al. (1999) and Zawojewski and Shaughnessy (2000) reported larger percents of responses classified as "high" as opposed to "low" across the different versions of the task. In the current study "low" was more common than "high" in both the list and choice versions of the task. Like Shaughnessy et al. (1999), however, "wide" was preferred to "narrow" on all versions of the task. In particular the highest percents of "wide" responses in the range version of the task (Q4a)), were reported for the prep and grade 3 students, whereas for Shaughnessy et al., this was prevalent throughout all grades.

Reasoning levels. The graphing component of the Lollies Task was not given the same weight as verbal reasoning when classifying responses into the four levels described. Because it was felt that some students lacked graphing ability (especially those in prep, grades 3 and 5), their explanations of the predicted outcomes in terms of centre and spread were weighted more heavily than technical graphing ability. For others, however, the graphing component acted as a supplement to help clarify uncertain or confused responses.

In contrast to Torok and Watson's (2000) observation of younger students being easily swayed by experiments, here most students seemed unaware that the experiments should be considered. Similar to the earlier study, students at Level 1 could not produce meaningful graphs (one pictograph that was drawn could not be explained) and there was no evidence of proportional or even majority reasoning. At Level 2 in the current study, reasoning expressed as "more reds" reflected the weak understanding of proportional ideas found earlier. Again, inconsistency was a common feature of responses about chance outcomes. For many, producing graphs was still difficult. At Level 3 students were more likely to suggest proportional ideas, such as "half", in both studies, although in the current study, responses were stronger in clustering about the middle, in both verbal suggestions and graphical representations. At Level 4, students in both studies had firm ideas of proportion and variation with clustering about the middle, but spread was often too great. Although a few students reflected the strict probability views also expressed by the grade 12 student of Reading and Shaughnessy (2000), all displayed an appreciation of sampling variation elsewhere in the protocol. Overall the increasing structural complexity with higher levels observed in earlier studies was confirmed in the current data.

This study adds to the evidence about students' understanding of sampling variation in a chance context, particularly in allowing for reasoning to be expressed, supplemented in some cases by graphs, in justifying numerical predictions. Further research is needed to consider the effect of different proportions of red lollies in the container, possible cognitive conflict if the outcomes do not match the proportions students are *told* are in the container, and the effect of collaboration if pairs or triples of students engage in the task.

Acknowledgments

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Textbooks: An Investigation of their Visual Attractiveness

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The visual attractiveness of two Queensland mathematics textbooks is examined. Diagrams were categorized according to their type, and each type was examined in terms of perceptual precedence. Series A textbook was found to have one third of diagrams of a cosmetic type, commanding 40% of the perceptual precedence. Such diagrams have no relevance to the subject matter, and therefore the potential to distract a student. Series B textbook was found to have combination diagrams in abundance that demanded perceptual precedence. Each diagram of this type has two or more functions and the potential to be effective in developing mathematical understanding. It is recommended that further research be conducted into student uses and perception of diagrams so as to inform authors and illustrators.

Recently, a State Primary School in Queensland abandoned one series of mathematics textbook, Series A (Boswell, 1998) appropriate for Years 1 to 7 inclusive, because the teachers felt the series was not meeting its claims of promoting mathematical understanding, knowledge and skills appropriate to the learner. In the views of the teachers, most students were not stimulated by the series, and as a result, additional materials (over and above those one would normally expect a classroom teacher to employ) had to be generated to compensate for the shortfalls of the series. Series B (Parker, McSeveny, Collard & Thomas, 2000) was selected to replace Series A as the teacher's felt both series covered the same curriculum content from the Year 1-10 Mathematics Syllabus. However, Series B had clearer explanations and a more visually attractive format than Series A.

The purpose of this paper is to investigate this notion of the visual attractiveness of the two series. I will concentrate my attention on the function of the diagrams as primary transmitters of visual information in their own right. Both teachers and textbook authors and illustrators use diagrams frequently on the assumption they make things easier for students to understand (Henderson, 1999). It is worth a close investigation of the use of diagrams in textbooks, as they may not be as useful as one may think.

Diagrams as Visual Information Transmitters

In many circumstances in mathematics (such as in early numeration and geometry), diagrams are traditionally used as the main transmitters of information with the "written text being absent or sparingly used in a support role as captions or labels" (Lowe, 1986, p. 8). Diagrams are a fundamental component of today's textbooks. As outlined by Hannus and Hyönä (1999), diagrams serve a number of functions: (a) attention guiding – to attract a student to pay more attention to the material in general, and specifically to the illustrated contents, (b) affective – to develop interest and motivation towards the textbook material, (c) cognitive – to improve learning by enhancing comprehension and memory, and (d) compensatory – to improve the learning achievement of poor readers. Diagrams may also be particulary efficient in providing spatial information difficult to express in words (Levie & Lentz, 1982). Levie and Lentz (1982) suggested, on the basis of their seminal review of the literature, that in order for diagrams to be effective and to obtain improvement in

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B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 374-381). Sydney: MERGA. ©2002 MERGA Inc.

learning, they needed to be directly relevant to the text content. Gombrich (1974) pointed out that the educational intention of the illustrator is not always matched by the viewer. It may not be a simple question of comprehension by the viewer, but rather a question of the generated meaning matching the intended meaning.

A review of the research literature indicates that diagrams can play at least four interrelated roles:

1. *Identification* - Diagrams that point out or identify parts of things (Cook & Mayer, 1988, Charles & Nason, 2001) (e.g., geometry diagrams of a circle upon which the names of the diameter, radius and circumference are labelled (Iding, 2000; Kidman, 2000; Lowe, 1993)).
2. *Comparison* - Diagrams that compare one kind of thing to another (Lemke, 1999; Hunter, Crismore & Pearson, 1987; Winn, 1989). (e.g., two different kinds of time-pieces, analogue and digital, might appear next to each other).
3. *Sequence* - Diagrams that point out stages in a chain of events (Cook & Mayer, 1988; Hunter, Crismore & Pearson, 1987). Typical examples would be diagrams of a sporting race showing the ordinal finish.
4. *Combination* - Diagrams that combine two or more of the above functions (Cook & Mayer, 1988; Hunter, Crismore & Pearson, 1987; Iding, 2000; Lowe, 1993; Winn, 1989). (e.g., a scaled map provides more than one view - an additional cross-section or enlarged view (i.e., comparison). Aspects of the map can be labeled (i.e., identification) and the pathway to follow can be indicated via the use of arrows (i.e., sequence).

As outlined above, diagrams have different purposes and roles. Along with each purpose and role, a different interpretation and set of spatial skills for understanding also needs to be considered. Hannus and Hyönä (1999) outlined five cognitive processes necessary for the comprehension of an illustrated textbook: (a) to comprehend abstract concepts and principles, (b) to decide in which sequence the text and diagrams are studied, (c) to distinguish between relevant and irrelevant diagrams, (d) to decide which diagrams are related to each other, and (e) to integrate related pieces of diagrams and text into a coherent internal representation. It can be seen from this list, that an intellectually high-achieving child is more likely to carry out such integrative processes than a low-achieving child.

In a study investigating both high and low-ability children, Reid and Beverage (1990) found the less able children spent more time studying diagrams and accessed them more frequently than the higher ability children. This was particularly so for low-spatial-ability children presumably because they had more difficulty forming spatial representations (Hegarty, Carpenter, & Just, 1991). Low-ability learners did not seem to spontaneously construct elaborate mental structures of text materials, but did so in the presence of a diagram (Reid & Beverage, 1990). In terms of an authentic textbook with several diagrams (i.e., 3-5) per page, the diagrams distracted children's attention away from the content, and thus hampered successful learning (Harber, 1983).

The value of a diagram, or a set of diagrams, may increase with increasing detail, but a point has to be reached where any additional detail or diagrams, becomes a distraction to the viewer from the key concept of the author (Henderson, 1999). Because a learner often learns as much from a diagram as from text, it is essential that the visual components of textbooks are not merely insertions, but actually relate facts or understandings (Dowd, 1990). Cox (1996) found that logically inappropriate material, (i.e., analogies and illustrations) should be eliminated, as such material is distracting. Such distractions "reduce the impact of a lesson, and increase the chance of misunderstandings developing" (Cox, 1996, p. 24). Excessive and unrelated diagrams are both a handicap and a disincentive to continue the study of the subject, something we obviously want to avoid in mathematics. Students experience problems in the processing of diagrams when (a) all aspects of a diagram and text have equal perceptual status, and (b) "visual prominence to information that is actually of low salience with respect to the presented content" (Lowe, 1996, 240).

Textbooks as Learning Materials

The textbooks being considered in this paper are authentic textbooks, as classified by Willows, Borwick and Hayvern (1981), in that they typically contain more than one diagram per page and the diagrams often cover a significant proportion of the page. In order to effectively use such a textbook, a child would need to decide as to which diagram they should attend to and which diagram they ought to ignore, when to study each diagram and what information they need to extract from each diagram (Hannus & Hyönä, 1999). A textbook can therefore be a highly complex teaching item for a child, especially if there is an absence of text to diagram links.

Methods and Data Analysis

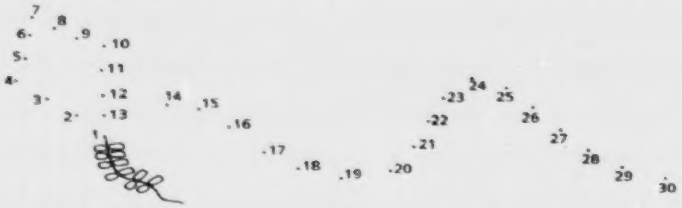
Two methodologies were developed for investigating the visual attractiveness of the two series of textbooks. The data comprised the entire set of diagrams contained in eight mathematics textbooks, four from each of Series A and B (Years 1, 3, 5 and 7). All diagrams were classified according to the role each played. The frequency of diagrams across the Years 1, 3, 5 and 7 textbooks were then averaged and tabulated for each series. Averages were calculated in order to obtain an overview of each series for comparison purposes, as it was Series A in general, the teachers were dissatisfied with, rather than individual year levels.

In a second methodology, four classes of children, one class from Years 1, 3, 5 and 7 (a total of 115 students), participated in an 'initial perception' experiment for each series of text book. Each child was asked to randomly select a page in the Series A textbook, relevant to their year level, and to 'tick' the diagram that initially caught their attention. This was to identify the pre-attentive stage, a stage that occurs "within the very first few moments of seeing the picture before we start to process it in a deliberate, systematic way" (Lowe, 1996, 239). The child then placed a 'postit note' on the page as a marker. This was then repeated nine times giving a total of 10 random pages per child, per class for Series A. The 'initial perception' experiment was then repeated for Series B. Each 'ticked' diagram in each textbook in each series was then classified according to the role each played. The frequency of diagrams across the Years 1, 3, 5 and 7 textbooks were then tabulated for each series.

Results and Discussion

It was necessary to create two additional classifications in order to accommodate *all* diagrams. It was found that the four types of diagrams (identification, comparison, sequence and combination) compiled from the literature related only to those diagrams comprising a pedagogical role. Diagrams in the forms of novel presentation of numbers and cartoon drawings also needed to be accommodated, so the classifications of illustrative and cosmetic diagrams were included. Figure 1 provides examples of diagrams in each classification.

Join the dots to complete the picture. Name, colour and decorate the shape.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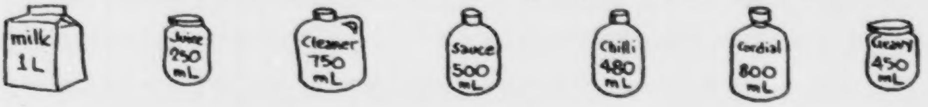
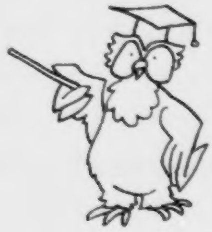
(a) Sequence (Series A)

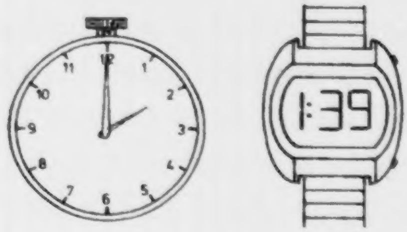
Arrange these containers in order from **largest** capacity to **smallest** capacity.


milk 1L
Juice 250 mL
Cleaner 750 mL
Sauce 500 mL
Chilli 480 mL
Cordial 800 mL
Gravy 450 mL

(b) Combination (Series A)



(c) Cosmetic (Series A)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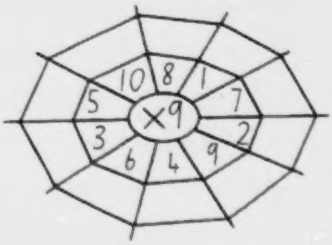
(d) Comparison (Series B)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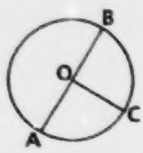
(e) Illustration (Series B)



The **radius** is the distance from the centre to any point on the circumference.
AO = OB = CO

(f) Identification (Series B)

Figure 1. Examples of diagram types.

Table 1 shows the average frequency and percentage of each type of diagram for Series A and B. Table 1 indicates a clear difference in the average number of diagrams per series. Series A has, on average, an additional 137 diagrams to Series B. When one considers the average number of pages per textbook in Series A (165 pages), this equates to approximately 1 extra diagram per page. Having one more or one less diagram per page ought not to be a major deciding factor of the visual attractiveness between two series, except when the type of diagram is considered.

A difference in the proportion of cosmetic diagrams between the two series was found. Nearly one third of all diagrams in Series A were of a cosmetic type. Figure 1(c) gives an example of such a diagram. This cartoon of an owl has no relationship to the mathematical activity of completing a multiplication grid, yet they (the owl and the grid) are presented side by side on the page, with equal space being allocated to each. Series B has only 18% of diagrams falling into this 'cosmetic' category.

Table 1

Frequency of Each Diagram Type for Series A and B

		Identification	Comparison	Sequence	Combination	Illustration	Cosmetic	Total
Series A	f	122	104	18	120	70	201	635
	%	19	16.5	3	19	11	31.5	100
Series B	f	66	98	34	160	50	90	498
	%	13	20	7	32	10	18	100

Whilst Series B has its share of cartoon characters, the majority are similar to the one shown in Figure 2. Such cartoons carry an educational message relating to the mathematical activity on the page, and so have been classified into one of the other categories depending on the message. Series B has fewer irrelevant cosmetic diagrams than Series A, and this may have been an influencing factor of the visual attractiveness between the two series.

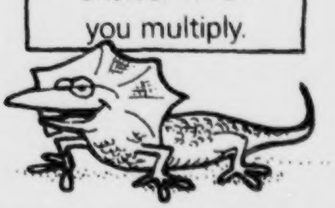

The product is the answer when you multiply.

Figure 2. Example of a cartoon from Series B.

Neither series seems to include many diagrams of the sequence type. Perhaps the authors and illustrators do not think mathematical concepts like ordering or chain of events are suited to diagrammatical instruction. Of the 52 diagrams of the sequence type presented in the two series, only one was identified as to having any form of saliency in the 'initial perception' experiment. Table 2 shows the frequency and percentage of each type of diagram for Series A and B for the 'initial perception' experiment.

In order to capture the pre-attentive precedence of diagram types, an 'initial perception' experiment was conducted with 115 students on both series of textbooks. The experiment exposed the types of diagrams that had displays, symbols or characteristics that took perceptual precedence over other diagrams due to distinctive visual characteristics. As Table 2 indicates, 40% of perceptual precedence in Series A was allocated to the cosmetic type of diagrams. As the cosmetic diagrams are not related to the subject matter in the textbook, such a high perceptual precedence is of concern. For a child to turn to a page in a textbook and be visually drawn to a diagram of no relevance to the lesson, it has to be distracting. It appears that the cosmetic diagrams have the potential to "reduce the impact of a lesson, and increase the chance of misunderstandings developing" (Cox, 1996, p. 24). Excessive and logically inappropriate diagrams are both a handicap and a disincentive to begin work, something to be avoided in mathematics. Series B has a far lower percentage of cosmetic diagrams, and would therefore have more potential than Series A in getting a lesson off to a reasonable start. The diagrams that present a combination of functions appear to capture the pre-attentive precedence in Series B. Why this is the case is unclear from the study, but it is interesting to speculate that the detail contained in such diagrams improves their perceptual precedence in a similar way that the value of a diagram, may increase with increasing detail (Henderson, 1999).

Table 2

Frequency and Percentage of each Diagram Type for Series A and B for the 'Initial Perception' Experiment

		Identification	Comparison	Sequence	Combination	Illustration	Cosmetic	Total
Series A	f	225	120	1	300	54	450	1150
	%	19.5	10.5	0	26	5	39	100
Series B	f	370	243	0	467	30	40	1150
	%	32.5	20	0	41	3	3.5	100

There are a number of interesting observations between Table 1 and Table 2. Table 1 indicates that for both series', the combination diagrams have approximately 20% to 30% frequency, yet they have a higher perceptual precedence, especially for Series B. A similar finding relates to the identification diagrams in Series B.

The inverse situation occurs for comparison (Series A) and illustrating (Series A and B) diagrams. For these two categories, there appears to a greater frequency of diagrams in either series', but they do not have great perceptual precedence. A possible explanation for this may lie in the placement of the diagrams on the page.

Conclusion and Implications for Further Research

The teachers may well have been justified in selecting Series B over Series A on the basis of visual attractiveness and stimulatory material for their students. Series A has almost one third of its diagrams in the cosmetic category, meaning these diagrams are totally unrelated to the content of the material. Another problem with Series A is that these cosmetic diagrams are commanding almost 40% of the initial perceptual precedence. Potentially, these cosmetic diagrams could distract many students and therefore have a negative impact on the start of the lesson. The teachers felt their students were not being stimulated mathematically, by Series A, and the distraction potential of the cosmetic diagrams may be responsible. If this is the case, then the teachers ought to be satisfied with their selection of Series B. Series B has a marked reduction in the number of cosmetic diagrams, and these few diagrams have low initial perceptual precedence. Combination diagrams rated very highly in initial perceptual precedence for Series B. Such diagrams have more than one purpose in portraying mathematical meaning, and so, coupled with high initial perceptual precedence, they have potential to be good aids in developing mathematical understanding. Thus it appears that the visual attractiveness of Series B, over Series A, may lie in the decreased number of cosmetic diagrams, and increased proportion and perceptual precedence of combination diagrams.

Further research is recommended. An analysis of the diagrams in textbooks is necessary to give authors and illustrators some understanding of what is thought to influence the way diagrammatic information is being processed by students. Authors and illustrators would be more able to design effective textbooks if they had some understanding of that information from the student perspective. Authors and illustrators need to be aware of what the students have to deal with when presented with a textbook. With this knowledge, an author and illustrator can be more analytical in the design of the textbooks. Therefore, an investigation into establishing the differences between pre-attentive and attentive (deliberate and systematic) processing of diagrams is recommended. Such an investigation could determine perceptual characteristics of a diagram such as the size, shape and overall arrangement favorable to its effectiveness, but that are not noticed in pre-attentive processing. Knowledge of these visual characteristics could result in improved diagrams so as to direct student attention in initial moments to salient aspects of a page and it's diagram thus improving subsequent progress of mathematical understanding.

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Pedagogical Content Knowledge Development of Preservice Secondary Mathematics Teachers

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This study investigated how pedagogical content knowledge developed while five preservice secondary mathematics teachers were enrolled in a mathematics methods course. In three phases, data collected were participants' concept maps of pedagogy, teaching and learning, and assigned content and related interviews. Themes of adjusting and reflecting often developed during interactions with students' misconceptions or when timing and motivation were pedagogical issues. Implications include the metacognitive benefits of focused reflection on pedagogical content knowledge during preservice education.

Only a century ago, majoring in a subject was considered sufficient for those becoming teachers. There was no concern for how subject matter was transformed from the teacher's knowledge into the content of instruction (Shulman, 1986). This raises questions today about the transition from expert student to novice teacher. A particular transformation of which little is known, is characterised by the development of pedagogical content knowledge. Shulman conceptualized pedagogical content knowledge as, "the ways of representing and formulating the subject that make it comprehensible to others as well as the most useful strategies likely to aid in the reorganization of the understanding of the learner" (Shulman, 1986, p. 9). Clarified by Ball (2000), pedagogical content knowledge includes knowledge of what is typically difficult for students, most useful for teaching a specific idea or procedure, and ways to develop a particular idea.

Researchers have studied many concepts related to pedagogical content knowledge such as teacher beliefs and conceptions (Cooney, Shealy & Arvold, 1998), teacher development (Ball, 2000; Day, 1996), teachers' reflection-in-action (Artzt, 1999; Goodell, 2000; Schon, 1987) and teacher knowledge (Ball 1990; Jones & Vesilind, 1996; Simon, 1995, Sullivan & Mousley, 1994). Results have indicated pedagogical content knowledge develops, in part, due to teachers' experiential processes, the permeable belief structures of teachers, and reflection on and within their practice. Few of the studies, however, are specific to novice secondary mathematics teachers. General and mathematics educators, working from a constructivist theory of learning, have not yet answered the question of *how* pedagogical content knowledge develops. Therefore, what is missing is a way to define the building process of pedagogical content knowledge of preservice mathematics teachers as it is "unique to the profession and relatively undeveloped in novice teachers" (Brown & Borko, 1992, p. 212). In this investigation therefore, we aimed to answer the following research question: How and when does pedagogical content knowledge develop in preservice mathematics teachers? Gaining deeper understanding of teachers' processes can be of great importance to teacher educators. Ducharme & Ducharme (1996) called for the application of such research findings. They claimed that "if teachers could be helped and persuaded to make explicit the knowledge which they implicitly use in their day-to-day work, teacher education could begin to achieve something of the practical relevance and theory-practice integration which it is still accused of lacking" (p. 1031). Through this

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study, we attempted to identify practical and applicable aspects of the development of pedagogical content knowledge for mathematics teacher educators.

Theoretical Framework

Teaching can be thought of as a continuous inquiry into content and ways to structure the learners' experiences to best facilitate understanding (Ball, 1990). Janvier (1996) considered preservice teachers' theories of teaching and learning as an important aspect of their evolutionary development. He claimed such evolution occurs through self-reflective processes. Indeed, Artzt (1999) found that individual and social construction of preservice teachers' knowledge changed through ongoing, reflective, self-assessment. We know from these studies that novice teachers' theories shift and change more often when encountering their initial teaching experiences, that is, when they take on the role of the teacher.

Social and interactive processes of communication occur continuously within the classroom as a dimension of teaching practices. The nature of the teacher-student interaction is interdependent but has been outlined by Steinbring (1998) as autonomous also. The teacher cannot be sure the student has comprehended what was the intended new knowledge, and must engage in continuous, informal assessment. As clarified by Steinbring,

> The teacher has to observe and diagnose the students' learning activities and achievements, then change and vary the new learning offers accordingly. In this way, there is an indirect relationship between two autonomous systems: the students' learning process and the interactive teaching process between teacher and student (p. 159).

It is in the relationship among the students' and teachers' systems where the in-action development of pedagogical content knowledge takes place (see Figure 1). Teachers have to analyse in-action students' mathematical knowledge construction, comparing those understandings to what was intended for students to learn.

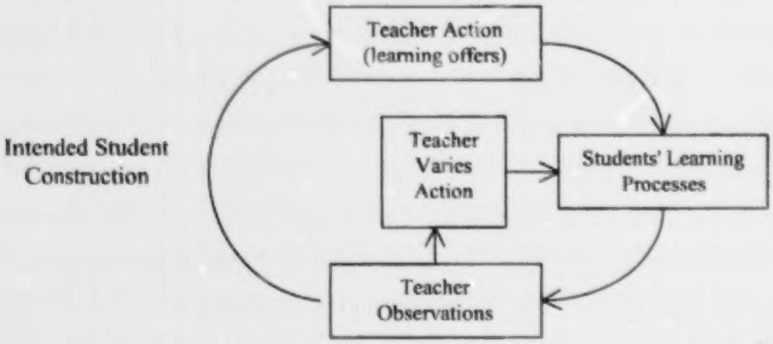
Teacher Action (learning offers)
Intended Student Construction
Teacher Varies Action
Students' Learning Processes
Teacher Observations

Figure 1. Adaptation of Steinbring's teaching and learning as autonomous systems.

It was precisely the observations and concurrent thoughts causing the varying action that we hypothesised to characterise pedagogical content knowledge in this study. In order to study the actions and possibly subsequent varying actions of preservice teachers, it was essential to be able to clearly document and analyse their metacognitive structures by utilising what cognitive psychology offers via concept mapping.

The study of mental events primarily concerned with the human mind (i.e., knowledge, beliefs) and the mental processes in which people engage (i.e., thinking, problem solving, planning), is what cognitive psychology encompasses (Brown & Borko, 1992). An important and relevant assumption of the cognitive psychologist is that within the human mind, knowledge is organized and stored in structures. "Research on learning to teach must examine how teachers learn to translate the knowledge stored in their teaching schemata into operational plans or agendas for classroom action, and how they learn to carry out these agendas in the classroom" (Brown & Borko, 1992, p. 212). Gaining access to such schema can be achieved by studying the concept maps of preservice teachers, which emerged from the seminal work of Novak in the early 1970's. One purpose of using concept maps was to provide a schematic summary by learners of what has been learned (Novak & Gowin, 1984). Since then, it has been adopted to diagnose and test, design curriculum and instruction, and be used as a metacognitive aid to help students learn. Concept maps reveal not only what the learner knows but also the ways he or she organises such knowledge. We used concept maps to serve as a tool for understanding preservice teachers' structural knowledge, especially the development of the novices' pedagogical content knowledge.

Methodology and Data Analysis

Our study involved a voluntary group of five preservice secondary mathematics teachers over a period of four months during their enrollment in a methods course and a concurrent, mid-semester, seven-week, student teaching experience. There were four undergraduate juniors: three white males and one white female, and one white female graduate student. During the seven-week student teaching experience, the two females were assigned to an urban high school where classes were block scheduled. They were assigned to remedial algebra classes and a geometry class. All three males were assigned to urban middle schools in 8th grade classrooms. The preservice teachers were asked to draw three concept maps during three phases. These three phases are outlined as follows: Phase One - *Preactive*, during the middle of the first six weeks in class and before student teaching, Phase Two - *Interactive*, during the middle or near to the end of seven weeks of student teaching, and Phase Three - *Postactive*, during the last three weeks in class after seven weeks in student teaching. Participants studied the concept map in course readings and were taught and given an opportunity to practice drawing a simple concept map prior to Phase One. Their maps represented the structure and organization of their knowledge in three areas: (a) theories of teaching and learning, (b) pedagogy, and (c) content they had been assigned for student teaching. During the second and third phases, the previous map the preservice teachers had drawn was returned. Participants were asked to modify or redraw any part of the map while being interviewed about their revisions. Each 60 to 90 minute session was audiorecorded and transcribed.

Data were analyzed using both qualitative and quantitative methods. The concept maps analyzed quantitatively were scored using a modified version of Novak and Gowin's (1984) method. One point was assigned for the each superordinate concept (S), taken as an indication of the extent of conceptual knowledge. Five points were given for hierarchies (H), indicating instances of knowledge subsumption. One point was given for each example (E) of a hierarchical or superordinate concept, representing specificity of knowledge. Finally, ten points were assigned for cross-links (CL), depicting integration of conceptual knowledge. See Figure 2, in flowchart form, a partial concept map of Rick's pedagogical knowledge, indicating the labeling method and subsequent scoring. The weighting of the scores indicated that (a) cognitive structure is hierarchically organized, from superordinate or general to subordinate or specific, (b) concepts undergo progressive differentiation, gaining greater specificity or more examples, and (c) integrative reconciliation occurs when two or more concepts are related (Novak & Gowin, 1984).

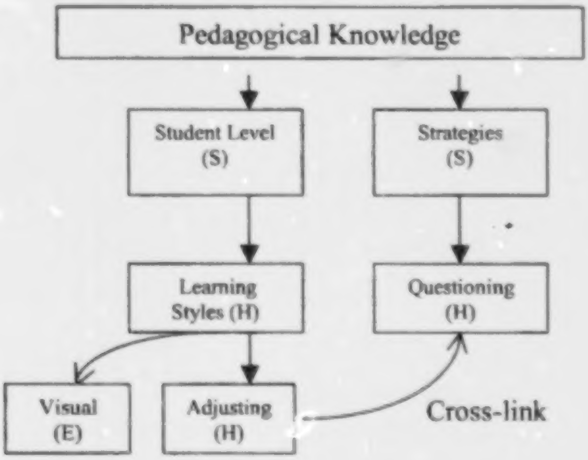

Figure 2. Partial concept map of Rick's pedagogical knowledge.

Analyzing the phases of the study included quantitative examination between-phase maps and within-phase maps for each of the four categories outlined above to first determine when changes occurred in the thinking of the preservice teachers. Qualitative analysis of the maps produced emerging themes of descriptive concepts between and among participants, illuminating how pedagogical content knowledge developed. The interviews provided deeper insight into such themes as the preservice teachers explained their rationale for including concepts as well as the concepts' importance and place in their respective hierarchical schema.

Discussion of Results

Quantitative analysis of the concept maps of the domains of pedagogy and theories of teaching and learning showed further evidence of depth of hierarchical subsumption. In the case of participants' theories of teaching and learning analyzed across phases, all five showed the greatest number of additions of superordinate concepts, hierarchical levels, and examples during the interactive phase of the study. Across that phase they collectively increased the number of cross-links but it was in the postactive phase that they collectively connected levels of hierarchies the most, using cross-links. Participants collectively gained holistic and tighter connections between these domains of content and their emerging theories of teaching and learning.

It was in the extended analysation of pedagogy maps that we were able to clarify the depth of the changes that took place during the interactive phase as well as across all phases. Using aggregated data of the five preservice teachers across the three phases, the weighted subscores were used to generate the descriptive statistics in Table 1.

Table 1
Pedagogy Concept Map Using Aggregated Data, Mean ± Standard Deviation

	Preactive Phase	Interactive Phase	Postactive Phase
Superordinate Concept	5.4 ± 0.9	6.4 ± 1.1	7.2 ± 1.3
Examples	8.4 ± 3.7	14.0 ± 4.9	18.8 ± 8.9
Hierarchy	21.0 ± 8.2	50.0 ± 26.2	69.0 ± 27.9
Cross-Links	10.0 ± 10.0	18.0 ± 10.9	38.0 ± 31.1

Since the data of interest within this table was *when* the greatest number of changes occurred, the large spread was considered important only to identify the range within each of the phases. The data indicated that the greatest initial number of changes in preservice teachers' knowledge structure occurred between the preactive and interactive phases for hierarchical subsumptions. This growth remained consistent between the interactive and postactive phases. For the cross-link connection between hierarchies in their schemata, the greatest change in preservice teachers' knowledge structure occurred between the interactive and postactive phases. This suggested that during the student teaching experience, the preservice teachers' schema consisted of detailed knowledge of concepts but lacked integration among them until the postactive phase when they reflected on the interconnectedness of their hierarchical structures.

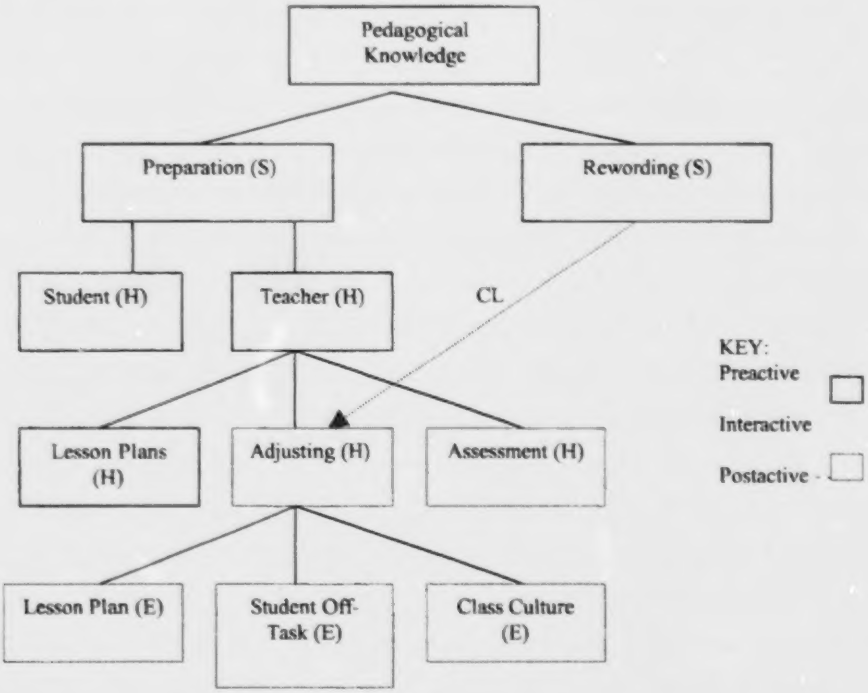
Pedagogical Knowledge
Preparation (S)
Rewording (S)
Student (H)
Teacher (H)
CL
Lesson Plans (H)
Adjusting (H)
Assessment (H)
Lesson Plan (E)
Student Off-Task (E)
Class Culture (E)
KEY:
Preactive
Interactive
Postactive -

Figure 3 Don's partial pedagogical map illustrating "adjusting".

Qualitative analysis of Don's schema of pedagogical knowledge over time indicated that the concept of adjusting was an important characteristic for him as the teacher.

As we analysed concepts across and between pedagogical maps, we gained deeper insights into participants' schema. Figure 3 above, shows that two examples of superordinate concepts from Don's preactive pedagogical map were "teacher preparation" and "rewording". Don continued by adding the example concepts "adjusting for students off task", " adjusting for the classroom culture" and "adjusting lesson plans". Don's conception of teaching became more refined as he practiced the role of teacher in the classroom. Additionally, during the postactive phase, Don revealed that his concepts of "adjusting" and "rewording" should be connected thus indicating a cross-link. His integration of these concepts occurred through reflection-on-action and indicated that he learned that what he said and did in the classroom were always interdependent, contextual and subject to refinement. He and his supervisor discussed that the timing of a lesson can affect whether or not students fully engage with content:

I: So what have you been learning about lesson planning?

D: First of all, you learn that the lesson doesn't go as you planned.

I: What is it that's important about that?

D: I think being able to adjust in certain situations.

I: OK, like what?

D: I thought I had a good plan when the supervisor came. I saw him before the lesson and told him I was having discipline problems. He said maybe I was giving too much time for tasks. And that was what was happening, we were taking too much time for tasks. He suggested taking a problem out of the book and telling them, "take five minutes".

I: So how do you think about timing during planning? This will take so long…

D: Everything is fine in theory, but when you get in the battle zone…

I: Battle zone! (laughs)

D: Yeah. Anything can change with one off-the-wall question and then you're off task.

Don paid particular attention to the time management of his lesson after some coaching by his supervisor. He found that adjusting his lesson plan was necessary when encountering students off task. Other factors that may have contributed to such knowledge growth were the classroom culture or the level of the mathematics task. As we found next, with Rick, misconceptions or skill gaps occurred during his lesson and he also needed to adjust his plans:

I: So you have your lesson plan and then you have what happened.

R: (laughs) I noticed that. I am *constantly* changing. Today we went to the auditorium for an activity. They went in three groups. They still had trouble with measuring and I wasn't expecting that. So when the next group came down, I checked to make sure we knew what we're doing. It seemed that everything that was wrong was altered. So by the time we were in the last group, it was amazing how the lesson went, not content-wise.

I: The pedagogy…

R: Yeah.

I: Does that happen every day, in every lesson?

R: Yeah, definitely.

Rick understood that feedback from students' interactions with content could force him to be reflective about his pedagogical style. That is, he varied his actions (learning offers) as necessary to best facilitate understanding of the content. Rick's expectations of student readiness were unfulfilled but he adjusted his strategy in order to teach the prerequisite skills for his intended lesson.

All five preservice teachers added the concept of "adjusting" as hierarchical knowledge during the interactive phase on the pedagogical maps. It seemed that regardless of their classroom setting or assigned curricular content, the need to vary their learning offers to students was encountered in action. The reflective process used by participants was described by Schon (1987), as "thinking on your feet" in the case of reflection in-action or constructing a new theory about an event after the fact as reflection on-action. For this group of novice teachers, when pedagogical content knowledge development occurred, it was not limited to particular realms of teachers' activity either in class or out of class. Via either kind of reflection, participants could describe classroom impediments to their intended outcomes. Rick seemed to simultaneously develop insight that he would need to perceive student meaning and be able to informally assess student knowledge based on his listening skills. He said, "It's almost like you have to think how they'll see it." Thus, for Rick it seemed that pedagogical content knowledge developed because he was thoughtful about his interactions as well as his plans and when he encountered skill gaps.

For Lauren, encountering unmotivated students gave her an opportunity to reflect on her efficacy. She realized that her questioning techniques regarding students' difficulties allowed her to assess what she could do differently to present the content more clearly and in more easily understandable ways:

L: I'm surprised more than I thought, that they would take a zero when they didn't complete homework. I thought it would be some but not so many. I would ask them, 'what do you have difficulty with?' These questions would assess both the students and myself.

I: How would your knowing what they had difficulty with assess you?

L: If one student said, I'm having difficulty, then I could work with that person one on one. If 50% or more of the class said, 'I have no clue', then I'd say, OK, I've got to go back and do this again. And I've got to find a different way to approach it because what I thought would be something they'd pick up by the method I used didn't work. So now I have to find a different way of teaching this lesson.

Lauren, although frustrated with student apathy, recognised her responsibility as a pedagogue to search for the most comprehensible ways to present the content for students, varying (or adjusting) the learning offers.

Analysis of these concept maps offered that pedagogical content knowledge developed within participants' cognitive framework of reflection in and on their interactions with students' learning processes and ongoing, informal assessment. Instances of considerable awareness on the part of the preservice teachers often occurred when skill gaps, timing, and motivation were issues of relevance in classroom experiences.

Conclusion

While planning lessons, preservice teachers should be prepared to adjust in-action for classroom idiosyncracies. Novice teachers may, over time, be able to reflect on-action in a firm grounding in knowledge, enabling reflection in anticipation rather than in response (Cooney et al., 1998). Future teachers must learn to anticipate students' learning processes, which requires ongoing assessment of the teaching and learning setting. In our study, reflective processes were critical components to the conceptualisation of teaching issues for the preservice teachers. We now know that a fruitful site for growth of the novices is in making adjustments while learning to teach. Since they often do so "on the fly", or in those stream of decision making instances so common in the complexity of classroom dynamics, we suggest that preservice teachers need to engage in continual reflective practice both in action and on action throughout their preparation program. Such metacognition aids in the reorganisation of ongoing constructed knowledge about teaching and learning for novice teachers. We suggest preservice mathematics teachers be required to write, speak and listen to each other in discussion that continuously reexamines and questions their knowledge and beliefs about specific instances of pedagogical content knowledge.

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A Poststructuralist Perspective on the Productive Power of *Process* in Mathematics Education: Practical Implications for Pedagogy and Research

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What would it mean for mathematics education in the twenty-first century were teachers and researchers to take even more seriously, to interrogate more carefully, the productive power of the learning *process*? While the importance of the *cognitive* aspects of process in the construction of robust mathematical understandings and relationships is well appreciated and articulated in teaching and research, its *constitutive* force has largely been ignored. In this paper I use the poststructuralist notion of the productive, constitutive force of the process of *coming to know* in mathematics to (a) extend current understandings of how the teaching/learning processes of the classroom influence participation, knowledge growth and mathematical identity, and (b) contemplate the practical implications of this potentially generative force with regard to instructional practice and further research. The paper is framed by the poststructuralist notion that the ability and inclination to engage in mathematical reasoning and inquiry is not a personal attribute or skill but discursively produced.

Introduction

In contemporary mathematics education and research learning processes are taken very seriously; it is generally well accepted that students actively engaged in processes of conjecture, reasoning and the communication of mathematical ideas and relationships will develop robust conceptual understandings and the ability to apply newly constructed knowledge in and beyond school [Australian Education Council, 1990; The National Council of Teachers of Mathematics (NCTM), (2000)]. For example, the NCTM (2000, p. 20) talks about *proficiency* as the ability to use knowledge flexibly, applying what is learned in one setting appropriately in another and sees a close relationship between understanding and proficiency: "One of the most robust findings of research is that conceptual understanding is an important component of proficiency, along with factual knowledge and procedural facility".

There are however, many students who do not reach appropriate levels of understanding, or proficiency, leaving school anxious and nervous about further involvement in any form of mathematical activity. As stated in *A National Statement on Mathematics for Australian Schools* (1990, p. 31) "children come to school enthusiastic and eager to learn mathematics and ...leave school with quite negative attitudes". While it is often assumed that this disaffection is a direct result of a school curriculum that has not concentrated well enough on the development of understanding, pedagogic practices may also be implicated in how they coercively position some learners. Willoughby (cited in Bourke & Curcio, 2000), for example, suggests that much of the mathematics taught in schools has been taught in such a way as to make students dislike both the mathematics and the learning of it; even if school leavers could use mathematics effectively they would be unlikely to do so. Confrey (2000) speaks up on behalf of those regularly marginalised in pedagogic

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practice to assert that certain groups are disproportionately filtered out by instructional practices and experiences that are impersonal and alienating. The task I set myself in this paper is to delve more deeply into the processes of *coming to know* in mathematics; to analyse *how* engagement in learning processes, while ostensibly leading to the construction of mathematical knowledge, is also dangerous because of its constitutive effect on identity and agency.

The Productive Power of Process

While teaching/learning processes are not commonly considered dangerous, poststructuralist theorising suggests that this is so because engagement in discursive practices, while producing intellectual knowledge also constitutes or produces the learner. The learner, "is the effect of a production, caught in the mutually constitutive web of social practices, discourses and subjectivity; its reality is the tissue of social relations" (Henriques et al., 1984, p. 117). For mathematics education this means that as students engage in those learning processes that educators recognise as powerful and productive, they are simultaneously *subject to* positioning within the familiar storylines and practices of the classroom. Subjectivity, in a poststructuralist sense, moves beyond notions of individual consciousness or perception about action, events and ideas to describe a "way of knowing" about ourselves in the world that is both intellectual and emotional; it describes who we are and how we understand ourselves and is both conscious and unconscious (McNaughton, 2000, p. 97). As students take themselves up within discursive practices (transmission teaching, group work) and storylines (females can't do geometry) their very subjectivity, and mathematical identity, is actively constituted. This act of positioning is not merely a psycho-social event but an embodied productive process whereby the learner *comes to know* (unconsciously, perhaps) about mathematics as a potentially powerful avenue of further work and study and about her/himself as a legitimate user of its authoritative ideas. A student's constituted *knowing* about mathematics invisibly influences her/his appreciation and application of it, in and beyond the classroom (Lather, 1991).

Clearly teachers (and the wider educational community) hope that all students will come to know mathematics as a potentially powerful and useful tool and themselves as competent and confident in its many applications. However, just how this might be realised in the classroom is a difficult and contentious issue. Where historically educators have taken for granted a rational individual able to act autonomously on constructed knowledge, poststructuralist theorisations, recognising the constitutive force of power relations in all discourses including mathematics education, render this a contentious issue. Proficiency, in the way it has been defined by the NCTM (2001) implying an easy relationship between understanding and application, is provisional and dependent on the extent to which learners have access to discursive practices that positively impact on their developing mathematical identities.

While the ability to apply constructed knowledge in new contexts is always provisional and problematic, a sense of agency with (the intellectual knowledge) and in (the discursive practices) can be constituted in teaching/learning interactions that celebrate the student's presence and ways of making sense of mathematics and (the pedagogic) experience. Power/agency/autonomy is not an attribute but a relation; to be able to act in powerful ways the student must be capable of, respected and valued in speaking/writing the accepted 'truths' of a discourse, in enacting established 'ways of being' and in going beyond these to forge something new (adapted from Davies, 1991). Thus the teacher cannot 'give' power to students but can make spaces for them to establish themselves in powerful ways in teaching/learning interactions as they construct mathematical ideas. Significantly for educators, this notion of the operation of power suggests that it can have libratory and/or limiting effects; for example, while a student is always *subject to* relations of power in the classroom s/he also constructs intellectual knowledge and is constituted as a speaking subject who may be able to act powerfully in that, and/or related discourses.

In this paper I rely on a compilation of recent research to demonstrate how power/knowledge/identity relationships render the classroom a productive though coercive space; here learners are *coming to know* about mathematics and themselves as learners as the teacher's already constituted *knowing* influences the extent to which they are able to realise themselves as competent and confident. While the teachers discussed in the analyses below clearly value inquiry-based, investigative learning processes that should lead to robust mathematical constructions, in most cases the ways in which they interact with students appear to render positive outcomes unlikely. This is because, in basing practice on humanist, psychological understandings of learners as essentially rational and autonomous they take agentic participation for granted and do not recognise the extent to which their uses of language and classroom practices tend to have a limiting effect on engagement. How classroom relationships of power/knowledge/identity intersect and operate to constitute identities, as constituted identities simultaneously shape the learning context (Walkerdine, 1988), is the object of analysis and speculation in the following sections of this paper.

Coming to Know…

Over the past several years I have worked extensively in classrooms, teaching and researching the ways in which mathematics education operates in a variety of contexts. One thing I have learned is that the operation of the mathematics education discourse is never innocent; it comprises discursive practices which maintain "socially organised frameworks of meaning that define categories and specify domains of what can be said and done" (Burman, 1994, p. 2). Unfortunately many students *come to know* a mathematics of low intellectual demand, and discursive practices insensitive to constituted socio-cultural identities that rob them of the possibility of making sense of mathematical experiences in personally meaningful ways (Willoughby, 2000). Such students may merely take up "the available discursive position of subordination and (in)difference" (Kelly, 1997, p. 43).

I (Klein, 1999; 2001a), for example, have analysed inquiry-based instructional practice from a poststructuralist perspective to show that *how* inquiry is realised in the classroom can have questionable effects on students' mathematical identity and their ability to (want to) apply constructed knowledge in new contexts. Indeed, the claim to inquiry is merely rhetorical in that it is commonly the case that the students do not engage fully in investigating the mathematical ideas and relationships but rather question whether or not they have undertaken the correct mathematical procedures and whether their answer is correct (Lave, 1997). Similarly, in 'problem-solving' lessons (Klein, 1999) several teachers maintained traditional authority relationships while actively engaging the children in low-level 'busy' work around mathematical ideas of mass, space and number. Though students eagerly engage in these teaching/learning processes, they come to know a mathematics of disparate facts and skills and themselves as dependent always on the teacher. This was also the case in a Year 6 classroom (Klein, 2001a) where the common assumptions on which inquiry based practice is based (Gregg, 1995) were not realised in the classroom; for example, there was no less teacher control, many students did not 'freely' engage in investigative processes nor develop powerful mathematical structures. Although teachers clearly value student engagement in mathematical inquiry with regard to the construction of mathematical knowledge, they are often unaware of the constitutive effects of all teaching/learning processes. Where autonomous action is taken for granted and the conditions of inquiry are not met, students do not come to know the richness and regularity of mathematics, nor are they constituted through discursive practices where they can engage on their own terms to powerful ends.

Zolkower (1996, p. 59), in a paper titled *Math Fictions*, shows how supposedly investigative classroom learning processes, while championing the 'open ended', 'realistic' and 'multicultural' operate in ways that deprive the students of the possibility of making sense of the mathematical situation or of acting in any sense autonomously. With reference to a low-income, multicultural , Latino neighbourhood, Zolkower (1996) demonstrates how a teacher's ignorance of her own cultural assumptions impacts with disastrous effects on her instructional practice and throws into question the possibility that the classroom could ever be free from the operation of relationships of power. More specifically, though the children have to place sense-making on hold to engage at all with the problem (where a person's pay doubles each day and s/he works every day), the teacher blames the students for lack of attention, poor knowledge of facts, confusion and difficulties in implementing problem solving strategies. While one reading of this situation could be that at least the children engaged in some 'doubling' activities, another could interrogate the pedagogic interactions for their constitutive force. Because the process of getting to the answer is teacher led, because the problem itself is 'not real' for the children, and because the teacher blames the students for lacking attention participation is likely to be superficial. As well, it might be assumed that the existing order will be maintained because students are unlikely to challenge a teacher's actions where they are positioned (have come to know themselves) as deficient, having only themselves to blame for imperfectly playing the game of classroom mathematics.

Knowing...

Lather (1991) makes the important point that in our action is our constituted *knowing*. She does not refer to the intellectual thoughts that guide our actions, nor the skills we have learned to implement these actions but the constituted knowledges that have invisibly formed us over the years and coercively, unconsciously influence our every action. In the classroom, a teacher's constituted *knowing* about learners and learning influences the qualitative status of the teaching/learning interactions where intellectual and social knowledges are produced. For example, in the past I (Klein, 2001,b) have framed my teaching on 'constructivist' principles assuming that preservice teachers actively engaged in investigative learning processes would construct robust mathematical ideas and easily implement similar instructional strategies in their classrooms. However, over time, I came to realise that in my teaching I maintained conventional power relations that, to some extent at least, create the mathematics teachers of the future. Preservice teachers come to know ways of being a student that (are constitutive of and) reflect the power relationships of previous schooling rather than interrupt or change them. I now find it surprising that I continued in my 'constructivist' practice without realising that as I encouraged them to construct their own mathematical ideas and relationships, I also positioned them as dependent learners. This may have serious implications for their ability to establish themselves as competent and generative teaching professionals in the future, given that how they are positioned, and are able to establish themselves as reflective learners, has implications for their developing professional identity.

There is also the problem of how socio-cultural assumptions about what is important and how things should be done permeate interactional relationships, leading to disaffection and alienation for some students. For example, I had an Aboriginal student in my class (Klein, 2001b) who came to tell me that she was extremely nervous and anxious about the work we were doing and her participation in my subject generally. Like all teachers, I acted from my constituted *knowing* of learners and learning (Lather, 1991). I set up an extra tutorial for her and other students who were having 'problems' and asked her to keep a journal documenting these problems. I deferred to humanist understandings of the individual, as essentially rational and autonomous, to classify the student as in need of help and 'remediation'. At no time was anything about my motives or instructional practice questioned – I was blind to the White, middle-class European appropriation of the 'real world' in the way I interacted with students and the Anglo-European genesis of journal writing as a means of making sense of experience (Klein, 2001b). Classroom teachers, too, can find themselves confronted by students such as Lenny (Klein, 2000a, p. 75) whose actions in the classroom do not make him recognisable as a legitimate student; Lenny sits awkwardly on his chair and calls out to the teacher who ignores him because of is inability to 'do' school properly. The teacher's constituted *knowing* about how students must disport themselves makes Lenny unrecognisable as a learner in this classroom. In a similar situation, a boy works out answers to mathematical problems in his own way, and then does them the teacher's way, so that he can get a mark (Klein, 2000a, p. 74). As educators we may not be sensitive enough to the alienating effect it can have on students' identity when they have to put sense-making on hold to be able to achieve themselves as legitimate students in the mathematics classroom.

From Humanist Agent to Subjection…

While contemporary research and teaching is based on notions of a humanist individual able to act rationally and autonomously in 'supportive' contexts, poststructuralist thought juxtaposes a timely interrogation. Through analyses such as those briefly sketched above, poststructuralists make visible the mutually constitutive force of individual and context. Teacher and students in the mathematics classroom weave threads of power/knowledge/identity that constitute or create the context while simultaneously they are (re)created in the discursive matrix. Thus pedagogic imperatives must not concentrate on the teacher alone, and how s/he can make teaching more investigative and effective, but rather they must be concerned that the conditions of agentic participation are met. In mathematics education, this means that the students are respected and valued as authors of their own constructions and meanings and ways of making sense of experience, that the mathematical knowledge fostered is robust and connected and that the skills needed for agentic participation are met (adapted from Davies, 1991). This does not imply any particular teaching method, but values highly students' ability to access the discursive threads of power/knowledge/identity in ways that support their collaborative participation and developing identities.

In the examples depicted above, the teacher in each case tends to act in ways that limit students' ability to establish themselves as knowledgeable participants in the mathematics classroom. From a poststructuralist perspective I take it that teachers act in these ways according to their already constituted *knowing* about how mathematics education is done and how they have come to know learners and learning; their instructional practice reflects psychological theories of socialisation that take agency and autonomy for granted and regard interaction as merely representative of the knowledge constructed rather than constitutive. These theoretical perspectives, in assuming a rational, autonomous human essence, allow the teacher to classify non-participants according to demeaning binaries (good/poor student; motivated/unmotivated; autonomous/dependent learners) and remain ignorant of pedagogical relationships that limit engagement and knowledge growth. However, on the other hand there are many teachers who appear to have a clear appreciation of the productive potential of pedagogic encounters; these teachers make explicit the regularity and pattern of mathematics, they make spaces for students to establish themselves as active participants in the classroom and they authorise student voice and ways of making sense of the mathematical experience. 'Gina' is one such teacher (Manouchehri & Goodman, 2000) who enlivens the interactional processes in ways that appear to invite participation and robust knowledge growth. Gina does not classify or coddle students but works with them in a collaborative partnership that facilitates the construction of mathematical knowledge and student authorship of the sense-making process. Gina works with her students in ways that suggest she sees them, and the knowledge they are constructing, as always evolving and *in process*, not static and set in stone as can often be the case, especially in mathematics education.

Implications for Practice

No doubt many educators and researchers are surprised to find an argument for the practical worth of poststructuralist theory in mathematics education (Constas, 1998). Here data do not stand as evidence of what is true but look to the ways in which sense is being made…and to the possibilities of personal sense making available . Interested in processes of subjection, and the position that can be taken by the speaking subject, poststructuralist theorising is profoundly sceptical of appearances and common sense, making visible the productive power of interactional patterns in classrooms and teacher education, working at the margins to challenge and disrupt dominant views. As Kilpatrick and Silver (2000) suggest, mathematics educators need to consider very carefully the unexamined and unexpressed assumptions that guide their work. In the section below I contemplate some of the practical implications a view of the discursive production of knowledge and identities might hold for mathematics education. New forms of research able to draw attention to previously unseen aspects of practice can be of interest (and constitutive) even though they may not sit comfortably with previously constituted subjectivities.

One implication has to do with the qualitative nature of pedagogic interaction. A first step might be to abandon the eternal search for that one best teaching 'method' to suit all children in all contexts; methods are based on essentialist notions of identity and take for granted rational teachers and learners able and willing to act autonomously. From a poststructuralist perspective it may be propitious to focus on the qualitative nature of all interactional processes; to ensure that they are as productive as possible for all learners. A coalition of many ways of interacting with students can be valuable as long as it is sensitive to the need for students to learn in ways that are personally meaningful and satisfying, that value and respect their voices and encourage them to go beyond the given to explore relationships and new ideas. Thus there is a need to de-emphasise stipulated practices in the classroom (doing worksheets, problem solving, uses of technology) to examine more carefully the productive nature of the interactions and relationships that mediate and produce meaning; group work and other technologies are only as useful as their ability to enhance the qualitative status of the teaching/learning partnership. Teaching is no longer something 'done' to students; rather any instructional act always comprises a teaching/learning relationship where relations of power/knowledge/identity constitute, and are constituted by, the teacher and students. In this way relevance, motivation and agency are seen to be discursively constituted rather than personal attributes or attitudes.

In the examples above, most teachers based their instructional practice on essentialist, humanist notions of individuals as capable of choosing to act rationally and autonomously, or not. For example, in the Zolkower (1996) article the teacher blames the students for lack of attention, for not engaging enthusiastically in the mathematical activities; the teacher does not recognise the coercion in her instructional practices that may lead to student disaffection with mathematics and the learning of it. In my teaching I, too (Klein, 2001b) categorise students into binary pairs (motivated/unmotivated; competent/incompetent) which diverts attention from my teaching and blames the student for non-performance. Similarly, socially constructed notions of 'ability' infiltrate classroom practices and support or suppress students' disposition to act in powerful ways in the classroom. This deferral to 'ability' and 'attitude' talk by educators traps them into a crippling stasis that they are trying to move beyond; in attributing blame to students for not learning or having the correct attitude, teachers remove any blame from themselves and are content and able to continue with teaching-as-usual. As Kilpatrick and Silver (2000, p. 225) state:

> As long as ability is taken as rock-solid property of the individual…it undermines a commitment to ensuring that all students receive an optimal education in mathematics.

Within poststructuralist thought students (and teachers) do not *have* 'ability' and 'attitude'; rather these are social constructions that are discursively produced and so could be (produced) otherwise. This is a second area with implications for further research in mathematics education.

Practical implications also pertain in the area of preservice teacher education. It has historically been taken for granted that preservice teachers, given new theoretical, disciplinary and pedagogical insights, could implement new, inquiry-based instructional practices in the classroom. Such notions were based on humanist notions of the agentic, freely choosing individual. An alternative poststructuralist reading of this situation might be that teachers do what they do in classrooms (they interact with students in certain ways), not because of theories and pedagogic knowledge they have constructed in teacher education programs, but largely because of what they have *come to know* of learners and learning mathematics through their formative years. A special effort has to be expended in teacher education to focus less on the thinking individual, and more on the operation of discourses and the implications for changed practice. It would appear that much more research needs to be carried out in this area towards a re-culturing of teacher education (at the moment students probably experience the same power relations they knew at school) as teacher educators come to recognise the productive potential of pedagogic relationships (Klein, 2000a). .

Conclusion

In this paper I attempt to reveal the intellectual and constitutive processes of mathematics education. I suggest that these processes are not separate but intersect and intertwine to support or suppress students' qualitative experience of mathematics and their ability to establish themselves as competent and confident users of its powerful ideas (NCTM, 2000). While many teachers and researchers will lament poststructuralist's troubling of knowledges that for so long have been known to be certain and true, and perhaps mourn the interrogation of the rational, autonomous agent of humanist thought, they might also consider its many possibilities. As Davies (2002, p.14) states:

> Poststructuralist theory, in its openness to meanings not yet thought of, and in its dedication to not getting stuck in old clichés and explanations, is often surprising, joyful and energising, bringing life to research and to teaching by breathing life into the educational institutions in which we are (always becoming) subjects.

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Teacher Development for Inquiry Based Instructional Practice in Mathematics: A Poststructuralist Postscript

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In this paper I argue that teachers of the twenty-first century need radically reformed professional development experiences if they are to teach in inquiry-based ways consistent with views of learning mathematics as an active, social and constructive process. By carrying out a comparative poststructuralist analysis of two teaching/learning interactions [from a paper authored by Manouchehri and Goodman (2000)], I go beyond the extant argument about the importance of teachers' mathematical and pedagogical knowledge and new images of teaching to suggest that additional factors may need to be considered as part of the professional development process. From a poststructuralist perspective I argue that the teacher's unspoken but influential *knowing* about learners and learning must also be considered and addressed as this impacts heavily, and often conservatively, on practice. Though contentious, the implications for professional development are contemplated.

Introduction

Although it has been argued that learning mathematics *without* understanding has long been an outcome of school mathematics instruction (NCTM, 2000), in the twenty-first century understanding has become imperative. This is so because well-connected, conceptually grounded mathematical ideas are seen to be more readily accessed and flexibly used in new situations. In the classroom, inquiry based instruction is advocated for its ability to actively engage students in tasks and experiences that lead to understanding through deepening and connecting their knowledge (Australian Education Council, 1990; Boaler, 1998; Brown, 1999). However, although inquiry based teaching/learning engagements are prized, the contextual and personal conditions of a teacher's acceptance and implementation of such practices are rarely researched beyond taken-for-granted understandings of the conditions of teacher change and development.

Manouchehri and Goodman (2000, p. 29) author a paper wherein they clearly demonstrate that a teacher's mathematical and pedagogical knowledge influence instructional practice; they show, with reference to two teachers Gina and Bonnie, that each teacher's content knowledge shaped expectations of students, their curriculum decision making and the teaching/learning interactions:

> In the absence of a detailed understanding of mathematics and knowledge about mathematical connections on the part of the teacher participants in this study, even sincere attempts at creating an inquiry-based instruction were futile.

While the authors make an important point, I am not convinced that such arguments tell the whole story of what needs to be considered relative to professional development. It may be that inquiry-based instructional practice is premised not only on the content knowledge a teacher has or does not have, and the images of teaching they carry with them or envision (Manouchehri and Goodman, 2000), but also on their constituted *knowing* of the nature of learners and learning mathematics (Lather, 1991). It is argued in this paper that these constituted knowledges are extremely difficult to interrupt as they are beyond the

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realms of cognitive reconstruction and require special attention in teacher development programs.

In adding a poststructuralist postscript to the analysis of interactions in Gina's and Bonnie's classrooms, I attempt to show that there is something in the *quality* of the teaching/learning interactions in the two classrooms that affords (or not) students the space to establish themselves as inquirers; this *qualitative difference I argue has as its foundation* differentially constituted views of learners and learning on the part of the two teachers in the study. I put the case that while Gina frames her practice with a view of mathematics and learners as always mobile, moving and *in process* Bonnie manages to inhibit inquiry by striving for the 'one correct answer' and her incorporation of romanticised notions of students and what it means to teach. In the concluding sections of this paper I contemplate the hurdles to be expected and overcome in having Bonnie, and others like her come to know mathematics, and how it is learned, differently.

Looking at Teaching and Learning Through a Poststructuralist Lens

Although institutionalised education is currently premised on notions of individuals as naturally autonomous and rational, this does not hold in poststructuralist thought where the person is the effect of a production, a multiple and often contradictory being. Each person is constituted, and actively constitutes him/herself through various discourses which promulgate certain 'truths' and suggest appropriate ways of acting and making meaning. In the mathematics classroom, Gina and Bonnie and their students are positioned in a three-dimensional constitutive space where relations of power relative to teacher/student identities, knowledge and authority intersect and overlap to produce, or not, inquiring habits of mind and robust mathematical understanding. In the first part of this paper I examine how, in this three-dimensional space, Gina's and Bonnie's invisible but powerful *knowing* about mathematics, about who can learn it and how it is taught, is played out and constitutes (a substantial part of) the students' experiences of what it means to learn and do mathematics.

In the second part of the paper I attempt to think about what it might mean to have teachers *come to know* learning and teaching mathematics differently; what might be the professional development implications of a view of teachers as also constituted through past and present discursive practices of the classroom and professional development programs. Teachers' implementation of inquiry based instructional practice requires new *ways of being* in mathematics education centred around new interactional patterns that are sensitive to student voice and experience and ways of making sense in/of mathematics. Such interactional patterns do not sit comfortably with many teachers who, through past experiences of school mathematics, have *come to know* learners as rational individuals who merely need to remember and reproduce the facts, skills and procedures the teacher demonstrates (Foss & Kleinsasser, 2001). How might teachers be encouraged to move 'outside the square' of established (constituted) comfort zones to embrace very different teaching/learning partnerships that generate robust mathematical understandings and student engagement in inquiry-based learning processes? Before addressing this question in more detail, I analyse and compare the interactions in two classrooms to reveal their constitutive (of teacher and learners) and constituted (by teacher and learners) nature.

Analysing Teaching/learning Interactions

From a poststructuralist perspective, inquiring habits of mind are constituted through classroom and other socio-cultural discursive practices; they are not personal attributes or attitudes as understood to be the case in humanist understandings of the individual. This being the case, it is important that discourses, in how they operate, position learners as respected, valued and competent in speaking and writing the commonly accepted 'truths' of the discourse and in going beyond these to forge something new (adapted from Davies, 1991). In mathematics education it is important that students, and as will be addressed later, the teachers, come to see themselves as agentic, able to act in powerful ways in the discourse. In the sections below I make the case that Gina realises the conditions of inquiry for students while Bonnie does not. I then attempt to show how each teacher's (previously constituted) relative understandings of learners and learning impact on classroom practice.

Gina's Class

Gina: What is the best way to create two congruent figures?

(*Many students raise their hands. Gina asks one student to respond.*)

T: I can trace the figure.

J: Yeah, like put a piece of paper on top and just copy it…we know that they are going to be the same thing.

Gina: Let's do that together and see what we get. (*She places a transparency slide on the overhead projector and draws a rectangle. Placing another transparency on top of it she traces the rectangle with a different color marker.*) Okay – Let's see – what do you notice about the two shapes that we have here? (*Pause*) How are they similar?

K: They look exactly the same –

Gina: What else? Tony?

Tony: I don't know – they are just the same – Like they are just copies of each other.

Gina: Let's go beyond that and try to describe what we see in terms of what we have talked about in the past – Like, what can we say about their angles? What can we say about their perimeters? What can we say about their areas? (*Students shout out that they are the same.*) Right you guys – this is really cool – Their areas are the same, their perimeters are the same, their angles are the same, and the length of their sides are the same – have I missed anything Jacquie?

J: Nope –

Gina: Good – Now I want you to think about situations when we have two things that look alike but are not exactly the same size – Let's forget the examples in the book and think of real life situations – What do you think? (*Pause*)

S: Like in the movies?

L: Oh, yeah – in the pictures too. (Other students nod their heads in agreement.)

Gina: Cool! So when we go to the movies what we see on the screen is actually similar to those images that we have on those little films… Why do you say they are similar?

H: They are the same people only bigger –

Gina: Tell us more Hillary –

H: Well, it is like Tom Cruise but a whole lot bigger – His face is bigger, but the same thing – His body is the same thing, only bigger –

Gina: Right – All the detail is there only bigger – But could we enlarge his legs say 10 times and his head, I don't know, say only 5 times? (Silence) What would be wrong with that?

Jenny: It won't be right – I mean, he won't look right – Not as good looking as Tom Cruise (laughs).

Gina: What else? Who else has something to say about this?

K: I think that it won't look right because if we make his legs bigger say ten times then we need to make his head also ten times – I guess that is what it is – If we don't enlarge it the same way for all parts it is not gonna look right – It won't look like him anymore –

Gina: What do you think Sam?

S: He would sure look funny – (laughs) I think some parts of him will be similar but not the whole thing –

Gina: How can we make sure that when we have two shapes, they are similar?

P: I guess we can check to see if it is bigger or smaller the same amount on all parts.

Gina: Good idea – Let's see if we can do what P said – (She turns on the overhead projector and projects the rectangle she had drawn on the transparency on the screen.) Do you think these two shapes are similar? (pointing at the one on the screen and the rectangle on the transparency)

D: I think so –

Gina: How can we be sure?

D: It is the same thing only bigger?

Gina: How much bigger do you think the one on the overhead projector is? (Students decide that they want to measure each rectangle first. Gina asks them to do so and also to look at the perimeters and areas of the two figures and to decide whether there is a relationship between the two sets of data. She also encourages students to move the overhead projector and consider other cases.) (Manouchehri and Goodman, 2000, pp. 24-26)

My reading of Gina's teaching is that she manages to orchestrate the conditions of inquiry (Davies, 1991). She creates a classroom space where both the mathematics and the students are valued, and where each is in the process of formation or construction. She attempts to authorise student voice and ways of making sense of experience. Gina begins the lesson: "Let's do that together and see what we get". She invites the students to join her on an intellectual and social/emotional journey that could go off in any direction. Gina demonstrates her agency with the mathematics and in the discursive conversation by stating: "Let's forget the examples in the book" and invites student participation: "Have I missed anything, Jacquie?" By largely avoiding closed questions, preferring rather questions of the form: "Tell us more, Hillary" there is no pressure for students to come up with the one 'correct' answer. Rather students are given the time and space to reason in a safe yet challenging context: K reasons: "I guess that is what it is – If we don't enlarge it the same way for all parts it is not gonna look right – It won't look like him anymore -; P: "I guess we can check to see if it is bigger or smaller the same amount on all parts". Students are also authorised to make sense of the mathematics in ways that are meaningful to them: "Students decide they want to measure each rectangle first". It would appear that Gina interacts with her students in such a way that they are developing inquiring habits of mind and coming to know themselves as competent doers and users of mathematics.

Bonnie's Class

> Problem: It is Karen's birthday and Karen's mother bakes 13 cookies. People at the party take turns picking cookies from the tray. If Karen takes the first cookie and the last, how many people were at Karen's party?

Bonnie had intended to use the 'cookie problem' as a part of her warm up segment and was prepared to begin a unit on fractions for the day. The events of the session, however, not only forced a change of plans on her part, but also created a situation in which she was uncomfortable with the mathematics discussion she had to lead in class.

Bonnie assigns the warm-up problems. Following an 8 minute work time, Bonnie asks students if they are ready to present their solution to the group. Several students raise their hands. S1 explains that his answer is 12 – it seems that the answer is consistent with what Bonnie considers as the correct answer. "Good work S1. Does everyone agree with the solution?" Bonnie asks. S2 suggests that her answer is 2 and not 12. Bonnie looks at the problem again – She asks the student to go to the board to explain her strategy. S2 draws a circle and places 2 lines around the circle…she counts starting from the line she identifies as Karen up to 12 going around the circle 6 times – Bonnie looks surprised. She is looking at me for assurance. "What do you think class? Which do you think is right?" S3 suggests that his answer is different from the others and that he thinks there are 3 people at the party. Students are quiet – Bonnie asks S3 to show his work – S3 using the same diagram S2 has used, suggests adding another line around the circle representing another guest and shows by the means of counting and going around the circle 4 times why he thinks the correct answer is 3. "Do you see what I have done?" he asks Bonnie – Bonnie nods her head in approval – S3 goes back to his chair – Bonnie looks puzzled and continues to read the problem. A few seconds of silence pass as students look at Bonnie to tell them what to do – In the meantime…[some data omitted] S5 and S6 simultaneously ask Bonnie why their answer of 12 could not be correct. Long pause by Bonnie as she looks at the solutions offered by the group. "I think this was a really good problem. We all have different answers and they all sound right…(Pause) This is real problem solving, you see! You all did very well class". S3 asks which answer he should record on his paper - "I don't understand why these answers all can be right – I still think mine is right! Should I write my answer or everybody else's?" "Just put down all the answers so you could think about it some more", Bonnie responds. All students begin recording the answers on their papers. *(Manouchehri and Goodman, 2000, pp. 19-20)*

Although Bonnie and her students are ostensibly involved in inquiry based practice, they together establish and maintain a competitive search for the 'one correct answer'. Once this culture is in place it is unlikely that students will want to engage substantively in exploratory processes, as they come to see their task as getting as expeditiously as possible to the answer. Though the students are asked to explain their answers, they are not able to experience themselves as in any way competent because they do not get to know whether what they have produced, or how they have produced it, is acceptable within the discourse. It could be argued that the substance (as argued by Manouchehri and Goodman, 2000) *and* the experience of the interactions in Bonnie's classroom militate against deep inquiry on the students' part. While Bonnie is clearly not aware of the powerful mathematics that could ensure from involvement in this activity, the quality of her interactions with students is also hampered because of her constituted *knowing* of what it means to learn (and teach) mathematics.

How the Teacher's Constituted *Knowing* Impacts on Practice

Gina's joy with/in mathematics (*with* mathematics as a discipline or discourse worthy of study and engaged investigation, and *in* the discursive practices as teacher) catches up the students and carries them along: "Right you guys, this is really cool". She does not shy away from taking a proactive role to facilitate knowledge growth; my guess is that Gina knows from personal experience the joy of intellectual and emotional challenge and resolving a problem. The students are formed in discursive practices that uphold mathematics as really 'cool'. As well, the students' already constituted identities are recognised and brought to bear on the task at hand; in this case they bring in Tom Cruise and create a comfortable relevance in the activity. However Gina does not subscribe to notions of learners as needing to be protected and coddled; while she is sensitive to their prior experiences and their needs to make sense for themselves, she states: "When they are in my mathematics class, they are there to learn mathematics – My job is to make it accessible to them and to help them to see that they can do it" (p. 10).

Gina recognises her students as mathematicians in-the-making and endeavours to have them see themselves as legitimate participants in the discursive community; her stated purpose is to help students grow mathematically and to communicate mathematics (Manouchehri and Goodman, 2000, p. 9). I would argue that these facilitative qualities in Gina's practice are not merely cognitive and conscious, but are based on constituted *knowing* of mathematical knowledge and learners as always growing and in process. Gina knows the mathematics and she is also comfortable with inquiry-based, investigative ways-of-being in mathematics that are based on new power relations. However, it could be, as most surely it is for many teachers, that although they know the mathematics, the new instructional patterns that celebrate student voice and ways of making sense of mathematics feel threatening and do not sit well with constituted knowledges of how mathematics is learned.

Bonnie, quite distinctly from Gina, appears to base her practice on understandings of learners that effectively preclude their engagement in extended inquiry. Bonnie seems to equate active engagement with learning mathematics. She says that she seeks to accommodate students both emotionally and intellectually (Manouchehri and Goodman, 2000, p. 4) and she is happy with a lesson "when she is able to engage nearly all the students and have them share their thinking" (p. 22). Bonnie often engages in 'ability' talk; she speaks of her "least able kids" (p. 4) and expresses "dissatisfaction with the students' inability to understand…"(p. 18). Bonnie's way of dealing with a perceived lack of understanding is to do more of the same or similar examples (p. 26). A humanist, individualistic notion of learners and romantic notions of teaching appears to cause Bonnie to take for granted that the growth of mathematical knowledge, and a positive disposition towards mathematics, will automatically grow out of activity, however poorly conceived. Bonnie, basing her instructional practice on notions of a learner with the inherent potential to be self-motivated and autonomous, sees her role as teacher to be one of 'bringing out' this potential (Usher and Edwards, 1994).

Throughout the lesson Bonnie maintains a position of authority, in the sense of authorship of classroom interactions and actions, even though she struggles with the content of the lesson herself. The students are actively involved, they do get up to explain what they have done and they do ask questions; however, these acts are invariably focused on the answer and not motivated by a wish to participate in and contribute to a continuing conversation around a mathematical idea. What happens is that any dilemmas or problems that arise for the students are about their performance rather than the mathematics (Lave, 1997). For example, S3 asks: "I don't understand why these answers all can be right – I still think mine is right!" and "Should I write my answer or everybody else's"? Students invariably defer to Bonnie's authority: "A few seconds of silence pass as students look at Bonnie to tell them what to do". Bonnie positions the students in such a way that they are dependent on her and she does not allow their ideas and experiences to broaden and strengthen the communal learning process; although she hears their ideas, these ideas are not used to nourish and enrich learning. Bonnie clearly takes for granted, in that she related to the researchers that she was happy with this lesson, that active engagement unproblematically advantages learning.

An interesting and perhaps controversial issue arises here concerning the affective dimensions of context and the effect on student engagement in inquiry based learning. For example, Bonnie appears to have a warm and caring nature and indulges in more student praise than Gina: she says "Good work S1". "I think this was a really good problem. We all have different answers and they all sound right…this is real problem solving, you see! You all did very well class". But a poststructuralist reading of identity constituted through classroom relationships of power would suggest that agency (*with* the mathematics, and *in* the discursive relations) is what matters. For students to act in agentic or investigative ways they have to feel confident with the mathematical content and their ways of speaking and representing mathematics and they have to have a sense of themselves as able to go beyond what is given to produce something new. One can only wonder at the effect on students when the mathematics is mystified yet they are constructed as somehow deficient; for example, when Bonnie suggests *they* need to "think about it some more". Bonnie does not pause to question the qualitative status or effects of her particular interpretation and application of inquiry-based instructional practice!

Implications for Teachers' Professional Development

Where relationships of power are not seen or are ignored in classrooms or professional development programs, little about the quality of interactional patterns is likely to change. This is because any blame or implied lack is readily sheeted home to the supposedly automous, rational individual who chooses not to act appropriately. In Bonnie's classroom she proclaims that the students need to 'think about it some more'; similarly, teacher development programs assume that teachers merely need to cognitively reconstruct the mathematics and pedagogy relative to the new pedagogical practice. For example, Manouchehri and Goodman (2000), proffering the view that learning is an active, social and constructive process for students and teachers, suggest that teachers *need* new images of teaching and enactment strategies; they *need* 'authentic' experiences that engage them in conceptual explorations of mathematics, and assist them in constructing a mathematical and pedagogical point of view that would serve as a foundation for successful implementation of reform-based curriculum and instruction (p. 31). While I agree that these experiences could assist teachers with the *substance* of what is taught in the classroom, from a poststructuralist perspective I make the point that this is, as Manouchehri and Goodman (2000) have stated, just a foundation. It may indeed be a shaky foundation or a robust one; depending on the *nature* of the professional development experience and the constitutive effects. Teachers will act in powerful or powerless ways depending on how they have been able to establish themselves as generative professionals in the classroom and professional development process. A too ready assumption of the easy implementation of change or application of (re)constructed knowledge traps practitioners and researchers alike into the very stasis we are trying to move beyond; in deferring to commonsense or taken-for-granted understandings of the efficacy of rational thought and the 'naturalness' of autonomy we may be blind to those power-related factors of instructional practice that militate against change.

Contemplating change from a poststructuralist perspective necessitates that we not only consider the thinking individual, but also the operation of discourse(s) (such as those of teacher development). Discourses comprise "historically, socially and institutionally specific structures of statements, terms, categories and beliefs" (Scott, cited in Adams St Pierre, 2000, p. 485) and organise ways of knowing into ways of acting in the world. A new discourse, with new 'truths' and ways of interacting is needed to bump into, to confront and interrupt the operation of existing discourses in the interests of redefining notions of what it means to learn and teach mathematics for the twenty-first century. New discourses can supposedly re-write the world as teachers are configured at the intersection of multiple intersecting discourses, living/acting in an between discourses finding comfortable spaces and investments (not necessarily conscious) in discourses that enact new truths and ways of operating. Change is accomplished as a result of contradictory positioning, due to the co-existence of the old and the new; every relation and every practice to some extent articulates such contradictions and therefore is a site of potential change as much as it is a site of reproduction (Hollway, 1984, p. 260).

Conclusion

My argument and analysis in this paper is founded on the idea that teaching/learning relationships are productive; they are productive of intellectual knowledges and simultaneously constitutive of participants. In relation to teacher development, it is important that the teachers not only construct new conceptual understandings and mathematical connections, and become acquainted with new images of teaching and learning (Manouchehri and Goodman, 2000) but also that they are positioned in ways that afford them the space to establish themselves as agentic and generative professionals. In other words, programs that operate solely on knowledge transmission, that cement traditional power relations, are probably not useful. A special effort must be expended to engage teachers in a 'border pedagogy' (Davies, 2000) that re-visions taken-for-granted understandings of what learning and teaching might be. This is undertaken not on the assumption that the teachers will act upon these new understandings, but rather that through engagement in this alternative discourse they will sense how the process of *coming to know* is socially produced, and contingent, tentative, provisional. A re-culturing of professional development would embrace the practices mooted by Manouchehri and Goodman (2000) and go beyond these to emphasise new dialogic forms celebrating teacher led inquiry; teachers' initiation of ideas and ways of making sense of experience. Over time, it may be constitutive of teachers who have a sense of themselves as able to go beyond the taken-for-granted to forge something new in mathematics education. As Kilpatrick and Silver (2000) make clear, changing the ways teaching and learning mathematics is done is not a technical problem - it involves a form of social change. This, I would add, may take some time and dedicated thought beyond the pale of contemporary and comfortable research methods!

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Preface

These are the proceedings of the Twenty-fifth Annual Conference of the Mathematics Education Group of Australasia. During the twenty-five years in which this group has met, research in mathematics education has grown markedly in this part of the world and is now a field of research in nearly every university in New Zealand and Australia.

The focus of this conference is Mathematics Education in the South Pacific, an appropriate focus for a conference for which authors come from Australia, New Zealand, Samoa, and several countries that border the northern, western and eastern Pacific. It is a particularly appropriate focus for this conference in Auckland, which is reported to be the city with the largest population of people who come from Pacific nations. We are pleased to welcome keynote speakers from France, the United States, Samoa and Dunedin to this city for this conference.

Papers in proceedings for different years reflect changing areas of concern in mathematics education. The first conference in 1977 included an early paper on problem solving which then became a major focus of mathematics education in the 1980s. In the 1980s and 1990s there were many papers on this topic in MERGA Proceedings. However, in 2002 there is only one paper on problem solving, and there are many papers related to numeracy in primary or elementary school. This change of focus reflects the emphasis that mathematics educators and several governments are placing on numeracy at this time.

Each of the 81 research papers accepted for these proceedings has been reviewed, without the author being identified, by two experienced researchers in mathematics education who followed strict guidelines. Where reviewers disagreed about the acceptability of the paper, a third reviewer read the paper without seeing the previous reviews. Only those papers that were accepted by two reviewers have been included.

Some papers were submitted as entries for the Practical Implications Award, and reviewed by a committee of four other researchers. In 2002, this has been awarded to the paper written by Len Sparrow and Sandra Frid. Other papers were submitted for consideration for the Early Achievement Awards. The winner of this award will be decided after the presentations at the conference.

A considerable amount of time and effort has gone into reviewing these papers, making arrangements for the conference, and preparing these proceedings. As chairperson of the Proceedings Committee, I would particularly like to thank Mike Thomas, Maxine Pfannkuch, and Bill Barton for their many hours of labour.

Kathryn C. Irwin
for the Proceedings Committee
Auckland, New Zealand
June 2002

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The Conference Committee would like to thank the following who reviewed papers for the MERGA 25 conference:

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Patterns of Misperceptions in Linear Transformations: Four Illustrations

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Student misperceptions (visual unawareness) constitute a powerful hindrance to successful mathematical problem solving. Using the study's computer software, a number of lower secondary, upper secondary mathematics students, their teachers and other pre-service teachers were required to perform a number of transformations. These exercises demonstrated the extent to which misperceptions confound the process of conceptualisation in problem solving activity and mathematical learning. This presentation highlights the misperception phenomenon and describes remediation processes that are being developed in an effort to counteract the negative impact of the phenomenon.

Background

At the previous MERGA conference in Sydney, the authors described an ongoing study of misperceptions in music and mathematics (Malone, Leong & Lamb, 2001). The term 'misperception' was defined as the act of perceiving via a single sensory modality (e.g. seeing in maths or hearing in music) something that is different from reality or an imagined reality (e.g. visualising a reflected object/shape in maths, or auditing a transposed sound pattern in music). The presentation this year focuses on one small component of the larger study – namely, a demonstration that misperceptions do indeed exist in mathematics, and the production of evidence that these misconceptions can be overcome through training in the use of software developed specifically for the purpose. The mathematics topic selected to be the vehicle for this demonstration was one involving linear transformations. Brief explanations of the status of this topic in the Western Australian mathematics curriculum and the reasons for its choice in this study are appropriate at this point.

The Space Strand of the *WA Curriculum Framework* constitutes one of seven mathematics learning areas. The strand descriptor reads: 'Students will describe and analyse mathematically the spatial features of objects, environments and movements. In particular, they should visualise, draw and model shapes, locations and arrangements and reason about ways to solve related problems and justify solutions'. The strand is divided into four sub-strands, one of which is entitled 'Representing Transformations'. This sub-strand's objective reads: 'Students should visualise and show the effect of transformations on shapes and arrangements'.

Eight levels of outcome statements are associated with the Representing Transformations (RT) sub-strand, these levels representing standards of output considered achievable by students ranging from the lower and upper primary levels to the lower and upper secondary levels of schooling. For example, the Level 4 outcome for the RT sub-strand (roughly equating to the expected achievement of 12 to 13 year old students in Grades 6 and/or 7) reads: 'Students will recognise rotations, reflections and translations in arrangements and patterns, and translate, rotate and reflect figures and objects systematically'. At Levels 7 and 8 (equating to the expectations from Year 11/12 students),

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the RT outcome reads 'Students can identify the transformations needed to produce a given image from an original and applies transformations to problems including those involving congruent and similar shapes'. From these statements, the outcomes expectations from students are quite clear, and a range of assessment practices is now used by schools to gauge the attainment of these outcomes.

The topic of transformations lends itself conveniently to displaying the misperception phenomenon. The negative effects of misperceptions on learning have been largely unappreciated, and learning problems could have been misdiagnosed to be the result of other student difficulties. Recent cases involving students demonstrating various forms of mathematical misperceptions were found by Shaw, Durden and Baker (1998), Bottage (1999), and Malmer (2000). In one case, a student perceived a 90-degree angle correctly when aligned horizontally and vertically but perceived it as 140 degrees when tilted. A second case described a student misperceiving objects as reflections of these objects. In other scientific research, positrons were misperceived in data collected over many years, resulting in the non-identification of a new elementary particle. This misperception delayed an entire paradigm shift in quantum mechanics (Solomey, 1998).

Our own informal studies with school and university students and maths teachers (reported in this presentation) have revealed disturbing misperceptions involving reflections. Although these linear transformations are fundamental to maths, misperceptions involving them have not been studied in regular classrooms. There are many other situations in which we suspect that mathematics students are being hampered by misperception – for example, perceptual errors occur when students observe the vertex position while comparing two graphs (Goldenberg, 1988). Perceptual errors with respect to curved properties such as smoothness and concavity and slope were also displayed routinely by students in the above studies.

It is our contention that misperceptions seriously affect students' efforts to achieve a level of outcome appropriate for their age in this strand of the mathematics syllabus, and that the problem affects not only upper primary and lower secondary school students, but pre-service and practising teachers also. Teachers who perceive in their mind's eye something that is different to reality are likely to mis-teach. Students who misperceive will fail to see what the teacher is presenting (be it correct or incorrect) and hence cannot learn properly. The identification of these students and the diagnosis and remediation of their problems will result in a more efficient and effective learning environment. By producing a battery of crystallised examples of typical student misperceptions, teachers can be trained to identify them.

Another important aspect of the study is the interaction between the disciplines of mathematics and IT. The inter-disciplinary approach is in line with "several influential documents advocating the integration of mathematics, technology and other content areas" (Venville et al., 1999, p. 2). There are definite educational benefits in engaging student learning through integration, and some evidence of this is beginning to appear. This cross-pollination would bring new insights into the benefits of inter-disciplinary learning and professional collaboration.

Aims of the Presentation

To demonstrate that school and tertiary students, along with pre-service and practicing mathematics teachers experience misperceptions when working with linear transformations; to explore the effect of remediation via a software tool, and to examine some of the implications of the results.

Method

Participants

A sample of convenience was employed since there is no evidence to suggest that any particular group of students or teachers suffer more from learning or teaching problems caused by misperception. Six upper primary school and six lower secondary school students were drawn from primary and secondary schools in the Perth metropolitan area. Also, four undergraduate mathematics students along with four full-time pre-service mathematics teachers and four part-time practicing mathematics teachers enrolled in graduate level courses at a Perth university were also selected. There was no attempt on our part to seek out students who possessed a disposition to misperceive. Teachers in the primary and secondary schools were simply requested to allow students to participate in an activity with the software, and the tertiary participants were approached directly and requested to participate. From this sample of 24, four participants are described in this paper because they displayed the typical misperception phenomenon to a significant degree. Another seven participants displayed misperceptions to a lesser or trivial degree.

The Software

The instrument identified to possess the required functionality is *Mathemagic* (Lamb, 1995). By providing an interactive environment that is instantaneously responsive, this software is able to help us identify misperceptions demonstrated in the workings of individual learners. The *Mathemagic* software allows students to explore the spatial transformations of reflection, enlargement, translation and rotation. The transformations can be applied individually or in combination, and the software provides opportunities for problem-solving using the following operations:

(a) Translation (sliding) up or down by one unit; left or right by one unit.
(b) Rotation through a quarter turn (90°) clockwise or anti-clockwise.
(c) Dilatation (scaling) by a factor of 2 or 1/2.
(d) Reflection in the x- or y-axis, or in the line $y = x$, or in the line $y = -x$.

A typical transformation task set for problem-solving is presented as follows: the original picture on the screen (in black), is shown together with the end result (in blue) of several transformations. The challenge for the student is to find a combination of operations that turns the (original) black picture into the (transformed) blue one. The solution may be entered in one of two ways: (1) each of the student's moves may be shown when it is selected, or (2) the moves may only be viewed *after* all the moves have been chosen, thus requiring the student to visualize the entire sequence of transformations. The software has a feature that challenges the student in visualising the least number of correct moves possible.

The Approach

In contrast to most other subjects where the computer can only simulate reality, this software permits us to work with real mathematics. Virtual reality is a pale substitute for the real thing. Furthermore, virtual reality is perceived by many students as 'fictitious', thus it is not always believable to them (Yeo, Loss, Treagust & Zadnick 1999; Gorsky & Finegold, 1994). Because technology provides the explicit reality corresponding to mathematical models, students can directly perceive the validity of the models which they manipulate on the computer screen and thereby make the essential connections to effect authentic learning. In this portion of our study, the *Mathemagic* software has been used to diagnose several cases of misperceptions. Participants were first allowed time to become familiar with this software, manipulating a bird object (Figure 1 – used in the problem presented to them later) according to the four transformation operations listed above. A bird object was chosen to manipulate because of its lack of symmetry when viewed from the side. Any shape could be used and is easily created with the software.

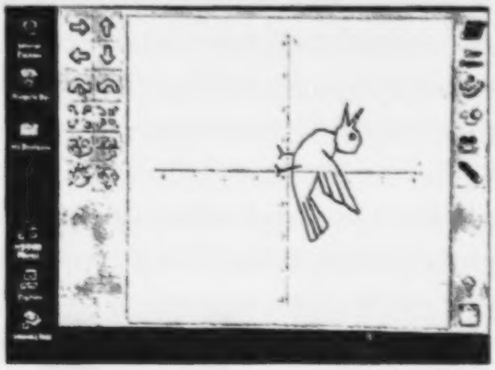

Figure 1. The bird object.

The steps in the procedure used with each student were as follows: After 30 minutes practice with the *Mathemagic* software operations, the participants watched as a sketch of the bird was drawn on paper. They were then told a transformation to apply to the sketch (the reality), mentally carry out the steps involved in the transformation (the imagined reality), and then sketch the result on the same piece of paper. Any discrepancy could be due to faulty visualisation or faulty drawing skills, so the significant parts of the picture and its transformation were examined to deduce the occurrence of misperceptions. The correct orientation of the bird was shown to the participant. A record was kept of the participants' comments and drawing attempts, and their reactions on being presented with the computer-generated solution were also encouraged and recorded. We were interested in seeing if the participants were able to predict the transformation that they we knew they understood both conceptually and cognitively. The task confronted the participants' powers of visualisation or perception, with the program providing the reality of the task.

Examples of Misperceptions

Reproduced below are the drawing efforts of four of the participants Each example is accompanied by: (a) details of the task, (b) details of the participant, and (c) their reactions on being presented with the correct transformation.

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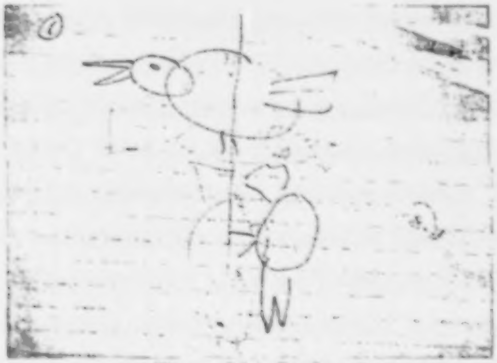


Figure 2(a). Student A.

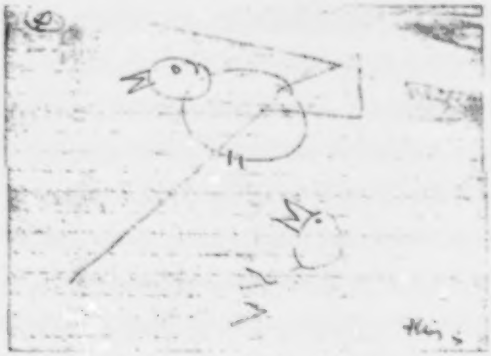
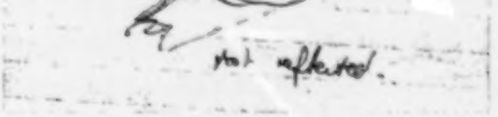


Figure 2(b). Student A's re-try.

Figures 2(a) and (b) are the work of Student A, a male teacher of mathematics and science studying full time for a postgraduate degree at a Perth University. These were A's second and third attempts at this procedure – he had gone through the sketching activity two weeks earlier (without much success). In Figure 2(a), the bird object was sketched at the top of the page and A was requested to reflect the bird about the vertical axis that was drawn in on the figure. After a few moments thought, A sketched the bird as being rotated through 90°, and then reflected about the y-axis. He then used the software to perform the requested translation and was surprised at the outcome. In order to check if A had misunderstood what he had been asked to do, he was then asked if he would like to re-try with a second example. In Fig. 2(b) the uppermost sketch was drawn and A was asked to sketch the bird if it was reflected in the line $y = x$.

Again after several minutes thought, A sketched the figure in the lower half of the Figure which represented a rotation through 90° followed by a translation to the right. On using the software to carry out the transformation requested, A was incredulous at the result. He spent some time afterwards experimenting with the software, striving to perform each transformation again. He was eventually successful with both after some 10 minutes – indicative of the value of practice – but he continued to experience trouble dealing with the actual and the imagined reality of the activity.



not reflected.

Figure 3. Student B.

Figure 3 represents the work of Student B, a male pre-service teacher of computing. He was asked to reflect the bird at the top of Figure 3 about the line $y = -x$, drawn on the diagram. B sketched the figure in the lower portion of Fig 3 after only a few moments thought. It represents a rotation through 180° about the line $y = -x$ (correct), followed by a reflection in the x-axis (incorrect). He then used the software to reproduce his thought sequence. He recognised the requested transformation after the first translation when he used the correct software operation. B was annoyed with himself, saying that he could not understand why he had gone on to perform the additional incorrect operation. He said that the task became clearer when he had the

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options of the various computer operations to follow. Student B had no wish to spend more time gaining experience with similar activities, saying that he was confident that with practice he would be able to deal with any task given to him.

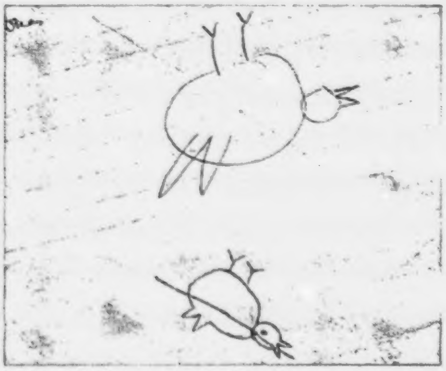

Figure 4. Student C.

Figure 4 represents the work of Student C, a Grade 7 primary school female. After checking that she knew the meaning of the term 'rotate' and could used the software correctly to carry out this transformation, C was asked to rotate the bird through the line y = –x drawn on the bird in the uppermost section of Figure 4.

Her response, carried out after several minutes consideration, represents a rotation through 45° in the plane of the line y = –x. On using the software to carry out the requested exercise C exclaimed 'of course! – It's nothing like it, is it?'' Student C described to us what should have occurred – pointing out where the correct position of the tail feathers should have been, and why. She practised with other positions of the bird, checking with the software each time for a considerable period afterwards and became very adept at all manner of transformations before she terminated the activity. At one point student C made the comment: "it's good to see the right picture at the end. It's like checking your answers from the back of the book". Practice coupled with the conflict situation had clarified the difference between the actual and the imagined reality for C.

It is worth recording that student C was the single female to be included among the four participants demonstrating the most pronounced examples of misperceptions. The original sample for this component of the larger study consisted of twenty-four students – the four described herein, and another seven demonstrating misperceptions to a lesser degree. Student C was the only female among this group of eleven participants. We propose to investigate the typicality of this situation in the larger study.

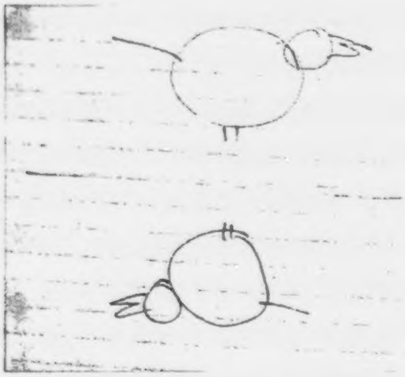

Figure 5. Student D.

Figure 5 depicts the effort of Student D, a male first year university student enrolled in a BSc (Mathematics) program. He was asked to reflect the bird sketched in the uppermost section of the Figure about the x-axis. It can be seen from his sketch in the bottom half of the Figure that he first reflected the bird about the x-axis as required, but then rotated the bird through 180°. On using the software to carry out the requested transformation he quickly recognised the error he had made, but could not immediately decide what he had done incorrectly when imagining /visualising the steps that he had to take. "I imagined that the reflection would also turn the bird" he said. "I don't know why". Student D asked to spend further time practicing and then checking his efforts against the software, and after almost one hour, was performing a variety of transformations confidently and successfully.

Discussion

The four illustrations described in this paper suggest that the outcomes of the larger study could have far-reaching implications for educational practice. The existence of the misperception phenomenon in the teaching and learning of linear transformations cannot be disputed, and it is probable that it would have an effect in other subject areas too. It is of some concern that both teacher and learner may experience the same problem, but because of the lack of attention given to this phenomenon in the past, the problem may have been attributed to other forms of teaching/learning difficulty or even passed unnoticed.

It was interesting that more male than female participants from our sample seemed to experience misperceptions in linear transformation tasks. This seemed to be also true when considering the age groups represented. However, more research would be needed to verify this possibility that there is a gender factor in the misperception phenomenon.

While some of the participants had the opportunity to spend time on the software for remedial purposes, most of them were unable to devote the time for this activity. It is gratifying that all of those who took part in remedial activities improved in carrying out the correct transformations.

We hope to produce a battery of examples of typical student misperceptions, the existence of which will assist teachers to better understand student learning difficulties resulting from misperceptions and facilitate the training of teachers to identify misperceptions when they occur.

The development and customisation of diagnostic and remediation software tools of the type used here has the potential to apply our approach to other mathematical topics. Such development will also facilitate the testing of large groups of students, enabling those who do experience misperception to be identified early and be assisted in overcoming some of the negative outcomes of the phenomenon. The computer-aided approach enabled the participants described in this paper to be presented visually with the discrepancies between their imagined reality of the transformations requested of them and the actual reality of the transformation that the software presented to them. This was a real 'conflict teaching' situation that, particularly in the case of Student A, forcefully brought home to the students the extent of their misperceptions. We are hopeful that the outcomes of our research will assist in overcoming the problems associated with the misperception phenomenon and consequently contribute to an improvement in the quality of teaching and learning in mathematics at all levels.

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Differential Pathways of Learning: How Four Low-Achieving Undergraduate Students used a Graphics Calculator

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This paper reports on a year-long technology-rich teaching experiment conducted with four low-achieving undergraduate students. Using questionnaires, classroom assessment tasks, self-report journals, interviews and field notes, the study identified three stages of cognitive development – *computational*, *technician*, and *multirepresentational* – which reflect successively higher levels of tool-based engagement and performance. Four tool-based pathways or "journeys" were also identified, illustrating how these mathematically weak students navigated their way through the stages of the model. They are: a restricted learning pathway, a linear learning pathway, a non-linear learning pathway, and multiple learning pathways.

The past two decades has witnessed substantial increases in the number of students, particularly low-achieving ones, undertaking technology-rich mathematics-based undergraduate degrees for which they are poorly prepared (Hillel et al., 1992). Secondary schools and to a lesser extent post-secondary institutions have been keen to include technology into their learning programs, the basic assumption being that the integration of tools such as Computer Algebra Systems and graphics calculators has the potential to better assist students with their learning, and are superior to the more traditional methods of course delivery. Furthermore, the majority of studies that have reported these findings have been carried out with average to high-achieving students. However, little research has been done with low achievers, particularly at the post-secondary level (Dunham & Dick, 1994).

I began integrating technology-rich activities into a mathematics "bridging" subject in the mid-1990s with low-achieving students who were embarking on engineering, business and education degrees. Most students in my charge were poorly prepared to embark on courses that required a thorough grounding in Year 12 mathematics. The courses in which I was involved prior to the study were typical of many undergraduate programs found in Australian universities at that time: syllabuses were content laden, teaching strategies emphasised information transmission as a means of course delivery, and assessment practices valued the importance of mid-semester tests and end-of-semester examinations as measures of performance. Consequently, classroom learning remained a passive process of knowledge and skill acquisition with the attainment of high grades as indicators of success (Confrey, 1990).

The Study

This paper reports on the final of three technology-rich teaching experiments that were conducted over a period of three academic years (Lindsay, 2002). The developmental models of Pirie and Kieren (1994) and the SOLO Taxonomy (Biggs & Collis, 1982), together with a constructivist view of learning mathematics (Cobb, Yackel & Wood, 1990) provided the cognitive and pedagogical orientation for the study. The evidence that emerged from the first two phases of the study suggested that low-achieving students were

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 415-424). Sydney: MERGA. ©2002 MERGA Inc.

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not readily embracing the technology as was originally expected. In fact, quite the contrary: many avoided using the tool and when they did engage with it they used it incorrectly and/or inappropriately. The findings that emerged from these teaching experiments informed and provided a direction for the final teaching experiment. Using the graphics calculator as the preferred technology, and conducted over a period of two semesters, it consisted of the detailed quantitative and qualitative analyses of ten students' data sets. The data collection instruments included questionnaires, self-report journals, interviews, field notes and classroom assessment tasks. The latter were evaluated using a 5-level performance-based scoring rubric – developed in an earlier phase of the study – that provided a clearer picture of the qualitatively different ways in which students were engaging with tool-based tasks (Lindsay, 1999, 2000). The remainder of this paper will focus on the cognitive development of, and learning pathways taken by four of these students as they responded to assessment tasks that were given to them during this final phase.

The Four Case Studies

Dean: Computational User; Restricted Learning Pathway

Dean (age 19) was case-studied because he performed poorly in his entry-level test, and he had not used a graphics calculator before. His entry-level test responses revealed that both his graphing and basic algebraic skills were very weak. Assessment tasks administered to students over the course of the year (as will be seen later) relied predominantly on tool-based strategies although supporting pencil-and-paper activities were considered important components as well. By year's-end Dean had made limited progress and his graphics calculator (ninth) and pencil-and-paper (ninth) rankings at the end of each semester, relative to the cohort of ten students, were consistently low. His tool-based responses suggested that he was operating at a very basic level with respect to his procedural and conceptual knowledge (Hiebert & Carpenter, 1984). The *computational* stage to which Dean's responses were classified, is characterised by the following rubric indicators of performance:

- Output is being copied regardless of its accuracy or relevance
- Limited or no progress towards completion of the task, or no response to the question at all
- Incorrect and inaccurate use of mathematics
- Mathematics explanations brief or absent; limited technical ability in using the graphics calculator
- Employs graphics calculator-generated solutions instead of algebraically derived solutions
- No evidence of correct use of mathematics language, symbols and conventions
- No links between graphical, symbolic and numerical representations
- Makes no attempt at moving to generalisations.

For instance, Dean's response to a trigonometry question that required him to plot and describe the graphs of $f(x) = \cos x$ and $g(x) = 3\cos(x/2)$, including the location of amplitudes, axial intercepts and periods is shown below in Figure 1.

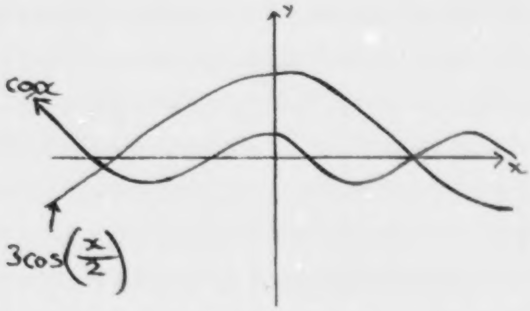


Figure 1. Dean's incomplete response to a trigonometry question.

Dean's graph drawing lacked all detail, the graphics calculator output being copied with little regard given to the properties of the images being generated by the calculator. He appeared to be using the tool as a "button-pushing" device. When asked to compare the functions' periods he said this:

> *g(x)* doesn't look like a zigzag. It's more like an oval mountain shape and is much higher than *f(x)*. *f(x)* is a more spread out graph and is a zigzag curve shape.

Computational users display limited use of mathematical language which restricts their ability to explain graph plots. Progress with tool-based tasks appears to be contingent upon their possesson of appropriate pencil-and-paper skills and knowledge. Their learning pathways remain restricted, and it is suggested that only by continually moving back-and-forth – or "folding back" (Pirie & Kieren, 1994) – between their newly acquired pencil-and-paper (P/P) skills and knowledge and their existing tool-based skills and knowledge (shown in Figure 2 below by a circular broken line) will students be able to achieve tool-related success at the most basic level.

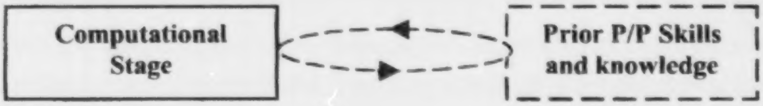


Figure 2. The restricted learning pathway of Dean, a *computational* user.

Stuart: Technician; Linear Learning Pathway

Stuart (age 17) was selected for the study on the basis of three attributes: he was not a prior user of the graphics calculator, he performed poorly in his entry-level test, and he displayed an unusually high level of interest in the graphics calculator early on in the course. His graphics calculator ranking within the sample of ten students improved (from sixth to fourth) whilst his pencil-and-paper ranking dropped from sixth to seventh between the end of Semester 1 and the end of Semester 2.

Stuart's enthusiasm for the graphics calculator was evident in his interview responses. For instance, his successful sketch of $f(x)=2x^3+10x^2-2x+50$ early in the year is shown in Figure 3 below.

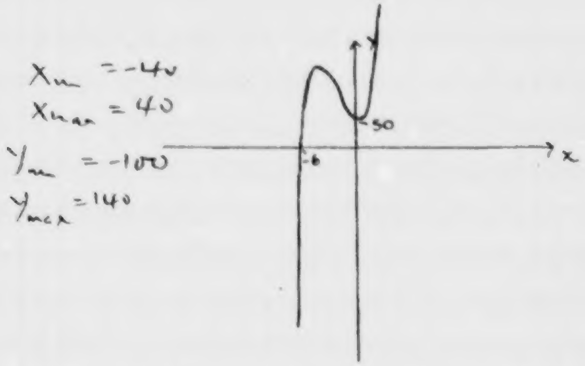


Figure 3. Stuart's technical proficiency in sketching a cubic function.

Stuart's proficiency in carefully manipulating the "windows" key is illustrated above. The following interview excerpt described how he achieved this graph plot:

> Yeah, I just used the graphics calculator and the "y = ", pressed the "y = " and then just typed in the y ... just graphed that using the "windows", and then just used the "trace" key to find where it touched the X-intercept and the Y-intercept, the Y (intercept) is 50, and the negative ... is 5.89, and ... just used the "trace" key again to go to one of the turning points which is about (-3.40, 93.79). (Interview 1, Semester 1)

Technicians, to which Stuart's responses are typical, are characterised by the following indicators of performance:

- Makes some progress towards completing the task
- Makes some progress towards correctly and accurately using mathematics
- Limited, though purposeful explanations and interpretations
- Is usually technically competent in using a graphics calculator
- Uses graphics calculator-generated solutions instead of algebraically derived solutions
- Some mathematics language, symbols and conventions used correctly
- Partial but limited links between graphical, symbolic and numerical representations
- Unable to move to generalisations

Technicians have more control of and familiarity with the technical aspects of the tool, employing it as a scaffolding device to build understanding in new ways (Kutzler, 1997). Drawing on the *computational* stage partial links are evident between graphical, symbolic and numerical representations and the graphics calculator is being used – with limited success – to explain and interpret screen images such as graph-plots. This is the stage at which students are trying, for the first time, to engage in an intellectual partnership with the tool (Salomon et al., 1991). As with *computational* users, gaps in *technicians'* pencil-and-paper skills and knowledge appear to compel them to use the tool more as the means by which they can achieve success in their mathematics tasks. The learning pathway taken by Stuart is linear: it begins at the *computational* stage and progresses to the *technician* stage, with brief task-based learning episodes suggesting that he was beginning to engage in *multirepresentational* thinking (see broken-lined unidirectional arrow in Figure 4 below). Throughout, Stuart "folded back" at appropriate times to retrieve pencil-and-paper skills and knowledge to support and enrich his tool-based learning.

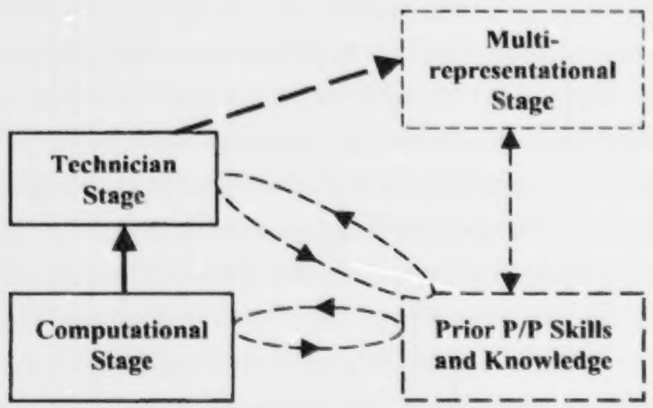


Figure 4. The linear learning pathway taken by Stuart, a *technician*, who was beginning to engage in *multirepresentational* thinking (see broken-lined, unidirectional arrow).

Lois: Multirepresentational User; Multiple Learning Pathways

Lois (age 18) was case studied because she had studied the more advanced Year 12 "Maths Methods" subject and was thus more mathematically experienced than her fellow students; and secondly, she was a prior graphics calculator user. Her pencil-and-paper ranking relative to the cohort of ten students (end-of-Semester 1: second; end-of-Semester 2: first) remained high whilst her graphics calculator ranking fell from second in Semester 1 to sixth in Semester 2.

Lois had few difficulties with tool-based tasks. For instance, her response to the task shown in Figure 5 demonstrates that she was technically proficient in using the graphics calculator and was correctly linking symbolic and graphical representations.

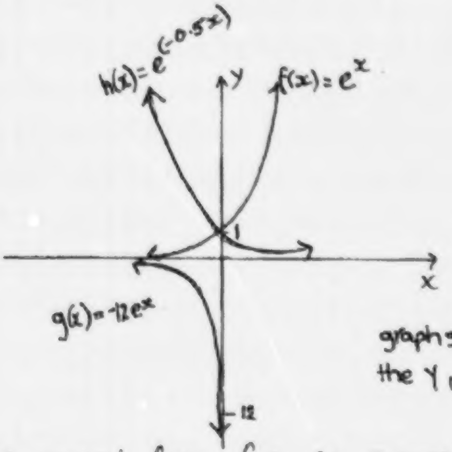
The responses of *multirepresentational* users such as Lois are characterised by the following performance indicators:

- Substantial progress towards completion of tasks
- Substantial progress towards the correct and accurate use of mathematics
- Substantial progress towards full and detailed explanations and interpretations
- Is generally technically competent in using the graphics calculator
- Prefers graphics calculator-generated solutions to algebraically derived solutions
- May or may not use the graphics calculator to check algebraically derived solutions
- Substantial progress towards correctly linking graphical, symbolic and numerical representations
- Has difficulty moving to generalisations

Whilst *technicians* are working with images, users at the *multirepresentational* stage are attempting to extract meaning from the properties of these images. Even though Lois engaged successfully with the graphics calculator she preferred, wherever possible, to employ pencil-and-paper strategies. When asked why, she said this:

> I just think of it as an aid, like a tool to help us through some problems and to verify it … putting it on the calculator you are not achieving anything, you have to know why it is like that, so when you showed me that maximum height reached (a non-routine problem), I am just not satisfied with

> getting the answer off the calculator, it is like cheating. You know, it is like I haven't really worked anything out. It is like the calculator has done all the work for me, like I am not really learning anything. (Interview 1, Semester 1)



graphs are and three cross the Y intercept at y = 1

(b) The general form for an exponential function is

$$f(x) = A\,e^{(kx)}$$

Explain what 'A' and 'k' do to

f(x A = translates the graph ↑ or ↓ on the y axis
Ⓚ dilates the size of graph / range on the x axis
g(x) Ⓐ because ⊖12 is negative the graph becomes inverted below the x axis
h(x Ⓚ dilate the size of graph ~~One~~ x axis
Ⓐ translates graph as previously stated
Ⓚ Ⓚ the negative produces the graph to become a reflection above the x axis + also dilates due to the 0.5

Figure 5. Lois' proficiency in answering a *multirepresentational* task.

In a later interview she again spoke about how the graphics calculator assisted her with her mathematics, but was still reluctant to use it other than for straightforward tasks such as graph plotting, the execution of computations and as a checking instrument:

> I think that basically that the calculator is there so you know that you are on the right track. You know, you can do everything by hand, but I think it is good so you know you have got another tool, so you make sure that you are on the right track and your graph is looking okay. If I didn't have the calculator I probably wouldn't be able to draw the graphs by hand any more, because I rely on it a lot. (Interview 3, Semester 2)

Lois searched for alternative pathways of learning to achieve competence at the *multirepresentational* stage. For instance, tasks that compelled her to use the tool resulted in a the linear pathway: *computational* → *technician* → *multirepresentational*. However, if the task could be accomplished with minimal use of the graphics calculator she employed a *computational* → *multirepresentational* pathway, bypassing the *technician* stage (see Figure 6).

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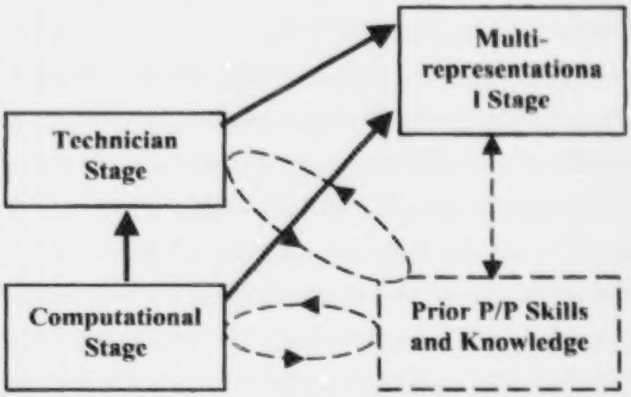

Figure 6. The multiple learning pathways taken by Lois, a *multirepresentational* user.

Nicky: Multirepresentational User; Non-Linear Learning Pathway

Nicky (age 23) was chosen for the study for three reasons: she was a mature-age student, she performed poorly in her entry-level test, and she had not used a graphics calculator prior to commencing her university studies. Like Lois, her task responses suggested that she was a *multirepresentational* user. Her search for meaningful links between the graphics calculator and her knowledge of mathematics, together with her views concerning the graphics calculator, are illustrated in the following comments:

> I am not learning the maths deeply. I feel that I need more exercises so that I can develop a more confident memory of the processes. I feel I am rote learning ... Remembering! Deep learning all of the topics. I still feel that I am rote learning. (Self-report journal, Semester 1)

> This (graphics calculator) doesn't help me to remember. If I was to do everything on a calculator I wouldn't remember as well as I remember this because I actually find meaning in that, whereas I don't have any sense of meaning with this. Last night I just discovered out of the blue how to factorise and everyone was saying just have a look at it and you should be able to estimate, but I don't understand which sections that I was supposed to estimate or what I was supposed to use, until I had actual meaning of what the equation was I couldn't understand it and I think that is the same with the calculator, until I can actually understand exactly what the calculator is doing, until I can read the calculator, I don't think I will understand it. (Interview 3, Semester 2)

Nicky was not as proficient with the tool as Lois. Consequently, her learning pathway reflected tool avoidance wherever possible and resulted in non-linear journeys (see Figure 7). For instance, for some tasks she began at the *computational* stage, progressed to the *multrepresentational* stage, briefly returned to the *technician* stage, and finally moved back up to the *multirepresentational* stage again. There was no evidence to suggest that tool proficiency at the *computational* stage was necessarily assisting her at the *technician* stage (shown by a broken-lined box to indicate that her proficiency at this level was incomplete). Hence, a unidirectional arrow between these two stages is not present in Figure 7. She possessed the prerequisite pencil-and-paper skills and knowledge to operate successfully at the *multrepresentational* stage but appeared to be unable and unwilling to engage with the tool sufficiently to demonstrate proficiency at the *technician* stage.

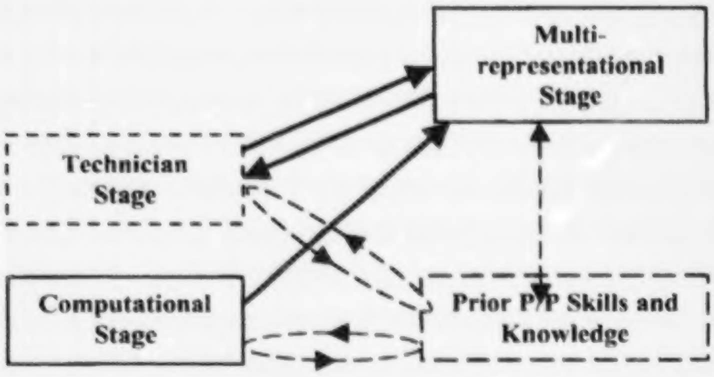

Figure 7. The non-linear learning pathway taken by Nicky, a *multirepresentational* user.

Discussion

An analysis of the tool-based assessment task responses of low-achieving students such as Dean, Stuart, Lois and Nicky revealed that they appeared to be operating at three levels of cognitive development. The lowermost level – the *computational* stage – represents the stage at which the user's engagement and performance with the tool is at its most primitive. This is followed by the *technician* and *multirepresentational* stages which depict increasing levels of successful user-tool engagement and performance respectively. The stages are qualitatively different and appear to be hierarchical, each characterised by indicators of performance.

For satisfactory performance at each stage students must possess prerequisite pencil-and-paper (P/P) skills and knowledge appropriate to that stage. This is represented diagrammatically in Figure 8 by the "Prior P/P Skills and Knowledge" box. The circular and bi-directional broken-lined arrows represent a dynamic interaction in which students are continually "folding back" (Pirie & Kieren, 1994) from the *computational*, *technician*, and *multirepresentational* boxes, to collect, reflect on, improve and reconstruct their prior skills and knowledge so that tool-based understanding can be supported, enhanced and enriched.

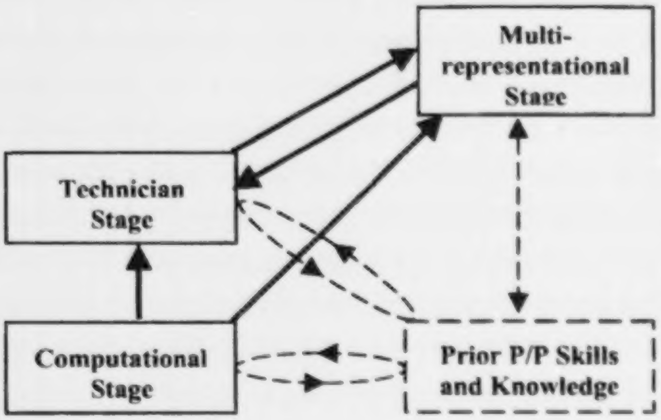

Figure 8. The three stages – *computational, technician and multirepresentational* – to which task responses for four low-achieving students were classified, and their differential pathways of learning.

Figure 8 also illustrates four tool-based pathways or journeys by which the four low-achieving students progressed as they navigated their way through successively higher stages of the model: a restricted learning pathway, a linear learning pathway, a non-linear learning pathway, and multiple learning pathways. The evidence presented here, along with the six other case studies conducted during the year, suggests that educators need to be far more aware of the qualitatively different ways in which low-achieving students engage with technology-rich tasks. While further research is needed to test the utility of the model, it is hoped that it will act as a powerful framework in assisting teachers understand and work with mathematically weak students as they engage with technological tools.

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Characterising Secondary School Mathematics Lessons Using Teachers' Pedagogical Concept Maps

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The construction of concept maps is increasingly seen as a useful tool for mathematics educators. The maps have usually been built either directly by the individual themselves or from their comments in a discussion or an interview. In this paper we describe how a novel kind of concept map was constructed based on questions teachers use in their classroom lessons. An analysis of what we call *pedagogical concept maps* may assist teachers to construct and determine the goals of a lesson. We present here the results of analysing four lessons from a conceptual and procedural perspective which show their value in discriminating between these kinds of lessons.

Introduction

One of the major roles of teachers is to act as facilitators of student knowledge, assisting them to build the relational or conceptual understanding which will enhance learning prospects (Skemp, 1976, 1986; Hiebert & Lefevre, 1986). However, it is evident that the quality of this teacher guidance will be determined by the teacher's own content knowledge, their pedagogical content knowledge (Shulman, 1986), and by how they structure their classroom teaching environment (see e.g., Bromme, 1994; Chinnappan & Thomas, 1999; Ball, 2000). This study used the hypothesis that the planning of classroom activities and the informal questions asked by teachers in the delivery of a lesson are likely to be influenced by their content and pedagogical knowledge structures. This assumption is based on the idea that teachers follow a hypothetical learning trajectory in their teaching (Simon, 1995) based on the content knowledge they have built themselves. Moreover, it also seems reasonable to assume that the relationships exhibited by the teachers' knowledge structures are very close to the ones that they want to encourage students to build in their own minds.

We believe that in creating a classroom environment teachers' informal questioning plays a vital role. We have previously reported on a framework that we have constructed to help identify and classify the types of informal questions that teachers employ (Liyanage, Irwin, & Thomas, 2000). Our intention here is to identify the informal assessment questions (and other teacher utterances) used in mathematics lessons with the teacher's schematic content knowledge in that particular domain, and hence with the pedagogical content knowledge garnered to deliver the lesson. Thus our aim was to represent this knowledge by means of novel concept maps (Novak & Gowin, 1985, Shavelson, 1998), which we have designated teachers' pedagogical concept maps, and to use these to provide information on the goals of the lesson, especially from the perspective of the procedural or conceptual basis of the nature of the teaching strategy.

Concept Maps: Assessing Internal Organisation of Knowledge

Concept maps have been considered by a number of researchers (e.g., Novak & Gowin, 1985; Shavelson, Lang, & Lewin, 1993; Ruiz-Primo, Schultz, Li, & Shavelson, 1998;

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 425-432). Sydney: MERGA. ©2002 MERGA Inc.

Nicoll, Francisco, & Nakhleh, 2001) as an important systematic device for depicting internal organisation or structure of an individual's knowledge. These representations display relationships between concepts, ideas, theories, procedures, and other mathematical entities. They are usually represented as comprising nodes with connecting lines, which may be labelled, with the labels describing the basis on which the connections have been made. They have been used in diverse ways, such as to assess students' conceptual knowledge (Williams, 1998), and to characterise teachers' mathematical content knowledge (Chinnappan, Lawson, & Nason, 1999), and in a number of subject areas, including physics (Austin & Shore, 1995), political science (Parkes, Suen, Zimmaro, & Zappe, 1999), statistics (Roberts, 1999), and mathematics (Bolte, 1999). In each case they have proven to be an effective way of looking at the organisation and the structure of a person's knowledge in a particular subject area, and are considered to be able to reflect conceptual understanding, or its lack (Williams, 1998).

Following a review of the literature, Shavelson, Lang, and Lewin (1993) have documented wide variations in concept mapping techniques, with no less than 128 identified variations. One important variation is that concept maps can be drawn by the person him/herself using techniques taught by the researcher in a training session (e.g., Williams, 1998) or by a second person using the information collected in clinical interviews (e.g., Chinnappan, Lawson & Nason, 1999).

Our motivation for the construction of the concept maps was to enable teaches to ascertain the extent to which they have been presenting mathematics procedurally or conceptually. It is important that as teachers we get a feel for the balance between these two vital parts of learning mathematics, and while we could give many examples of why this is so, the research on derivatives by Ferrini-Mundy and Graham (1994) clearly supports it. They concluded that "Startling inconsistencies exist between performance, particularly on procedural items, and conceptual understanding." (p. 43), giving a graphic example of a student who described in detail an algorithm for sketching the graph of a function including how the first derivative may be used. However, less than one minute later when asked how derivatives are linked to tangents to curves she replied "I'm not exactly sure… I can't remember exactly how it's related to the derivative…I remember doing it, but I can't remember exactly how." (p. 44).

Method

The method used to construct the teachers' *pedagogical concept maps* in this study differs in a major way from the maps used in previous studies. We have developed a method based on our hypothesis that the planning of teaching activities and the informal questions asked during a lesson are highly likely to be influenced by teachers' content and pedagogical content knowledge. Hence we identified teachers' informal assessment questions (and other statements) from the transcripts of videotaped lessons, and employed these to construct concept maps of the mathematical ideas being developed in the lesson. In our construction the labelling of connections between nodes was not carried out because it was not required in our analysis. We were primarily interested in assessing the richness of the resulting schematic structure rather than the manner in which it had been constructed. Hence only quantitative measures were applied to the links and to the resulting structure.

We took four lessons from four different teachers to construct the maps. The teachers, one male (V) and two females (B and E) from New Zealand, and a male (M) from Sri Lanka, are part of a larger study on the use of informal questions in the classroom. The three New Zealand teachers all taught in co-educational state schools, while the Sri Lankan taught in a private boys' school. The lessons were on logarithms, the laws of logarithms, the normal distribution and rounding numbers to significant figures.

Results

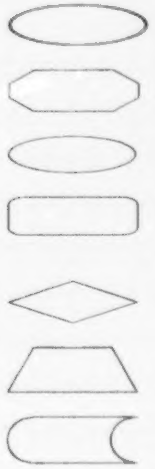
The following algorithm was followed in drawing the pedagogical concept maps.

Step 1: The main theme of the lesson was identified by looking at the title that the teacher put up on the board or his/her declarations, and this was then drawn in the centre of the map.

Step 2: The concepts, procedures, processes, objects, definitions, statements and the other ideas that the teacher tried to relate *directly* to the main theme during the lesson were identified from the informal questions asked and other classroom activities. For example, by asking the question, "what do we do when we have to multiply numbers with indices" during a lesson on "the laws of logarithms" the teacher is attempting to a make a connection between laws of logarithms and laws of indices in the students' minds. Likewise there can be many other ideas that the teacher tries to connect *directly* to the main theme of the lesson. These were drawn as nodes surrounding the main theme of the lesson and the links identified were drawn using arrows.

Step 3: Other concepts, definitions etc. that were mentioned in the lesson but were related to the nodes surrounding the main theme rather than the main theme itself, were identified. For example, standard form for numbers, mathematical operations, and square root of numbers are some of the nodes secondary to the main theme of logarithms. These nodes were linked to other appropriate nodes.

Step 4: Practical applications and mathematical solutions to given problems were identified and linked to other appropriate nodes.



- The main theme of the lesson

- Concepts directly related to the main theme of the lesson

-Concepts that are remotely related to the main theme.

- Operational concepts: Concepts that could be considered as requisites for operations or calculations in problem solving in the area of the main theme.

- Conventions

- Examples or practical applications in the real world context

- Solutions to problems in the mathematical context

Figure 1. The meaning of symbols used in drawing the concept maps.

In concert with the above steps, different shapes were used to denote the main theme, concepts directly related to the main theme, conventions, solutions in the real world context, and the solutions in the mathematical context etc. (see Figure 1). However, these different symbols are purely for ease of reading of the maps and do not play a direct role in the quantitative analysis. Using the process described above, pedagogical concept maps for lessons from four teachers were drawn, two of which are shown in Figures 2 and 3 (where the curved lines represent cross-links). The following examples of teacher comments and questions from the lessons are presented to illustrate how the maps were constructed. The extraction of the data proved a relatively difficult process, and one in which we tried to remain objective, taking only explicit linkings in context rather than attempting to read implied connections into the teacher comments and questions. Ideally we would want to interview the teachers and obtain their comments on the lesson and the concept maps, but on this occasion this was not possible.

Mr M's lesson (S is used to represent a student comment)

M: When we have to write 8 or 100 as logarithms, what do we need?	
S: A base	Relates base to logarithms
M: Yes, a base. What is the base we normally use?	

S: 25.83	
M: Tell it to me in standard form	Relates standard form to logarithms
S: 10^1x2.583	
M: What is the log of it?	

M: What should we do when we divide numbers with indices?	
S: We subtract the indices	Use of laws of indices in division of numbers
M: Yes. According to the laws of indices we should subtract them.	

Mr V's lesson

V: What about 0.00085. Is it a decimal?	Concept of a decimal number

V: 35.5283. There are how many decimal places in this number?	
S: Four	Concept of decimal places linked to rounding
V: Four. How would you round this number to one decimal place? What is the answer?	

Ms E's lesson

E: Some of these other ones [graphs on the board] are quite spread out. What is our measure of spread?	Relates standard deviation to spread
S: Standard deviation	
E: Standard deviation. What would you tell me about the SD of this one [pointing to a graph]?	

E: You know the total area is one. What is the area here? [points to the shaded RHS of a normal distribution curve]	Relates area to probability and SD
S: 0.5	
E: Yes. The table shows all the probabilities of the right hand side. They go up to 0.5. If they go one standard deviation either side the total area would be how much? [shades the area]	

Quantitative Analysis of the Concept Maps

While even a cursory examination of Figures 2 and 3 will reveal the differential richness of the two maps in terms of numbers of elements and connections, in our analysis we wanted to construct a more objective quantitative score for the concept maps for future analysis, and in order to be able to form a validation comparison with the results obtained from the Framework for Informal Assessment Questions (FIAQ) previously developed (see Liyanage, Irwin, & Thomas, 2000). The scoring procedure used was based on that formulated by Chinnappan, Lawson, and Nason (1998, 1999) to characterise teachers' mathematical content knowledge, since it has been shown to be useful in representing the connectedness of teacher knowledge. Following them, two main themes were considered.

1. The 'Richness' of the concept maps: Two basic aspects were considered in defining 'richness', namely, the number of nodes, and the number of links emanating from each node. The higher the number of nodes and links, the more the connections in the concept map and hence richness as a function of connectivity. By also taking the relative importance of each concepts and its relationships into account, a score was awarded to each node type and each type of link (see Table 1).
2. 'Connectedness' in the concept map: Apart from direct links between nodes, three further aspects were considered in defining the connectedness of the concept maps.
   - Cross-links: The cross-links are the 'horizontal' links between nodes that are already connected 'vertically' (by extended connections from the main theme) to other nodes. The importance of cross-links lies in the potential they have for connecting mathematical concepts and ideas, which otherwise could remain separate. Hence they make the schemata richer in inter-conceptual connections.
   - Depth: Some (e.g., Liu, 1994) refer to this as the hierarchies of nodes. However, we take depth as the vertical extension of a node from the main theme on a single path. This shows how far the teacher moves from the main theme in order to make connections between relevant (and closely related) mathematical concepts and the main theme of the lesson.
   - Cycles: A cycle in a concept map is a closed trail along links passing through nodes. They are mediated by a combination of cross-links and depth of links and the directed nature of the links. We refer to the cyclic connectedness of nodes, where cycles can be 3-cycle (3C), 4-cycle (4C), …etc, according to the number of nodes they pass through, and they are scored thus.

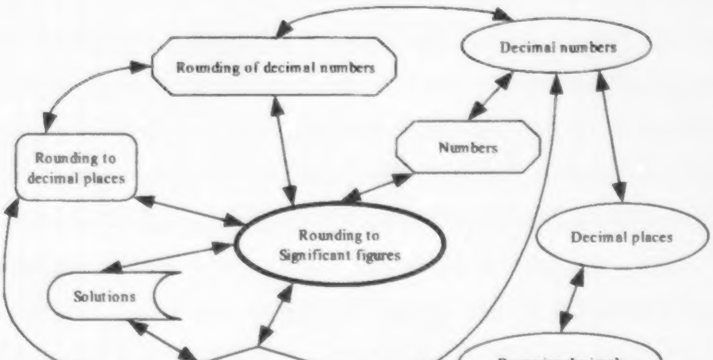

Decimal numbers
Rounding of decimal numbers
Numbers
Rounding to decimal places
Rounding to Significant figures
Decimal places
Solutions
Recurring decimals
Conventions

Figure 2. The pedagogical concept map of Mr. V for the lesson on rounding.

Taking into account the above criteria, a scoring procedure generating a numerical value to represent the richness of the concept maps was established. The number of nodes and links was multiplied by a base score to give a weighted total. The base scores applying to nodes, links, cross-links and depth are described in Table 1. The main theme was given the highest base score of 5 since having a focus is a fundamental requirement of any

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lesson. Similarly the concepts directly related to the main theme and the practical applications were considered to be next in importance and hence rated a weighting of 3.

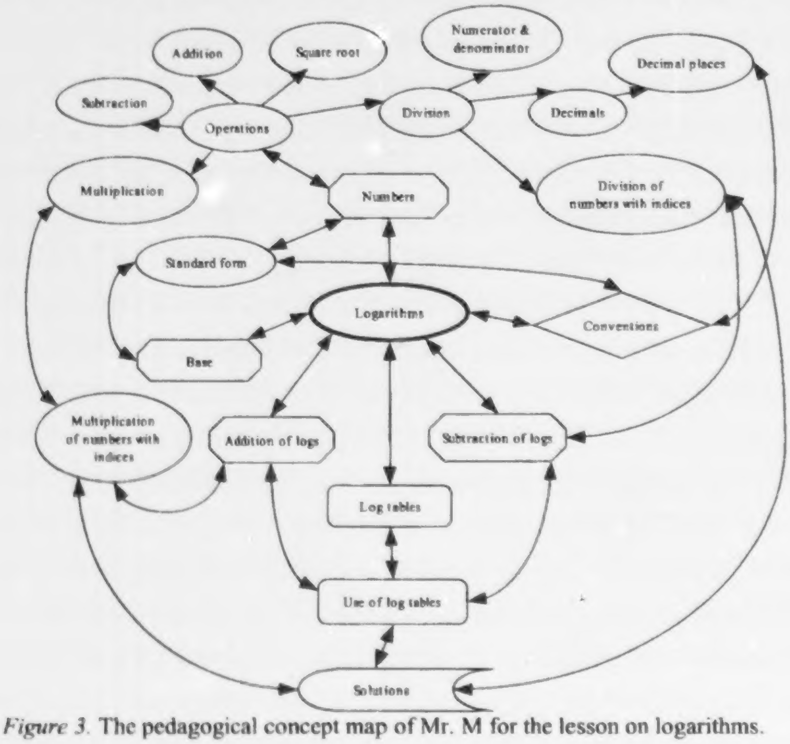
Addition
Square root
Numerator & denominator
Decimal places
Subtraction
Operations
Division
Decimals
Multiplication
Numbers
Division of numbers with indices
Standard form
Logarithms
Conventions
Base
Multiplication of numbers with indices
Addition of logs
Subtraction of logs
Log tables
Use of log tables
Solutions

Figure 3. The pedagogical concept map of Mr. M for the lesson on logarithms.

Mathematical procedures (weighting 2) placed next, followed by concepts only remotely related to the main theme (base score weighting 1). The base scores for links were obtained by using the same order of priority (see Table 1). As stated above, the 'connectedness' of the concept maps was considered in two distinct ways. The 'depth' of the map was scored according to its level, one point for each, cycles (mediated by cross-links) were granted a score according to the number of nodes they pass through.

We can see from Table 1 that the quantitative measures obtained from the lessons of Mr V and Mr M were 139 and 236 respectively, and the corresponding values for Ms. B's and Ms. E's lessons were 58 and 178 respectively. On this evidence it seems to be possible to make a distinction between the lessons of Mr M and Ms E and those of Mr V and Ms B. Although it may not be easy to establish a clear-cut dividing line, Mr M and Ms E have higher scores and those above a guideline of around 150 may indicate a more conceptual lesson, and lower scores a more procedural one. The conclusions reached by examining the scores generated by this process were consistent with those from the FIAQ, suggesting that this is a viable method of discriminating between lesson types. It should be remembered that the FIAQ was constructed on the basis of procedural-conceptual distinctions in questioning, as well as inter-representational linking (Liyanage, Irwin, & Thomas, 2000). The FIAQ employs Levels 1 and 2 to measure knowledge questions and those involving simple procedures, while Levels 3, 4 and 5 consider connections between concepts and representations. Mr V had 12 Level 1 and 35 Level 2 questions in his lesson, but only one question at Level 3, while Ms B had 2 Level 1 and 28 Level 2, with 2 Level 3 and only 3 Level 4 questions. In contrast Mr M's totals were 20 Level 1, 7 Level 2, 24 Level 3 and 4

Level 4 and Ms E's were 21 Level 1, 26 Level 2, 11 Level 3 and 7 Level 4. These latter two lessons are clearly differentiated as having a stronger conceptual base by their Level 3 and 4 questions on the framework, as confirmed by the concept maps.

Table 1

The Scoring Procedure and Scores for Teachers' Pedagogical Concept Maps

Features of the lesson	**Richness**			**Connectedness**										
	Nodes	Links	Cross links	Depth						Cycles				
				L2	L3	L4	L5	L6	L7	3C	4C	5C	6C	≥ 7C
1. The theme of the lesson	***5*** (5,5)	***3*** (15,18)	-											
2. Concepts/procedures directly related to the main theme	***3*** (6,12)	***2*** (6,12)	***2*** (4,10)											
3. Operational concepts or processes	***2*** (2,4)	***1*** (1,4)	***2*** (4,4)	***1***	***2***	***3***	***4***	***6***	***6***	***1***	***2***	***3***	***4***	***5***
4. Other related concepts or procedures	***1*** (3,12)	***1*** (5,11)	***1*** (3,9)	0 1	0 10	3 6	0 4	0 0	0 0	3 0	10 8	21 0	16 24	20 70
5. Conventions considered in the lesson	***2*** (2,2)	***1*** (2,1)	***2*** (3,4)											
6. Practical applications with solutions	***3*** (0,0)	***2*** (0,0)	***1*** (0,0)											
7. Solutions in the mathematical context	***2*** (2,2)	***1*** (2,1)	***1*** (1,2)											
Totals	(20,37)	(31,47)	(15,29)	03,21						70,102				

Numbers in bold italics give the base score weightings. Other numbers in parentheses and below italic weightings give the scoring for the maps of Mr V and Mr M respectively.

Of course there are still steps necessary to refine the process of constructing *pedagogical concept maps* and using them to gather information about the pedagogy of classroom lessons. One issue is checking the validity of the maps drawn, for example by using a reliability assessor to construct them independently. Also the weightings applied to the various measures of richness could be argued, affecting the differentiation of lessons in the middle ground. In addition it needs to be said that the approach may not prove suitable for all types of lessons. Those we have observed were primarily instructive in nature and it has to be recognised that many teachers value a constructivist approach which may lead them away from giving direct guidance even in the form of the type of questioning seen here, preferring to allow students to interact in investigative situations. It will be interesting to see if the method can be usefully applied to the teacher's role in these types of lessons. Another consideration is that the content of some lessons might better lend itself to the introduction of concepts than others, thus skewing the richness of the maps. This is an area that needs some investigation. Having said this we believe that many teachers value both the importance of teaching concepts as well as procedures and the opportunity to engage in constructive self-analysis of their teaching practice. It may be that the concept mapping techniques described here will give them another instrument to be able to do so.

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Interpreting and Constructing Models in Computer Environments: Lego™ Blocks and Spatial Reasoning

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This paper examines the problem-solving approaches employed by eight Year 6 students (aged 11 or 12) as they completed a task that involved the construction of a model using computer software. The participants were required to interpret two pictorial elevations of a pre-designed model and then construct the model using the simulation software. The students who were able to effectively use the perspective and orientation functions from the software package were more likely to accurately complete the task.

Computers make it possible to represent visual and spatial mathematics with an amount of flexibility not offered by other media. Construction-based programs, for example, allow the user to represent and manipulate images accurately as either two- or three-dimensional (2D -3D) representations. These images can then be transformed or rotated with precision. With appropriate software the computer can become a powerful tool that enables students to manipulate spatial arrangements and construct visual images that would usually be limited by their own drawing capabilities. Construction-based software allows the user to develop new mental tools (Clements and Battista, 1992) and establish more sophisticated spatial reasoning skills (Amory, Naicker, Vincent, & Adams, 1999; Nemirovosky and Noble, 1997). The interactive nature of such software ensures "computer screen representations of visual objects and even visualised relationships can be acted upon directly and we can observe the ensuing changes in the represented relationships" (Noss, Healy & Hoyles, 1997, p. 210).

In studies away from computer environments, strong links between construction play and the development of visual and spatial skills have been established. Brosnan (1998), for example, found that children with high spatial ability were more likely to be able to construct a specific 3D model using Lego blocks than children who did not possess these high spatial skills. Other studies (including Lowrie, 2000) have found that differences in student's ability to solve spatial tasks have been attributed to the extent to which they play with construction toys and building blocks. Similarly, other studies have found that representations of the physical world (Trawick-Smith, 1998) and play in naturalistic settings (Clements & Sarama, 2000) are also influential in spatial development.

As our society becomes increasingly reliant on visual stimulus, new technologies will push the boundaries between "real life" environments and 2D representations of 3D space. Links between foundation spatial skills (including depth perception) and engagement in simulated environments will almost become transparent (Lowrie, in press). The impact these inevitable changes will have on the children's spatial development is uncertain. Computer simulation programs, despite their obvious advances, do not always make 2D-3D representations easy to interpret or understand. Children often find it difficult to transform their understanding of real-life experiences to the representations created on the screen (Lowrie, 1998). This is particularly the case when children have not yet established

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 433-440). Sydney: MERGA. ©2002 MERGA Inc.

foundation spatial and visual concepts through lived experiences in a 3D world (Lowrie, 2000).

The Purpose of the Study

The central concern of this study was to investigate the extent to which the participants were able to construct a model using a software program from a pictorial (2D) representation of the object. The following research questions were posed:

- Were the participants able to make connections between pictorial and screen-based images?
- What spatial skills did the students employ to complete the task?
- To what extent could the children highlight possible difficulties that may arise when completing the task?

Method

The Participants

The investigation took place in a primary school in a large rural city in New South Wales. Eight Year 6 children (aged 11 or 12) were involved in the study. All of the participants were competent technology users in the sense that they used computers for a variety of purposes in both home and school contexts. Moreover, each student regularly played entertainment games in technology-rich environments at home. Importantly, all of the children had played with traditional Lego constructions in the past, although some of the children had not played with these blocks for some time. The students selected for the investigation had varying mathematical abilities.

The Activity

Each participant completed an activity involving the Lego™ (2000) construction program titled *The Knight's Kingdom*. The program allows the user to move building blocks onto a background and construct designs of their choice. There are a number of design facilities available in the program including buttons that could rotate blocks, change the orientation constructions, change perspective, move between close-up and wide-angle views, and change the colour of blocks (see Figure 1).

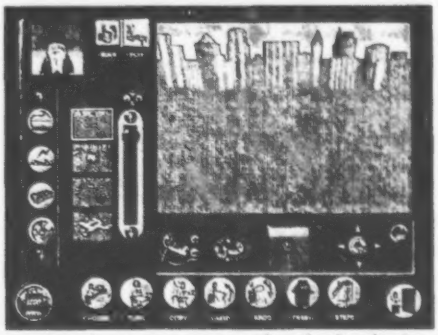

Figure 1. A representation of the screen image encountered at the beginning of the problem-solving activity.

The activity was conducted in an hour-long session on a one-to-one basis with the author. The session included: 1) a 30 minute instructional program that scaffolded the students' understanding of the program; 2) an explanation of the problem; and 3) adequate time for the students to complete the problem-solving activity. The students were asked whether or not they had played this game or similar construction games with a Lego theme in the past—with all participants indicating that this was the first time they had seen such a program. An instructional tutorial was developed in order to provide the students with opportunities to use key functions of the game. The instructional component of the session was interactive in the sense that the students were encouraged to ask questions about the program's features and build their own models using the construction facilities of the program. This is in line with Lesh's (1999) notion of *representational fluency* which encourages the user to develop increasingly powerful conceptual understandings by adapting (refining, revising, extending, integrating and differentiating) existing ideas and schemes. All of the students became familiar with the design elements of the program including the perspective and orientation features before being given the problem to solve.

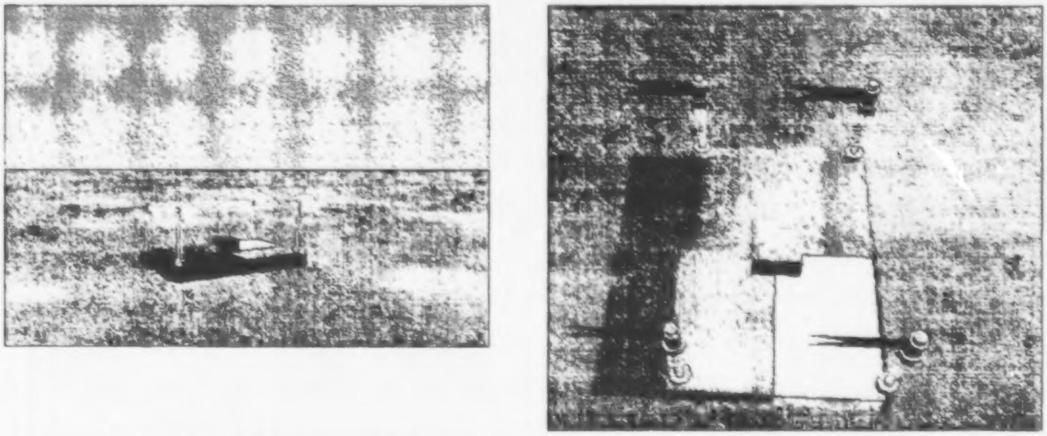

Figure 2. The front-view and bird's-eye view elevations of the pictorial representations.

The students were given two elevation views of a model (in pictorial form) which the author had constructed using *The Knight's Kingdom* and were asked to replicate the model using the program. The two colour prints of the elevations are presented in Figure 2. Consequently, the students were challenged to construct an exact replica of the model using the information presented from the elevation views. The students were informed that the two representations were of the same model and were asked to explain how this was possible. This allowed the author to ascertain whether or not the students understood what the task involved. Once any misconceptions were addressed, the students were encouraged to construct the model.

Results

The time taken to complete the construction activity ranged from seven to thirty minutes. Three of the eight students who engaged in the activity were able to complete the task accurately. The other five participants made one or more errors associated with the correct position or orientation of the blocks, the selection of the appropriate sized block or the correct block colour. Not surprisingly, the students who were prepared to use the orientation and perspective facilities from the computer program were more likely to complete the activity in an accurate manner. Nevertheless, the eight participants used a variety of approaches and techniques to construct the model on the screen (see Table 1 for a summary of these approaches).

Table 1

Different Elements of Orientation and Perspective used by Participants in both 2D and Screen-based Contexts

Student	Orientation and Perspective Techniques Used
Alex Found task difficult	Predominantly worked from above (bird's eye view) Never worked from a different orientation (remained in front-view)
Simone Found task difficult	Screen orientation limited to front view but changed perspective to a bird's eye view on occasions
Ebonee Initially, only correct from front view	Only worked from front-view picture. Initially worked from this view but began to change the screen orientation as confidence grew
Laura Unable to complete task successfully	Only consulted the bird's-eye picture. Wanted to work from this perspective on screen but became frustrated when the orientations were not identical.
Carli Completed task with incorrect blocks	Constantly moved between different perspectives (above, wide view and close up) Frequently changed orientation using shadow techniques for accuracy
Bailey Accurately completed task	Used both pictures as reference for design. Tended to limit screen orientation to a front view but moved between close-up and far-away perspectives
Joe Accurately completed task	Predominantly worked from front-view picture. Worked from a range of perspectives and orientations.
Percy Accurately completed task	Used a range of orientations and perspectives to complete task Made explicit reference to one picture or the other depending on the orientation of the screen object

Three of the five students who were unable to accurately construct the model viewed the screen from only one perspective—either a bird's eye or front view. By contrast, all of the students who accurately constructed the model moved between these perspectives, with one of the students using close-up and far-away perspectives in addition to the bird's eye or front views. The students who were able to replicate the model on the screen were also more inclined to alter the orientation of their model in order to make sure that the blocks and flags were positioned correctly. On occasions these students would rotate their model through 360^0 as they made connections between the pictures and their screen-based model.

Case Study Analysis

Some of the students were able to make strong connections between the 2D images (see Figure 2) and their screen-based designs whereas others found it difficult to interpret the different representations. Those students who were unable to make these connections either a) found it difficult to appreciate that the pictorial representations were in fact the same object and/or b) did not understand how the computer program represented images in a 3D-like form. This was the case even though the students had been assured the elevation views were of the same object before attempting to complete the task.

Alex. It was evident that Alex used computers on a regular basis. Some of the software she used included paint and drawing programs. She quickly learnt how to use a range of functions and was quite prepared to explore the environment in an exploratory manner. Alex was able to explain why some objects looked smaller than others despite the fact that they were the same size and appeared to have a good understanding of perspective and diminishing line. She had trouble placing the flags in the correct position—fundamentally because she was reluctant to rotate her design. As she commented, "it's confusing to change the angle because it makes it look like a different shape…mak[ing] the blocks look larger than they are." Here reluctance to move from a font view of her model resulted in a considerable amount of frustration. "Getting the flags with the right angle is the hardest part."

Simone. Simone did not have as sound an understanding of perspective as Alex did. Although she was able to navigate her way around the program quite well, it was evident that she had not been exposed to many 3D-like programs in the past. Without support, she was unable to make sense of the pictorial representations—for example, felt that the blocks at the back of the design were a different shape to those at the front. Consequently, she had considerable difficult selecting the appropriate blocks to make the design. On occasions it seemed that the combinations of blocks grouped in colour were puzzling her because her block selection matched the size and shape of colours rather than individual pieces. For example, Simone used two 8 x 4 block rather than a number of 8 x 2 blocks. Although she commented that she played Lego with her brother it was evident that she was not making connections between real and simulated contexts.

Ebonee. Ebonee was able to navigate her way around the computer environment quite effectively. She indicated that she was familiar with a range of software programs which were used for both entertainment and homework purposes. Although she was quite willing to experiment with orientation and perspective functions in the tutorial mode of the session, she did not use the rotation functions when constructing her model. Her completed design was accurate from the front view but incorrect from all other orientations. The block on the top of her design was not positioned in the centre of the model—it was too far forward (as could be interpreted in the front-view design).

Laura. Although Laura played entertainment games at home (eg. Playstation) she did not use the computer on a regular basis for any purposes except wordprocessing. She struggled to come to terms with many of the programs functions and had considerable difficulty in using the mouse in a coordinated manner. Instead of building her design from the base up, as all other participants did, she constructed her model from the top down. Laura used 4 x 2 blocks (and not the correct 8 x 2 blocks) for her design and continued to do so until she completed the base. It was only when she began to colour in the design that she recognised that she had used blocks that were too small. On her second attempt, Laura still persisted with a top-down approach. It could be the case that she was pre-occupied with making sure that the flags were positioned carefully. When asked to comment on which aspects of the problem she found difficult to complete she suggested "it was really hard to get all the angles right because they looked different on the paper than on the screen even though they were the same…the angles on the flags were the most difficult to work out." Laura found it difficult to make connections between the two pictorial representations, compounding difficulties faced in making the links to the simulated environment.

Carli. Carli played with traditional Lego blocks, with her brother, on a regular basis. She was quite adept at using all of the function keys demonstrated in the tutorial and was able to effectively use orientation and perspective functions to navigate around the screen. Carlie was able to make connections between the two pictorial representations and referred to each as she constructed her model. She had no difficulty moving between the two environments. As she commented, "You need to have two pictures so that you can build a 3D model. Without the top view you would not be able to tell how deep it was." Although Carlie completed the task in less than ten minutes, the model was not in the correct proportion. She had used blocks that were too large. Initially she insisted, "It looks too long because it looks more like a real thing on the screen than it does on paper." After showing Carlie that she had used the wrong size blocks she maintained, "When I play Lego I am able to count the little circles on top of a block to make sure it is the right size…but it was hard to tell here." Although she had completed the task quickly, a considerable amount of time was devoted to selecting the blocks. Laura had realised that her selections were inappropriate once she began to colour the blocks. Carli more than likely realised her mistake at this stage of the problem too, but reasoned that the screen representation was different to the 2D form.

Bailey. Bailey accurately completed the entire task in just over five minutes. She continually made reference to both pictorial representations when selecting and positioning blocks. Although she predominantly viewed her model from a front perspective, she continually moved between close-up and wide-angle views as she constructed her model. She suggested "moving between close-up and far-away angles lets you see your work in different ways and makes it easier to understand if you have the depth right." The "depth" she referred to was associated with the 3D-like nature of the object.

Joe. Joe was a very capable young mathematician who frequently used computers for a variety of purposes. He moved between different orientations and perspectives very quickly, with both precision and confidence. He was quite accustomed to efficiently navigating around computer environments and was able to quickly apply this knowledge to the problem-solving context. His initial interpretations of the pictorial representations (Figure 2) revealed that he was aware of the elements of the problem that would be most difficult to interpret. "You cannot tell what shape and colour all the blocks are unless you look at the information in both pictures…even then you need to predict what some of the blocks might be and then make sure it matches both pictures." Although Joe predominantly worked from the front-view picture he made reference to the bird's-eye perspective at critical times during this construction phase. What he did do more than most other participants was move between different orientations (through 360^0) in order to check that his model accurately matched the pictorial representations. Joe completed the activity in less than ten minutes.

Percy. Percy was another capable student who had extensive experience in using entertainment and educational programs on the computer. He was able to "click and drag" icons easily and played with Lego constructions far more than any of the other participants. In fact, he had an extensive collection of Star Wars Lego and other Technics Lego blocks. He ordered blocks from magazines and always asked his grandparents to purchase Lego for him at Christmas and on his birthday. Although he had not used this software before, he navigated his away around the environment intuitively—using tools that had not been demonstrated in the tutorial session. For example, he used the Control key to orientate his object through rotations of five degrees. He recognised that the base "was a Lego mat [and was] still a square even though it looks narrower and longer than a square." Interestingly, he made more connections between the game's theme and the actual activity than the other students. "This could be one part of a castle." Moreover, he was more inclined to respond to questions with a particular context. For example he commented, "the flags are at 45^0 angles because the wind is blowing from the East." He also indicated that the software program was more challenging that used the traditional block constructions. "This is more difficult than normal Lego because you have to make sure that you have planned it out in your mind first, otherwise it might look funny…even with directions it would be hard because you need to see that it looks right from 360^0 because some images are still flat." The notion that "flat" screen images require careful planning was, in part, overcome by his ability to use the perspective and orientation functions so well. Nevertheless, Percy was aware of the relationships between 2D and simulated 3D space.

Conclusion and Implications

Generally, the participants were able to make strong connections between the elevation representations and their screen-based constructions. Although only three of the eight students were able to accurately complete the task, all participants were able to make some links between the 2D and simulated 3D environment. In most instances, the students were able to appreciate that the screen-based images were representative of both the pictorial representations and actual Lego blocks. Moreover, the students were able to develop increasingly powerful conceptual understandings by modifying their computer-based constructions and relating these new experiences to existing ideas. The successful participants were able to develop powerful ideas about links between the 2D, the simulated 3D computer environment and the traditional concrete manipulatives.

Although a range of spatial skills were required to complete the task—including concepts associated with position, location, arrangement and diminishing line—conceptual understandings of perspective and orientation were essential. The students who were able to interpret dimension and manipulate objects in both front-view and bird's-eye perspectives were more likely to successfully complete the task than those individuals who worked only from one perspective. Similarly, those students who continually rotated their construction in order to view it from different angles found the task easier to complete. The students who had most difficulty completing the task tended to keep their construction in a front-on view. This became problematic if an individual was not able to rotate the model "in the mind's-eye."

Most of the students were able to recognise that it was worthwhile to have front- and bird's-eye elevations of the model. This was particularly the case when they were attempting to place the flags in the correct position. The activity certainly provided opportunities for the students to develop increasingly powerful spatial understandings (Amory et al., 1999) and more sophisticated mental tools (Clements & Battista, 1992).

Several implications for teachers emerged from the study:

- Investigative tasks that challenge students to simultaneously interpret information from different perspectives and orientations provide opportunities for more sophisticated spatial development.

- These students were able to efficiently navigate their way through a novel technology-rich environment because they were able to contextualise the problem by relating it to a familiar context.
- Children are exposed to a range of technology rich activities outside school settings that are more spatially rich than activities typically presented in the classroom.

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Making Connections Between Simulated and "Real" Worlds: Young Children Interpreting Computer Representations

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This paper explores the way in which six young children (6 year olds) interpreted and made sense of screen-based images on a computer. The participants were asked to interpret a screen image as they were introduced to a construction-based software program. Some of the children could not make links between the screen images and the intended three-dimensional (3D) representations. The participants who were able to make these connections tended to relate the screen objects to life-like images or used analogies to describe concepts associated with depth perception, perspective and orientation.

Recent innovations in Information Communication Technology (ICT) have placed an increasing emphasis on video technology as animation and three-dimensional (3D) simulations become more "lifelike." In this information age students are required to process and interpret pictures and images with increasing regularity. Since most aspects of mathematics are image-based (Wheatley & Brown, 1997) it is important to evaluate the effectiveness of new technologies on students' mathematical development as they spend increasing amounts of time on computers (and playing computer games).

It is evident that these technologies are playing an influential role in school-based curricula, information retrieval, communication and home entertainment. As the technology becomes more sophisticated, greater demands will be placed on the user's ability to interpret information and construct understandings from the visual displays generated from the computer-based environments. Since various types of imagery play a significant role in the learning of school mathematics (Presmeg, 1995) we would expect that such "developments" might have an impact on the way in which mathematics is delivered in the future. To date, most research associated with way students interpret visual stimulus, whilst engaged in computer-based activities, has been restricted to mathematical understandings associated with 2D representations. Moreover, these studies have not involved young children who are establishing foundation understandings of 2D and 3D space. Since many young children are being exposed to—and are motivated to play with—computer-based environments on a regular basis, more attention needs to be focused on the impact such technologies have on these students' learning.

There has been an extensive body of literature that highlights the benefits computer-based activities have on children's problem-solving skills (Betz, 1995-96), cognitive development (Fatouros, 1995) and creative thinking (Rieber, Smith & Noah, 1998). Other researchers have argued that computer games improve spatial visualisation (McGlurg, 1992) and mathematical reasoning (Stone & Stone, 1998). As a result of the changing habits of children's leisure and entertainment preferences, considerable attention has also been devoted to the type of skills that are developed by playing computer games. Greenfield (1984), for example, proposed that the use of computer games promoted hand-eye coordination and spatial skills.

Not surprisingly, Amory, Naicker, Vincent and Adams (1999) found that students preferred to play with 3-D adventure games that required the use of visualisation and

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challenged the user to employ a range of problem-solving skills. Such games involve 3D-like representations of background and foreground scenes with the user manipulating keyboard commands or other peripherals to navigate a character through scenes that are becoming increasingly lifelike. We could speculate that current trends in advertising could influence young children to play entertainment games more than educational games in the future. Today, popular television shows and movies are closely linked to entertainment games (eg., the Pokemon cartoon series is linked to Nintendo games while the latest Star Wars movie has software programs that simulate battle scenes from Episode 1).

Even with today's technology, which may be obsolete within two years, the user is able to move a character with precision in a simulated environment that is quite detailed. These images have structure, depth, perspective and flow seamlessly from one scene to another. We could hypothesise that some of the realistic flight-simulations developed for the training of pilots will be available to the general public (and will be inexpensive) within a few short years. As these developments become available there will be a need to monitor the extent to which these technologies influence the user's spatial development. Since young children are becoming increasingly exposed to simulated environments that require the interpretation of visual displays—more research is required in this area. At present, we are trying to fit new technologies to existing curricula rather than reconceptualizing curricula with the addition of technology.

Most studies that have attempted to gain insights into the way in which users engage in ICT contexts have been with high-school students or adults (Lowrie, in press). Few studies have been conducted with young children who are developing foundation spatial skills. One study that did involve young children (Lowrie, 1998) revealed that some students had difficulty relating the screen-based representations to real contexts. These students were not able to interpret the cues that provided information about the pictorial depth of the image and consequently could not go beyond 2D interpretations on the screen. This study extends our understanding of how young children make sense of 3D-like images in computer-based environments that require a high degree of visual processing.

Method

The Participants

Six students (6 year olds) were selected to participate in a problem-solving task that was intended to monitor their spatial and visual understandings as they completed an activity that required the interpretation of 3D-like representations on a computer screen. The investigation took place in a primary school in a large rural city in New South Wales

The Activity

The activity required the participants to interpret the template from a Lego construction program (Lego Group, 1998) titled *The Knight's Kingdom*. For approximately 20 minutes each participant was able to move building blocks onto a background and construct designs of their choice. The children were encouraged to click on icons (including buttons that could rotate blocks, change the orientation constructions, change perspective, move between close-up and wide-angle views, and change the colour of blocks). The present study is concerned with the manner in which the students made sense of the screen template and not their success in constructing models. At all times the children were encouraged to verbalise their thinking as they attempted to understand how to move through the program and construct models. The author took the view that the children should learn to use the program by promoting guided discovery and modelling through active inquiry.

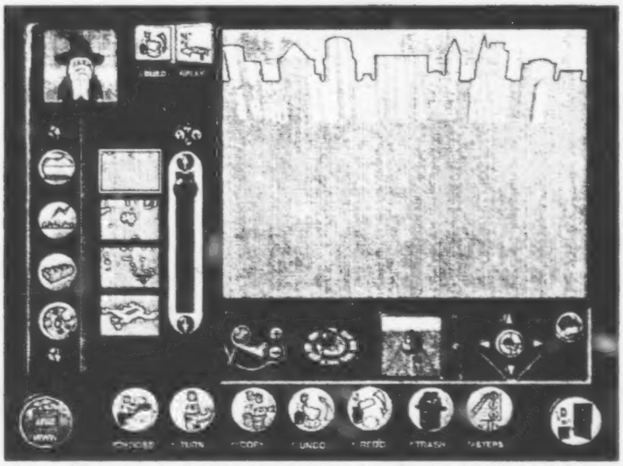

Figure 1. A representation of the screen image encountered at the beginning of the problem-solving activity

This screen image was quite dynamic in the sense that an object could be manipulated in a number of ways that provided the user with a 360 degree view of the scene. The opportunities for *active play* allowed the participants to gain an understanding of how background scenes and objects were represented. It also allowed the participants the opportunity to "play" first and to experiment with different screen views and construction options in a supportive manner. The instructional component of the session was interactive in the sense that the students were encouraged to ask questions about the program's features and build their own models using the construction facilities of the program. This is in line with Lesh's (2000) notion of *representational fluency* which encourages the user to develop increasingly powerful conceptual understandings by adapting (refining, revising, extending, integrating and differentiating) existing ideas and schemes. Although the interviews were relatively open-ended in nature, questions were posed in order to monitor the students' understandings of depth perception and the structure of 2D-3D shape. Consequently, the major focus of the study was to determine the extent the students were able to interpret screen-based images and relate these images to objects in their environment.

Results

The children's interpretations of the screen-based representation were quite diverse. Three of the participants had considerable difficulty in recognising that the images had depth and were intended to represent a life-like scene. The children had difficulty interpreting depth cues on the screen and tended to recognise the objects as 2D shapes. By contrast, the other three participants were able to relate the objects to life-like images and relate the images to personal lived experiences. These children had little trouble recognising that the images represented 3D objects.

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Difficulties in Interpreting Images.

Three of the six children involved in this activity found it difficult to make links between the screen images and the intended representations. When initially questioned about the screen representation presented in Figure 1 Carmel was unable to interpret that the background images represented buildings. When asked to describe what these objects were she responded, "they are [pointing to the objects] shapes. Some are rectangles and some are pointy triangles." She went on to comment "some of the shapes look like they are floating in the air." Even when Carmel was encouraged to examine other parts of the screen (for example, the Lego base that represented the ground) she did not appreciate that the shapes represented buildings. Carmel appeared to be viewing all objects and images without perspective or depth and thus only "viewing" the screen in two dimensions.

Carmel was the only student who was unable to recognise that the background objects were buildings in a city. Neil immediately identified these background images as a cityscape that had "some buildings taller than others." Neil was able to identify the ground and the sky and also recognised that it would be possible to build other structures in front of the buildings. Nevertheless, he was not able to justify why the buildings appeared to be so small.

> If I build something in front of these buildings they will go to the top [of the screen] and will be bigger than these buildings [pointing to the background]. (Neil)

With the exception of Carmel, all of the other children interviewed were able to reason that the buildings appeared to be small because they were so far away. Neil, however, was unable to consider depth or notions of diminishing line in his reasoning. When Neil began to explore the functions of the program he used the mouse to hit the arrow keys. These keys allowed the user to view objects from a number of perspectives. The students were particularly fascinated with the "up" and "down" arrow as they allowed the user to view buildings from varying heights. Neil appeared to become quite confused when using this particular function. He commented that:

> Everything [the buildings] changes shape. If I click here [the down arrow] it becomes high but if I hit here [the up arrow] it becomes so small. The more I click the smaller it comes. (Neil)

He did not seem to appreciate that the view and not the objects were changing.

The other student in this cohort who struggled to make sense of the images presented on the screen was Sandy. Sandy was not able to reason that the buildings could possibly be a considerable distance away. Like Neil, Sandy had difficulty in appreciating the conventions that take the flatness away from the computer screen. In another situation, she was encouraged to click on the rotation button and comment on what happened. The screen view could be rotated 30^0 clockwise or counter-clockwise with one "click" of this icon. She too seemed to think that the objects were changing rather than the manner in which they were viewed. After some time using this function she commented, "Now I see what it does—It lets you walk around the building." Interestingly, this interpretation did not require Sandy to interpret any depth perception—it involved the recognition that the particular object remained the same despite the fact that it looked different from various elevations.

Sophisticated Interpretations of Images.

The other three students who engaged in this activity immediately recognised that the background images represented buildings from a cityscape. They were also able to make judgements about the size of the buildings and why they appeared to be smaller than constructions they were creating. Both Rachael and Carl were able to articulate why the buildings looked like they did.

> You can only see the front of the buildings [because] they are so far away. They look small but they are probably very tall. If you got closer you would be able to see some parts of the sides [of the buildings]. It's like when you draw a picture you have to be careful about how tall you draw things. (Rachael)

> It's like drawing a picture. This is the front of the picture taken from above the ground. If you drew it from the side it would look different. (Carl)

It seems to be the case that Rachael is indicating that the template is in perspective and has taken for granted some of the assumed conventions that are required to interpret links between 2D and 3D representations. Interestingly, Rachael was quite anxious to test her theory and soon clicked on one of the icon buttons that enabled her to view the screen in a more interactive manner. Similarly, Carl wanted to see if you could see what the buildings would look like from side on. Both of these children recognised that the background scene represented a life-like scene and were anxious to explore the "world." Interestingly, these two children scored highly on the spatial ability measure and commented that they had played with Lego Blocks at home.

Within a short time Rachael was able to achieve "close-up" and "far-away" views of the cityscape by moving the mouse and using the appropriate function keys. She commented that:

> The building looked tall when you get up close. They do not get bigger even though they look big now [as she moved to a close-up view]. It's like real life. If you get close to a tall building and look up it looks really high but smaller from a long way off. (Rachael)

This response indicated that Rachael was developing an understanding of perspective. She appreciated that the screen images resembled life-like images by using directional functions from the program to explain her reasoning.

Samual also used these functions to explain his reasoning about the position of objects on the screen. He used analogies to describe the changing images of the background scene.

> When you go up it is like flying in a helicopter. You get to look down from in the sky like you are a bird. (Samual)

Samual enjoyed moving between the different screen elevations available in the program (different bird's-eye and front-view elevations). He reasoned, "things look different from far away and close up" and was quite motivated to test the limits of the program. For example, he wanted to see how high he would be able to go.

> If I can go high enough everything look(s) really small. They would look like ants…but really the just look smaller because you are so far away. (Samual)

It was evident that he was able to appreciate that the screen representations were dynamic in nature. By contrast, Neil had become confused when attempting to use the directional arrows on the screen. He had assumed that he was modifying the structure of his design because the image on the screen continually changed. The children who were able to relate the screen representations to life-like experiences were more successful in addressing concepts associated with perspective and orientation. Samual's analogies that were related to helicopters, birds and ants articulated a high level of spatial reasoning.

Conclusion

Generally, the students who participated in the activity were able to interpret at least some of the spatial and depth cues embedded in the screen image. Only one student was unable to recognise that the background scene represented a cityscape. Nevertheless, it needs to be recognised that three of the six children had difficulty providing any detail about the spatial orientation of the background scene. It may be the case that the students' familiarity of using Lego Blocks or other concrete manipulatives helped them to decipher spatial information more easily than those students who had difficulty interpreting the screen images. Another possibility is that these children had been exposed to computer or entertainment games that required the spatial interpretation of information on a more regular basis. It was evident that some of the children were able to perceive and interpret the screen images more easily than others. These findings are consistent with earlier studies that examined the way in which photographs are perceived. Serpell (1976) argued that particular depth cues needed to be addressed and the absence of others ignored when viewing both drawings and photographs. Serpell argued that it is only a western convention that teaches us to suspend the information that indicates flatness, and focus our attention on weaker pictorial depth cues (Presmeg, 1985). Such conventions take time to develop and need to be adjusted regularly when attempting to perceive images represented on computer screens. It could be argued that at least two of the children described in this study were unable to perceive the cues that provided information about the pictorial depth of the image—and that these children had not be challenged to consider such representations adequately in the past. In other words, they had not been specifically taught these conventions or had not been exposed to situations that allowed them to construct ideas about links between 2D and 3D representations.

The present study has revealed that some young children are unable to make connections between familiar real-life objects and the way in which these objects are represented on a computer screen. At this point in their physical and cognitive development they cannot appreciate that 3D objects need to be represented in a particular 2D form in order to look "3D in nature." Moreover, these students' understanding of diminishing line and depth perception have not been sufficiently developed to interpret depth cues.

Importantly, the children who were able to interpret images and relationships between objects within the screen environment often used specific function keys on the program (including arrow keys to project close-up or wide-angle perspectives) in order to describe or reinforce their reasoning. These facilities provided opportunities for the children to explain relatively sophisticated concepts associated with direction, position and perspective. On the other hand, these program function keys confused the children who were having difficulty interpreting the screen images. It seemed to be the case that these children were unable to appreciate that the objects had not changed but rather the perspective from which the objects were viewed. Consequently, the function keys either supported or hindered the participants depending on their capacity to interpret the simulated computer environment.

The participants who were able to make connections between 2D and 3D space also used analogies associated with familiar real-life experiences to describe highly visual and spatial arrangements. These analogies and concrete life-like representations helped the children make sense of these relatively unfamiliar screen environments. These findings further support the notion that young children need to develop and establish connections in the 3D world before effectively making connections across simulated screen environments.

Implications

From a teaching and learning perspective a number of implications can be drawn from the present study. First, screen-based representations of "real" objects are not easily interpreted by young children. Although the graphic images displayed through computers are becoming increasingly detailed and life-like, students with emerging concepts of depth perception cannot always make the connections between screen and 3D worlds. The explicit teaching of 2D and 3D concepts should be developed concurrently so that children are provided with opportunities to make connections between 2D and 3D space. For example, encouraging children to consider the restrictions involved in interpreting photographs and identifying cues that limit depth perception could be introduced from the early years of school—and may have direct links to limitations involved in interpreting images displayed on computer screens.

Second, construction-based computer programs provide opportunities for children to view and manipulate 2D and 3D objects with flexibility and precision. Moreover, these programs allow the user to change the perspective and orientation of objects with ease. Foundation spatial concepts can be enhanced in such environments. Finally, it seems advantageous to relate students' emerging development of perspective and elevation concepts to analogies in life-like contexts. Although teachers often use terms like "birds-eye view" to explain different viewpoints, relevant comparisons should be encouraged as new concepts are introduced.

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Measurement and its Relationship to Mathematics: Complexity within Young Children's Beliefs

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A qualitative investigation of young children's beliefs included the gaining of insights into beliefs about measurement and its relationship to mathematics, the focus of this paper. Through the use of a range of procedures in one-to-one interviews, conducted with eight children over a period of five months, beliefs were found to be complex and idiosyncratic. Children may see measurement differently from how adults see it, and may hold different meanings for measurement terminology. This has implications for teachers of mathematics in the primary school.

Based on a doctoral study of young children's beliefs about the nature of mathematics, the nature of learning, and helping factors for the learning of mathematics, this paper focuses on beliefs related to the nature of mathematics. The research question addressed is "What beliefs do children hold about the nature of mathematics?" Previously the author reported on one aspect of one child's beliefs about the nature of mathematics (McDonough, 1996). The present paper takes a different approach, focusing on a broader theme, beliefs about measurement and its relationship to mathematics, and drawing on findings gained from the indepth study of eight children's beliefs.

Why Investigate Children's Beliefs?

It is important that educators are aware of children's beliefs as they may influence children's interpretations of learning situations and may impact upon learning of mathematics. Beliefs may have more influence "than knowledge in determining how individuals organize and define tasks and problems and are stronger predictors of behaviour" (Pajares, 1992, p. 311). Perceptions of what mathematics is may influence approaches to the solving of problems in mathematics (Frank, 1988), may influence the nature of children's participation in meaningful mathematics learning (Franke & Carey, 1997), may be related to success in mathematics (Crawford, Gordon, Nicholas, & Prosser, 1993), may impact upon conceptions of specific topics in mathematics, and may affect attitudes, performance, confidence, perceived usefulness of mathematics, and choice of courses or careers (Kouba & McDonald, 1987). Similarly, the learner's everyday knowledge of what is mathematics can influence the learner's interpretation of what is taught (Lindenskov, 1993) and consequently influence what is learned.

According to Ernest (1996), little systematic research has been undertaken on images of mathematics, with limited investigation of the views of students. The research reported here has undertaken such investigation and provides insights not readily available in day-to-day workings of the classroom. As an aside, the procedures developed within the study are potentially of use to other researchers and to teachers wishing to seek insights into individual children's beliefs.

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Beliefs are generally stable (McLeod, 1992; Pajares, 1992), and therefore can have a consistent influence on learners, but there is the possibility also of change depending on the degree of non-centrality of beliefs (Rokeach, 1968). It is possible therefore that input from significant others, such as spoken views of teachers, impacts upon children's construction of beliefs. Although some studies, such as that reported in this paper, reveal differences in beliefs held by students taught the same program by the same teacher (e.g., McDonough, 2002; Rodd, 1993), it appears that curricula can impact upon the development of students' beliefs about mathematical activity (Franke & Carey, 1997). Other experiences or elements in students' lives may be of influence also. Having insights into the complexity and individuality of children's beliefs such as provided in the present study, or knowing the beliefs of individual learners in one's own mathematics classroom, can inform teachers for their communication with students and assist in making decisions for teaching. Although teachers may not be able to change the curriculum, they can cater for students in a more informed manner. In addition, it is of value to investigate beliefs about measurement as this is a key strand of the mathematics curriculum (e.g., Board of Studies, 2000) and, as such, potentially receives some emphasis in the primary mathematics classroom.

Theoretical Orientation of the Research

The study's focus on beliefs was based on a social constructivist view of learning, as expressed by Ernest (1991), where there is a "respect for each individual's rights, feelings and sense-making ... [and] children and other persons are seen as active and enquiring makers of meaning" (p. 198). Knowledge is considered to be individually constructed and socially determined. The research extended this view, making the assumption that beliefs, as well as understandings, are individually constructed (e.g., Yackel & Cobb, 1996), and that these, in turn, may affect learning.

The study was qualitative in purpose, that is, it had an "interest in human meaning in social life and its elucidation and exposition by the researcher" (Erickson, 1986, p. 119). Qualitative research, in contrast to the "'traditional' or 'scientific' paradigm, ... assumes there are multiple realities–that the world is not an objective thing out there but a function of personal interaction and perception" (Merriam, 1988, p. 17). Taking into account children's varying experiences and individual constructions, the research was underpinned by the expectation that beliefs differ from child to child.

Children's Beliefs about the Nature of Mathematics

Much research suggests that primary school children tend to have narrow conceptions of mathematics, focused mainly on number, including counting and the operations (e.g., Cotton, 1993; Frank, 1988; Garafolo, 1989; McDonald & Kouba, 1986; Spangler, 1992; Stodolsky et al., 1991). Students believe that mathematical activity at school includes memorising formulas and, in a rote fashion, carrying out computations and procedures (Garafolo, 1989; Schoenfeld, 1992; Spangler, 1992) that are "divorced from real life, from discovery and from problem solving" (Schoenfeld, 1987, p. 197). The emphasis on mathematics as a rule-governed activity appears present within students both at the primary and secondary levels (Cobb, 1985).

In contrast, different views of what it means to do mathematics were found in a study of the views of 36 first-grade children from six Cognitively Guided Instruction (CGI) classrooms (Franke & Carey, 1997). In these classrooms "students had opportunities to consistently engage in problem solving, discuss their solution strategies and build on their own informal strategies for solving problems" (Franke & Carey, 1997, p. 10). In discussing what it means to do mathematics, the children referred to "problem solving, use of manipulatives, talking about mathematics, and solving problems in a variety of ways" (p. 14). Beliefs differing from the traditional view of mathematics were found also among successful secondary mathematics students who saw mathematics as something in which one can be creative and discover things (Wood & Smith, 1993).

The reported focus on number within children's beliefs suggests limited recognition and/or suggestion by primary children of measurement as mathematical activity. Indeed, in one study of the beliefs of 1202 primary school children, teachers were surprised by their children's narrow perception of what is mathematics, including little reference to measurement (McDonald & Kouba, 1986).

The study reported in this paper gives insights into young children's beliefs that extend our knowledge of children's perspectives. This occurred through an indepth investigation.

Research Methods

The research took the form of individual case studies of eight children of eight to nine years of age from two schools in suburban Melbourne. The children were chosen by their teachers according to gender and achievement guidelines provided by the researcher. Table 1 gives summary details of the research participants, according to the pseudonyms given to the children.

Table 1
Summary of Details of Child Participants in the Study.

Child	Anna	Ben	Cara	David	Emily	Filip	Gina	Harry
School	S	S	S	S	I	I	I	I
Gender	female	male	female	male	female	male	female	male
Achievement (teacher rated)	high	high	low	low	high	high	low	low

As the main form of data collection, the children were each interviewed by the researcher for about 30 minutes on ten occasions over a five-month period using 30 semi-structured, creative interviewing procedures (Patton, 1990, p. 340) that were developed or adapted for the study. These included drawing, discussing scenarios presented through photographs, video snippets and other children's drawings, ordering of descriptors, writing, and responding to questionnaires presented verbally.

Of the 30 procedures used in the study, 14 were targetted specifically at investigating beliefs about the nature of mathematics. For example, in Task 5.2, adapted from Zevenbergen and Crowe (1992), the children were presented with 22 photographs representing school mathematics contexts, non-school mathematics contexts, and non-mathematical or ambiguous contexts (e.g., group posing in front of a mountain scene). Children were asked to describe what they believed was happening in each photograph and then asked whether they believed there was any maths in what the person(s) was doing.

Children were also given opportunities to proffer mathematical situations of their own choice. For example, in Task 4.2, children were asked to draw a picture of someone using or doing maths and to describe their picture. Discussion focused on what the person was doing, and in what way the activity was mathematical.

Analysis of each child's interview responses was undertaken through "criss-crossed" reflection (Stake, 1994) through which themes (van Manen, 1990) were drawn. Data on beliefs about the nature of mathematics came from a range of procedures increasing the credibility and authenticity of the findings, important elements of qualitative research (LeCompte & Goetz, 1982). For example, the number of procedures giving insights into beliefs about mathematics, that were drawn upon in the analysis and write-up in the thesis, ranged from 12 to 18 per child. Children's responses were not judged for a match to any predetermined categories and the researcher sought to take a stance of neutrality to the phenomena under study, again increasing the credibility of the study. Further themes, such as those related to beliefs about measurement as discussed below, were then identified across the data from the eight children.

Results and Discussion

In presentation and discussion of results the term *maths*, the common abbreviation of *mathematics* (Wilkes & Krebs, 1982), is used to mirror the terminology use in the interviews. Not all children held the same meaning for the terms *maths* and *mathematics* but the former was a term familiar to all eight children.

Themes that emerged from the children's responses regarding their perceptions of measurement as maths included the following:

- children vary in their beliefs as to whether aspects of formal measurement are maths;
- children vary in their beliefs as to whether informal measurement is maths;
- when speaking of measuring, children are not necessarily speaking of maths;
- children may hold idiosyncratic meanings for commonly used terms related to measurement.

Each of the above points is considered in the following discussion.

At the time of the study, the state mathematics curriculum document recommended that Grade 3 children be introduced to some formal units of measure preceded by informal measurement activities (Board of Studies, 1995). During the year of the data collection, measurement was included in the mathematics programs of the two classes from which the children came. However, Ms S, the teacher of Anna, Ben, Cara and David reported that the children had been taught less measurement than she would have liked. Use of both formal and informal units was reported to have been included in the measurement lessons. When Ms I, the teacher of Emily, Filip, Gina and Harry, was asked how much emphasis she had given to different aspects of the mathematics program, choosing from *a lot*, *some*, *not much*, or *not at all*, she reported that she had given *some* emphasis to measurement.

Data indicate that for the children referred to as Anna, Cara, David, Emily and Filip, measuring was linked mostly to the use of formal units. For example, when discussing a photograph of a child measuring another child with string, Cara referred also to measurement of length in the school situation, but indicated that she used a tape measure. Formal units of measure for mathematical activity were mentioned or implied at times when speaking of her father making cakes such as in her second interview when Cara stated, "He gets a jug that goes up to 150 and he puts the cream in". She spoke also of him using a ruler to measure a circle for a wedding cake. Formal units were referred to specifically when Cara described a photograph presented to her that she interpreted as a girl reading a packet and baking a cake. She stated: "she's seeing on the back here to see if it's 1 mil (mL) of water or 2 mils (mL), and there she's putting the water in". Cara also referred to mass formal measurement situations as mathematical.

David provided a further example of the identification of measurement, and most specifically formal measurement, as mathematical. David included measure and measurement within his maths word wheel response (a type of brainstorm in response to the question "What is maths?") and stated in another interview that "in maths you add up things and measure them". Measuring objects with a ruler or a measuring tape was identified as mathematical activity. David said that in one situation there was no maths "because you use rulers and other things in your own pencil case [for maths]". Situations posed that involved informal measurement through direct comparison were judged as not mathematical; for one the justification was that "you usually use a ruler" for maths. David's responses suggested that rulers and tape measures, as well as a range of scales, were tools that for David indicated mathematical activity. The tools involved the use of formal units of measure.

However, there were instances where the presence of such measurement tools and/or formal units did not guarantee that situations would be considered mathematical. For example, two mass situations, presented in photographic form and involving the use of formal units, were not considered by Emily as mathematical activity. A photograph of a woman using a graduated circular scale to weigh fruit at a supermarket was described as "The lady is weighing something and seeing how many kilos or grams it is". When asked whether there was any maths in what she was doing Emily seemed undecided but tended towards the negative: "um, um, mmm, mmm no", explaining that it was not maths because "she's just weighing something". Similarly, another photograph was described as a "man caught a big fish, holding it up and weighing how many kilos". Emily thought that this also was not maths. Emily's statements suggest that she may not have considered weighing to constitute mathematical activity. However, it is possible that she might have been looking for the presence of counting or an operation with numbers. She had shown an appreciation of the measurement of length using formal units as mathematical activity, but seemingly mainly because of the application of number through counting or addition. The numbers on the supermarket scale were visible in the photograph, their purpose was understood by Emily, but it appears Emily did not perceive their use on this occasion as mathematical.

These examples show that differences in children's beliefs about formal measurement as maths relate to aspects including measurement attributes, units of measure, and measurement terminology.

Links between informal measurement and maths were complex. It is noted that the informal measurement situations presented to the children included direct comparison and indirect comparison through use of a third object. Measurement using informal units was not included. In retrospect, inclusion of such situations may have given further insights. However, children did have opportunities to refer to such tasks in the open questions requesting examples of mathematical activity but none chose to do so. The measurement-by-comparison tasks do give interesting results.

Situations that were classified by the researcher as involving informal measurement were considered by two children, Gina and Ben, as measurement and as maths. These included an item presented verbally regarding a child laying a pencil next to a book to see which was longer, and a photograph of a child measuring the height of another using a piece of string. Gina and Ben also saw formal measurement as maths.

Anna held mixed views about the informal measurement activities posed to her: using string as a tool for comparing lengths was considered maths, but informal measurement through direct comparison was seen as guessing and not maths.

Cara referred to informal measurement-by-comparison situations as *maths* but not as *measurement*. It appears that *measurement*, as Cara saw it, was a subset of maths that included formal measures and thus numbers. Informal measurement-by-comparison situations presented by the researcher appeared to belong, for Cara, to a different part of *maths*, that did not include measurement, suggesting Cara believed measurement required numbers but maths did not.

Emily at times made an association between measurement and maths, but mainly when numbers, counting or adding were involved. Emily called informal measurement situations *measuring* but said they were not *maths*, the latter seemingly due to her need to identify number in a mathematical situation.

As discussed above, although Emily used terms such as measure, kilograms, grams, and weighing in describing situations, this did not ensure that she saw the activities as mathematical; she did not make an automatic association between measurement terms and maths. Harry also used the word measuring along with other measurement terms, including names of formal units such as "kilos", and measurement tools such as rulers, but did not indicate that he associated these with maths.

Responses such as those from Emily and Harry, where measurement terms were used but not associated with mathematics indicate that the understanding of what might be considered familiar language of mathematics is not common to all, although the same words, such as measuring or weighing, may be uttered. The results suggest that modelling and discussion of mathematical language is an important role for teachers of young children, and a role of which teachers can be conscious.

There are questions also about children's understandings of the relationship between informal and formal measurement. All of the children except Harry identified formal measurement situations as maths but six of the children either did not recognise informal measurement-by-comparison situations as maths or had mixed beliefs about the different situations posed by the researcher. As stated above, none proffered a situation with informal units as mathematical activity. This suggests that links recognised by adults should not be assumed for young children. A related point emerging from the study concerns the importance of informal measurement and the emphasis it should be given in the mathematics curriculum, a question of debate in the mathematics education community (e.g., Ainley, 1990). As this issue is not related directly to the research questions of the current study it will not be considered here but the responses from the children suggest investigation of the value of informal measurement in the mathematics classroom is worthwhile.

Conclusion

The teaching of measurement in the primary school involves the development of a range of concepts and skills. It may involve the use of problem solving processes if measurement problems are posed by the children or teacher. The present study indicates that although a teacher may include measurement in the mathematics program, children may not view this as mathematical activity. Children come to the learning situation with their own beliefs and, even for an aspect such as measurement, these may differ from child to child and from those of the teacher.

Two of the eight children in the study, Cara and Ben, experienced measurement in real life situations involving family members, and called on these experiences of food preparation and building when discussing maths. They did not see maths or measurement as only school-related activity. Cara also held a strong association between measurement and estimating and included the latter as mathematical activity. Indeed, these were the two activities she seemed to associate most with maths. The other children identified some situations posed by the researcher as measurement but proffered fewer instances than Cara or Ben.

These findings suggest the key role that families, particularly parents, can play in the development of children's beliefs. The research supports the message that it is desirable for parents to feel comfortable with their own use of mathematics and involve children in real, rather than contrived, mathematical activities in their daily lives. Schools might consider working with parents to examine appropriate mathematical activities to undertake with children in non-school environments in a relaxed and non-contrived manner.

The difference in beliefs regarding the relationship of informal measurement to maths, and of different measurement concepts to maths suggests value in teachers posing a range of problems to children and reflecting, with the children, on the solving of these as mathematical activity. The research findings suggest also that measurement attributes, units of measure, and measurement terminology might be considered when focusing children's reflections.

This research shows that young children's beliefs about measurement are idiosyncratic and that associations between measurement, its terminology, and mathematics, may be different from common understandings. Teachers should not assume the beliefs that children hold. As beliefs may impact upon children's reactions to, or interpretations of, what is stated, performed or produced in a mathematics learning situation and may affect many aspects of their learning, it is worthwhile to gain insights into children's beliefs. Results from the current study suggest that the use of more than one procedure on multiple occasions can show detail and complexity within young children's beliefs that might not otherwise be available and that can inform teaching and interactions with children.

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Common Errors in Mental Computation of Students in Grades 3 – 10

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Error patterns in computation have been well researched for written computation; but there is little research into the error patterns common among students computing mentally. The responses made by 3035 students in Grades 3 to 10 were analysed and the most common errors made at each grade level ascertained. Common patterns of errors are described. Possible reasons for these errors and implications for the classroom are discussed.

Mental computation is officially recognised as requiring at least equal attention in school mathematics with written computation. For example the *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991) states:

> People need to carry out straightforward calculations mentally, and students should regard mental arithmetic as a first resort in many situations where a calculation is needed. Strategies associated with mental computation should be developed explicitly throughout the schooling years and should not be restricted to the recall of basic facts…less emphasis should be given to standard paper-and-pencil algorithms and, to the extent that they continue to be taught, they should be taught at later stages in schooling. (p. 109)

A number of major attempts to develop coherent approaches to the teaching of mental computation are under way both in Australia and abroad. In the United Kingdom, the National Numeracy Policy (available on the website: http://www.standards.dfee.gov.uk/) states that "An ability to calculate mentally lies at the heart of numeracy". In the Netherlands a systematic attempt is being made to develop and link mental and informal written computation methods (see for example, Beishuizen, 1993). However much research is needed to secure these approaches on a sound footing. As one example, there is nothing in the mental computation literature that parallels Ashlock's *Error Patterns in Computation* (1994), which confines itself exclusively to errors in written computation. "This entire book is designed to help you learn as much as possible from the written work of children" (Ashlock, 1994, p. 13). Since children's focus of thought, and consequently their patterns of thinking, are often markedly different when they are engaged in mental computation from those they employ when calculating with pencil-and-paper, it is to be expected that the kinds of errors they make, and the reasons for these errors, may also sometimes differ.

Bana, Farrell, and McIntosh (1995) and Bana, Farrell, and McIntosh (1997) describe errors made in mental computation of whole numbers, fractions, decimals and percents by students in years 3, 5, 7, and 9, but there appear to have been no follow-up studies. Moreover the sample sizes were quite small (approximately 160 at each age level) and the number of questions very restricted (30 at each of grades 3 and 5 and 40 at grades 7 and 9).

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The Present Study

The current study used a considerably greater number of questions and a much larger student sample than that used in the research cited above. A total of 3035 students in grades 3 to 10 answered questions from a pool of 244 mental computation items, covering whole numbers, fractions, decimals and percents. Four parallel tests were developed at each of four levels: grades 3/4, 5/6, 7/8 and 9/10. The test for grades 3/4 contained 50 items; tests for all other grades contained 65 questions. There were overlapping questions between each pair of the four parallel tests at each grade, and between grade levels.

The items were developed and extended from tests used in a previous study (Callingham & McIntosh, 2001). In both studies, all tests were recorded on audiotape to ensure consistency of timing for each item. All items were administered orally via the cassette recorder, and were not seen by the students. Answers only were written on the answer sheet with no written working allowed. Class teachers administered the tests.

In the previous study students had either five seconds or fifteen seconds to answer each item. This was intended to distinguish Short items (five seconds) for which the students had instant recall from Long items (fifteen seconds) which they could calculate, given time. However it was clear that many students could calculate the answer in five seconds. In the present study therefore the time for Short items was reduced to three seconds, following the procedure used in New Zealand assessments (Flockton & Crooks, 1997). In addition some five-second items were included in order to enable comparisons to be made with the previous study.

The items were restricted to the range of calculations that it was considered desirable for most students up to year 10 to be able to calculate mentally. There was hence no attempt to include technically difficult calculations that might only be within the capability of a few gifted students. Rather the items were designed to be capable of calculation by a student who had conceptual understanding of the numbers and operation involved. For example the 'hardest' addition of whole numbers was 79 + 26; the hardest fraction addition was 1/2 + 1/3; the hardest decimal item was 0.2 ÷ 5 (Grades 9/10 only); whereas the hardest percent item was 33 1/3% of 600. It was expected that the majority of errors made would reveal weaknesses in understanding rather than inability to hold the complexity of the calculation in memory.

For each item, at each grade level, all incorrect answers were recorded, and note taken of the most common errors. These were then analysed for clusterings of error types; the most common of these error patterns for each number type (whole numbers, fractions, decimals, percents) are now described.

Error Patterns

Whole Number Errors

By far the most common error with addition and subtraction of whole numbers was an answer that differed by one from the correct answer; for example, for the 15-second item 27 − 9, 51 out of 98 incorrect answers given by Grade 3/4 students, and 21 out of 65 incorrect answers given by Grade 5/6 students, were either 17 or 19.

Table 1 gives the number of times an error of one was made for basic addition/subtraction fact items and for addition/subtraction of larger numbers. The table excludes all cases where no answer was given to a calculation.

For larger whole number addition/ subtraction items, the second most common error was an answer that was incorrect by 10. This error persisted through to Grade 9/10 students: for example, for the 15-second item 58 + 34, 21 out of 63 incorrect answers given by Grade 7/8 students, and 10 out of 28 incorrect answers given by Grade 9/10 students, were either 82 or 102.

Table 1

Errors of 1 for All Addition/Subtraction Items by Grade and Type of Calculation.

	Basic Fact Items				Larger Numbers			
	Grade 3/4	Grade 5/6	Grade 7/8	Grade 9/10	Grade 3/4	Grade 5/6	Grade 7/8	Grade 9/10
(a) Errors of 1	394	23	14	-	367	224	74	51
(b) All errors	1082	87	53	-	1506	1784	724	361
(a) as % of (b)	36.4	26.4	26.4	-	24.4	12.6	10.2	14.1

The most common error for multiplication and division basic facts was an answer that was wrong by one multiple: for example, for the item 21 ÷ 3, 21 out of 77 incorrect answers given by Grade 3/4 students, 21 out of 49 incorrect answers given by Grade 5/6 students, 6 out of 21 incorrect answers given by Grade 7/8 students, and 4 out of 15 incorrect answers given by Grade 9/10 students, were either 6 or 8.

Table 2 shows the number of times for each grade a wrong answer to a basic fact multiplication or division item was incorrect by one multiple, and expresses this as a percent of all errors for those items.

Table 2

Errors Incorrect by One Multiple for All Basic Fact Multiplication/Division Items by Grade.

	Basic Fact Items			
	Grade 3/4	Grade 5/6	Grade 7/8	Grade 9/10
(a) Errors of one multiple	279	319	229	80
(b) All errors	1294	1454	780	263
(a) as % of (b)	21.6	21.9	29.4	30.4

For multiplication and division of larger numbers, errors frequently occurred when one or both numbers was a multiple of 10. For example, of 143 students across grades 5 – 10 making errors for the item 9 x 200, 40 students gave the value 180.

Fraction Errors

Table 3 shows a range of the fraction items from the tests. Columns three to six show the percent of students at each grade answering the item correctly. The second column indicates the number of seconds given for each item. The most common incorrect answer across the grades is also given. Of 323 errors across grades 5-10 for the Long item 1 – 1/3, 101 gave either ¼ or 3/4. Of 121 errors across grades 7-10 for the item 3 ÷ 1/2, 81 gave 1 1/2.

The item 1 – 1/3 was given both as a Short (3-second) and as a Long (15-second) item. Table 3 shows that the change in the amount of time given made little difference to the percent of students answering the item correctly. The most common incorrect answers for this item for both timings were ¼ and ¾. However for the Short, but not the Long, item, the incorrect answer ½ was also common.

Table 3

Percent of Students Answering Correctly Selected Fraction Items by Grade, and Most Common Incorrect Answers

Item	Time (seconds)	Grade 3/4	Grade 5/6	Grade 7/8	Grade 9/10	Most Common Incorrect Answers
1/2 + 1/4	3	26	53	67	67	2/6 or 1/3
1 – 1/3	3	-	33	44	56	1/4, 3/4 or 1/2
1 – 1/3	15	-	30	51	52	1/4 or 3/4
1/2 + 1/3	15	-	-	8	10	3/4
1/2 of 1/2	15	-	35	55	78	1
1/2 ÷1/4	15	-	-	42	47	1/4
3 ÷ 1/2	15	-	-	46	474	1 1/2

Decimal Errors

Table 4 shows a range of the decimal calculations asked, together with the percent of students at each grade answering correctly. There were no decimal items in the tests for Grades 3/4. The second column indicates the number of seconds given for each item. The most common incorrect answer across the grades is also given.

Table 4

Percent of Students Answering Correctly Selected Decimal Items by Grade, and Most Common Incorrect Answers.

Item	Time (seconds)	Grade 5/6	Grade 7/8	Grade 9/10	Most Common Incorrect Answers
0.6 x 10	3	23	41	61	0.6
0.5 + 0.75	15	11	30	54	0.8
0.19 + 0.1	15	-	34	40	0.2
4.5 - 3	15	47	58	75	4.2
3 x 0.6	15	-	32	39	0.18
0.3 + 0.7	15	42	52	64	0.1

For the Short item 0.6 x 10, the incorrect answer 0.6 was given by a total of 270 students, and accounted for over 50% of all errors made at each of Grades 5/6, 7/8, and 9/10. For the Long item 0.3 + 0.7, the answer 0.1 constituted 221 of 296 incorrect answers.

Percent Errors

Table 5
Percent of Students Answering Correctly Selected Percent Items by Grade, and Most Common Incorrect Answers.

Item	Time (seconds)	Grade 5/6	Grade 7/8	Grade 9/10	Most Common Incorrect Answers
100% of 36	3	54	74	84	36% and 0
10% of 45	15	12	25	44	5
75% of 200	15	19	43	57	175
90% of 40	15	-	24	30	30
33 1/3% of 600	15	-	8	19	several
30% of 80	15	-	8	21	25, 20, 30, 50

Table 5 shows a range of the percent calculations asked, together with the percent of students at each grade answering correctly. There were no percent items in the tests for Grades 3/4. The second column indicates the number of seconds given for each item. The most common incorrect answer across the grades is also given.

Of 308 errors across grades 5 - 10 for the item 75% of 200, 59 gave 175. For most percent items, no one wrong answer was very common: instead a very wide range of incorrect answers was given. For example, for the item 30% of 80, the following, given in order of frequency, constituted 121 of the 177 incorrect answers: 25, 20, 30, 50, 15, 37.5, 27, 22, 28, 23, 3, 26, 60.

Discussion

A number of researchers have classified types of computational errors made by students, for example Brueckner (1930), Roberts (1968), Backman (1978), Engelhardt (1977), Brown and VanLehn (1982), Resnick (1984), and Ashlock (1994). However, except in discussing basic fact errors, the focus has been almost exclusively on categorising errors associated with written algorithms, errors that are often quite irrelevant to the study of mental computation. For example, 16 of 17 error patterns for whole number calculations described by Ashlock involve misunderstandings of the formats and procedures associated with the formal written algorithm for the operation.

Consideration of the errors associated with the present study suggests that the fundamental distinction that needs to be made for errors in mental computation is that between conceptual and procedural errors. A conceptual error is one made because the student does not understand sufficiently the nature of the numbers or the operation involved. A procedural error is one in which the student, although having an overall strategic understanding of what to do, makes either a careless error or other error in carrying out the strategy. For example, 0.1 x 0.1 = 0.1 and 3 ÷ ½ = 1 1/2 are likely to be conceptual errors whereas 58 + 34 = 82 and 3 x 5 = 18 are likely to be examples of procedural errors. (It is necessary to qualify the certainty of the classification, since one is placing an interpretation only on the child's thinking, however convinced one is of the validity of one's interpretation.). While procedural errors are associated with both written and mental computation, the procedures themselves, and therefore the types of errors, are often quite different. As an example, for the item 74 - 30, a quite common error at Grades 3/4, 5/6, and 7/8 in the current study was the answer 36. It is likely that students making this error had a correct overall procedure or strategy of taking the 4 off the 74, subtracting 30 from 70, and then replacing the 4; but a lack of control over the procedure led them to subtract rather than add the 4. It is interesting that this error did not occur with Grade 9/10 students, suggesting that this is a transitional error.

As observed in this study, the errors made by students with whole number calculations tended to be procedural, whereas those involving fractions, decimals, and percents were predominantly conceptual. For example, items such as 1 – 1/3, 0.3 + 0.7, and 30% of 80, which are typical of three types of items set in these categories, were very frequently answered incorrectly, and yet each depends on very simple arithmetical ability coupled with conceptual understanding of the type of number involved.

This study supports the finding of Kamii and Dominick (1997), that when children are performing calculations on whole numbers using their own, usually mental, strategies, their incorrect answers are usually reasonable (i.e., near the correct answer) in contrast to the answers given by students using written algorithms, which are often quite unreasonable.

Where an addition or subtraction was incorrect by one (and it is worth noting in passing that there were also frequent cases with larger numbers where the answer was wrong by two), it is difficult to avoid the conclusion that in many cases the children's strategy was to count up or down by ones; this is reinforced by the fact that this error occurred more often in basic fact calculations when the addend was larger. A recent study by the author has suggested that this error, after decreasing gradually in extent through Grades K to 3, appears to remain static at about 20% of errors in basic fact additions and subtractions in Grades 4 – 6.

A similar reason can be hypothesised for the number of multiplication/division errors that are wrong by one multiple. In both cases there appears to be an error of counting, whether by ones or by multiples of 2 to 10. However it is not clear how the failure to keep a correct tally of the count occurs. The author has been told anecdotally that in at least one Australian state skip counting traditionally starts with zero (e.g., 0, 3, 6, 9 …) and this can lead to an incorrect count if zero is then counted by the child as the first multiple. It would appear reasonable to suggest that the calculations that are incorrect by 10 are also often the result of a failure to keep an accurate record of counting in tens.

The most frequent errors associated with calculations involving fractions, decimals and percents appear to have a mainly conceptual basis.

The errors made in fraction computations scarcely overlap with those described by Ashlock (1994), who describes errors found with written computation, and ones in which the students could see the numbers written down, leading to more bizarre and intricate 'malgorithms' (Ruthven & Chaplin, 1998). The only exception is Ashlock's Error Pattern A-F-1 (p. 114), which is adding both numerators and denominators (e.g., 1/2 + ¼ = 2/6), and which was found frequently in the present study.

Not surprisingly, errors in mental computation of fractions appear to be much less intricate and, where their reasoning can be surmised, more conceptual. Three of the common errors given in Table 3 can be attributed to confusion of operations. However 1 – 1/3 = 1/4 or 3/4, and 1/2 + 1/3 = 3/4 appear linked to a tendency noticed by the author for children to intermingle and confuse thirds with quarters (1 – 1/3 = 1 – 1/4, or 1 – 3/4; 1/2 +

1/3 = 1/2 + 1/2): the result perhaps of hoping that thirds will be linked with halves and quarters with which children are more familiar.

Decimal computation errors appear to be predominantly associated with the common misunderstanding noted by Hart (1981) and Stacey and Steinle (1998), namely "thinking that the figures after the [decimal] point represented a 'different' number which also had tens, units etc" (Hart, 1981, pp. 51-52). Table 4 includes several examples of this error, including 0.5 + 0.75 = 0.8, and 3 x 0.6 = 0.18.

Errors with percents are less commonly analysed in the literature. Hart (1981) devotes only one page (p. 96) to percents, while Ashlock (1994) describes only two error patterns involving calculations of percents. Two observations may be made about the results given in Table 5: first, that the common errors are, for the most part, not significantly different from the correct answers, suggesting at least some basic number sense is present; second, that many students would appear not to move easily between percents and their fraction equivalents (75% = ¾, 30% = 3/10) as one way of simplifying calculations. Even at Grades 9/10, only 44% correctly calculated 20% of 15.

Implications for Teaching of Mental Computation

Major implications for teaching mental computation of whole numbers appear to concern development of efficient strategies, whereas for fractions, decimals, and percents the issue appears to be that of developing conceptual understanding.

For whole numbers, two issues should be addressed. First, when children are using counting strategies for computations, their teachers need to observe how they count and keep track of their counting, and need constantly to ask children how they arrived at their answers. Errors may be caused by inefficient use of their fingers or other procedural error, for example when adding 6 and 3, counting 6, 7, 8. If these incorrect procedures are not discovered and discussed early they can come ingrained and persist throughout primary school. Second, children need to be weaned off the increasingly inefficient strategy of counting on and back by ones, to more sophisticated, neater and simpler strategies: using doubles and near doubles, bridging ten, adding tens, using compatible numbers, using related known facts.

For fractions, decimals, and percents, the main remedy continues to lie in at least delaying the algorithmic teaching of procedures until children have a conceptual understanding of the objects and their operations. It appears that many children simply do not understand fraction, decimal and percent notation. For fractions, the work of Streefland (1991) is helpful. The website developed by Stacey and others at the University of Melbourne (http://online.edfac.unimelb.edu.au/485129/DecProj/index.htm) is an excellent resource for decimals. For percents, the approach developed by Dole (1999) provides a well-researched basis. One of the most urgent needs in mathematics education in Australia is to find effective ways to ensure that well developed approaches to number find their way particularly into the majority of middle school classrooms.

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Horse Power or Empowerment? Mathematics Curriculum for Maori – Trojan Horse Revisited

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> European mathematics is offered to indigenous people around the world, 'gift-like', as a passport to success and future development. Most Maori believe in this idea; but some ask at what cost, both culturally and intellectually. The promotion of mathematics may be ethnocentric, but it is not, we believe, offered with malice. The bearers are probably unaware of the inherent dangers as the receivers (Barton and Fairhall, 1995, p. 1).

This paper seeks to revisit the notion that on the one hand a mathematics curriculum for Maori in the Maori language can be a cause for celebration, progress in the journey towards empowerment. While on the other hand it has the potential, like the Trojan horse, to 'destroy the conceptual basis of Maori culture' and language (Barton and Fairhall, 1995, p. 1).

The publication of a national mathematics curriculum statement in Maori (Te Tahuhu o te Matauranga, 1996) heralded an historical event in the public education of Maori. This statement was followed by the development of curriculum statements in the other six essential learning areas and acknowledged the official status of Maori language in Aotearoa and also recognition of a 'parallel' system of education for Maori.

Following on from the effects of over 150 years of colonisation, Maori whanau (extended families) who wanted more for their children than the mainstream state system were providing began their own education system in the 1980s and 1990s. Kohanga Reo (early childhood learning nests), Kura Kaupapa (Maori total immersion schools), Wharekura (secondary schools), and Whare Wananga (tertiary institutions) were established by Maori outside the state system. Subsequently, after various lengthy negotiations on both sides, and Acts of parliament (eg the 1989 Education Act) these Maori total immersion educational institutions came under the umbrella of the Ministry of Education.

In this paper questions will be raised about the intentions and aspirations of both the state and Maori communities in supporting the national mathematics curriculum statement in Maori. Were these national curricula in Maori about supporting Maori aspirations for tino rangatiratanga (self determination) or was the intention a continued assimilation agenda? Also addressed will be the likely outcomes for Maori of having a mathematics curriculum in Maori.

These issues will be discussed on two levels – the political/economic level, and the cultural level. It will be argued that the state's intentions can be interpreted at a political/economic level and that the outcomes for Maori are likely to be a deepening of the assimilation agenda of the past one hundred and sixty years. Maori aspirations can be interpreted from a cultural level and while it may be argued that the outcomes for Maori will be renaissance and maintenance of their language, a dialectical tension exists that sees a potential Trojan horse situation that urges serious reconsideration of the following questions:

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> Is academic mathematics as taught in classrooms around the country an optimal path for contemporary Maori development and future aspirations? Or has the European techno-rational ethos been embraced, Trojan-like, as an inevitable part of modern life, and thereby destroyed the conceptual basis of Maori culture? (Barton and Fairhall, 1995, p. 3).

Development of Mathematics in Formal Settings

Various writings to date have discussed mathematics for Maori from pre-European times (Barton and Fairhall, 1995; Riini and Riini, 1993), the development of formal mathematics from early pre-European contact through to more contemporary bilingual mathematics teaching (Barton, Fairhall, & Trinick, 1995; Nathan, Trinick, Tobin, & Barton, 1993; Ohia, 1990, 1993; Te Puni Kokiri, 1993). Research on mathematics in total immersion settings however is still in its infancy (although see the work of Aspin, 1996; Trinick, 1999).

According to the research in the 1980s, findings on the outcomes of bilingual mathematics teaching revealed a major effect in an improvement in self esteem and attitudes of Maori students (Ohia, Moloney, Knight, 1990). This research also suggested that there was potential for the academic excellence by the Maori learners within the unit, "but this potential has yet to be realised" (*ibid*, p. 156). Maori parents' aspirations during this period appeared to be retaining students in secondary school education and the learning and maintenance of te reo Maori. If this was the outcome, with mathematics learning secondary then so be it (Wagemaker, 1988). In our experience, and according to Durie (2001), Maori parents who send their children to be educated in te reo Maori are no longer satisfied with merely achieving Maori language fluency, what they expect from education are the "best possible outcomes" and "zero tolerance of educational failure" (Durie, 2001, p. 6).

The question is then asked will mathematics teaching in te reo Maori both increase Maori language fluency and enhance the mathematical achievement of Maori learners?

Political/Economic Context

Education cannot be separated from culture, history, politics or the economy as they are bound together (Adams, Clark, Codd, O'Neill, Openshaw & Waitere-Ang, 2000). Education therefore becomes a site of struggle and compromise (Apple, 1996, Smith, 1997). Various groups compete for the power to define their knowledge as dominant or legitimate in the context of the school, with the result that minority groups' knowledge is usually devalued and silenced in the curriculum. The state plays a role in this process through the imposition and legitimation of the national curricula, national testing, and through its agents such as ERO, an inspectorial body of the state.

The neo-liberal-based reforms of education in the late 1980s and 1990s in Aotearoa New Zealand witnessed the withdrawal of the state from people's lives. This withdrawal was to enable freedom in the market place so individual consumer choice could be maximised. Devolution of responsibility for education to the community reinforced these notions of freedom and choice. While freedom and more choice could be seen as enabling Maori to gain more opportunities in their aspirations for self determination or empowerment, in reality argues Apple (1998) this could be giving 'false hope'.

The right to choose one's own school or education provider assumes that everyone is equal to make decisions in the education market place. Choice however is linked with ability to pay, as well as being informed about what is available. Maori, according to the

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statistics (Te Puni Kokiri, 2000), fall well below their non-Maori counterparts in areas such as education, health, housing, and in the job market. Therefore they are unequal in the market place thus limiting an ability to pay for the best choice.

Additional to this notion of choice is availability. According to Apple (1998:23) the "labour market will increasingly be dominated by low-paying, repetitive work in the retail, trade, and service sector". Among the top 10 occupations that account for the most job growth in the next decade will include positions that do not require high levels of education (eg truck-drivers). Given that mathematics is a requisite for many occupations (Ohia, 1993) the question could be asked why bother teaching high level mathematics to those who are unlikely to end up in high level occupations?

Despite this assertion about the labour market, and possibly as a result of on going submissions to the Minister of Education by Maori educators, a decision was made that a Maori mathematics curriculum document would be written following the mainstream mathematics document. Ohia (1993, p. 110) suggests this move "counters previous assimilationist policies".

However as Maori had previously exited the public education system in the 1980s, other interpretations are possible. It could well be argued that funding Maori to have a parallel mathematics curriculum could be seen as the more powerful dominant group legitimating its authority and buying the loyalty of Maori by giving them what 'they' want – curriculum documents written in Maori. The New Zealand Curriculum applies to "all New Zealand Schools, including kura kaupapa Maori…" (MOE, 1993, p. 3) and it "provides clear learning outcomes against which students' progress can be measured" (MOE, 1993, p. 24). Therefore it could also be interpreted that the curriculum documents in Maori are an important tool to "enable a form of accountability" "while at the same time providing mechanisms that aid managerial surveillance" (Carpenter, 2001, pp. 127, 128). In other words, to control teachers and make them accountable for public money.

> There is always tension. The bottom line is, public education is not just about how children learn. It's about what society wants out of its expenditure if you like, its investment (MOE Project Manager in McMurchy-Pilkington, in progress).

Apple (1998), in discussing the complexity of the reforms, claims that there has been an alliance with neo-conservatives who mourn the 'decline' of the traditional curriculum and stable communities with common values and people who 'know their place'. Neo-conservative policies include a 'return' to higher standards, national curricula, standardised testing and a revival of western traditions and patriotism. "Behind it as well – and this is essential – is a fear of the 'other'" (Apple, 1998, p. 26) and a sense of cultural pollution. Apple (1998, p. 26) claims that the conservative English-only movement pushes for a re-orientation of curricula and textbooks that characterises "a particular construction of the western tradition" of like-minded people who share common values and norms.

Under the contract to construct a mathematics curriculum document for Maori, one of the non-negotiable conditions was that the learning outcomes and the structure of the document were to be the same as the mainstream mathematics curriculum (McMurchy-Pilkington, work in progress; MOE, 1992). Therefore if the mainstream mathematics curriculum is based on a western tradition underpinned by assumptions of mathematics as the 'truth' with share norms and meanings, the same could be said of the Maori document. Both documents it could be argued view mathematics as apolitical and acultural. There was no allowance that there could be philosophical or structural differences in the Maori document. In fact the completed draft had aspects of it rewritten by Te Taura Whiri (Maori Language Commission) because the learning outcomes were not directly translated from the mainstream document. The writers had attempted to interpret and make more explicit the underlying mathematical concepts.

> Now when it got to the Ministry, the Ministry was unhappy that we had not directly translated the learning outcomes. We had translated the concept of each learning outcome. The Ministry then contracted the Taura Whiri to rewrite all the learning objectives, in other words translate the English ones. (Lead writer of Maori mathematics curriculum, in McMurchy-Pilkington, work in progress).

This could be seen as another example of the state's intention to control 'the other', while seeming to be giving over some power and autonomy to Maori. It is also evidence of the promotion of mathematics as ethnocentric, and an attempt to "destroy[ed] the conceptual basis of Maori culture" (Barton and Fairhall, 1995, p. 3). In the development of the mathematics curriculum Maori were tightly controlled by the state, even to the extent that deviations from the mainstream 'western' document were not tolerated. Certainly there was limited opportunity to even consider "the conceptual basis of Maori culture" let alone deviate from the language of the learning outcomes.

Cultural Context

Bishop (1996, p. 33) argues that western mathematics is "one of the most powerful weapons in the imposition of western culture". It can be seen from the above section that there was little opportunity for Maori to even consider a non-western cultural basis for mathematics. It appears that mathematics is still being used as a weapon in the imposition of western culture. This raises the question then, were there any gains for Maori in having a mathematics curriculum written in Maori? What were the costs? Were there any dangers?

There is no doubt that culturally and professionally there have been some gains for Maori. We have seen the development of Maori vocabulary and resources to support mathematics teaching. With the curriculum documents written in Maori language we have seen a shift in orientation to language and learning in bilingual or immersion educational settings away from a deficit model to seeing the language Maori children bring to class with them as a resource (Baker, 1993). Language is now viewed as a resource rather than a problem, "to be drawn on to facilitate meaning-making and access to new knowledge and/or a new language (Adler, 1998, pp. 1-9).

Many Maori teachers have been engaged in professional development surrounding the document. More recent initiatives have included Maori teacher professional development for national monitoring (Flockton & Crooks, 2000), exemplar development, and the early numeracy project for Maori medium schools and Kura Kaupapa Mäori.

There is now a reasonable pool of Maori people involved in Maori medium education discussing, critiquing and reflecting on Pangarau (mathematics). There is discussion about how we might best represent mathematical knowledge in Maori contexts and teachers are beginning to discuss what is traditional Maori mathematics and what does it look like? There is a focus on how Maori traditionally explained their world, and the differences and similarities from a Maori and a Western perspective, for example what mathematical elements are evident like locating, counting, explaining, measuring and designing (Bishop, 1996). These discussions are making mathematics educators look at mathematics not as an absolute body of truth but rather as a tool to explain how we view the world and to make our lives easier.

Maori educators are also starting to look at and understand how Maori children think mathematically, and where the language impinges on mathematics and vice versa. Having a curriculum document in Maori has legitimised and supported this process and there has been some financial resourcing for this purpose. The initiatives that support the implementation of the Pängarau curriculum statement are providing a common forum to talk about and critique pangarau (mathematics).

However there have been some costs and there are some dangers from a cultural perspective. Both of these centre generally centre on linguistics and the epistemology of mätauranga Maori.

Many Maori teachers in Maori medium contexts are second language learners of Maori. More often than not they have learnt school mathematics in English. It is important to note however there a few graduates coming through from Kura Kaupapa Mäori into preservice teacher education. It will be interesting to identify the mathematics discourse used by this group. For many teachers when communicating mathematics tend to think in English and translate the mathematical concepts using English language forms. Mathematics is frequently communicated in English in the active, i.e. 3 times 2. This form is followed in the Maori translation

toru whakarea rua (3 x 2) .

Whakarea however is an action and an instruction. Traditionally this language form was spoken in the passive i.e.

whakareatia te toru ki te rua

Modeling further complicates this language form when teachers adopt the English language mathematical model when using Maori. In English three times two is often demonstrated by saying;

show me 3 groups of 2 things

However in Maori *whakareatia te toru ki te rua* implies two groups of three things. Although the English and Maori word order is the same, the underlying mathematical model is the reverse.

Traditionally Maori number was seen as a verb or an adjective. Since colonisation Maori number has became a noun rather than a verb, primarily because of trade and the adoption of the Western mathematical system. For example when talking about number teachers often give numbers physical representations, i.e. pass me the two.

In many ways what Maori teachers are taking on board now as pangarau (mathematics) is in fact western knowledge translated from English. This brings into question the authenticity of the knowledge from a Maori epistemological perspective. Because a resource is in Maori it may be assumed by teachers as being authentic but this may not necessarily be so. For example in a diagram of traditional Maori compass the units of the compass are all uniform. For a Maori compass, positions would depend on geography, bird migrations, and climatic elements in particular winds. Therefore compasses differed from region to region. Another example is the Maori terms used in the calendar and days of the week. The terms for the months are authentic but are now used following the current Western calendar. Traditionally Maori months of the year were based on the rising and falling of stars and other celestial phenomenon. Terms for days of the week as such did not exit but Maori used and followed the nights of the moon.

This raises a big issue in regards to the place of authentic traditional knowledge in a national curriculum document. In writing such a document for Maori questions can be raised about the intellectual property and ownership of the knowledge. Questions to ask would centre around public or private domains. Can Maori knowledge located in a curriculum be bought and sold as a commodity? Who has rights to Maori knowledge? Is it all public or does some belong to the private domain of hapu or iwi? Traditionally in Maori society the 'right to know' was related to birth rather than everyone having a right to know. A further dilemma ensues from this and that is if Maori traditional knowledge is not taught in schools today where else would it be taught with extensive urbanisation of Maori whanau?

Where Are We Currently?

Currently Maori have a mathematics curriculum document in te reo Maori (Maori language). Does it have a Maori knowledge base or does it come from a western tradition? It would be true to say, that like the science (Putaiao) curriculum, what we have is a curriculum based on a western tradition that has been translated into Maori (McKinley & Waiti, 1995). Some of the contexts and the exemplars are from a Maori perspective, but the concepts and ideas are underpinned by western thinking.

While we might be getting development of te reo Maori vocabulary, traditional Maori knowledge and tikanga is still not being tapped in the teaching of pangarau.

We would argue that in mathematics teaching for Maori the two information systems are not being used or extended equally. Therefore children learning pangarau may be restricted in the way they can think, both culturally and mathematically about the world. We might be 'doing' mathematics but we are not using two knowledge systems or ways of thinking. Thinking mathematically about the world for Maori is essentially still from a western perspective. This it could be argued is therefore a deepening of an assimilation agenda. Was that the intention of the state in laying down the requirements for the writing team? It certainly was not the agenda of the Maori community.

Politically there has been a significant change in MOE with regards to Maori medium pangarau development. The policymaking and the practice have shifted. We as Maori are 'allowed' to have more freedom in our development – no longer were we obliged to have exact translations in the last two curriculum documents (eg Nga Toi and Hauora). But we believe attitudes within MOE have not shifted a great deal. While there is not the same 1992 requirement that everything has to be the same, albeit in another language, there is still a Eurocentric attitude. The talk has shifted from 'translating', to a new discourse. Requirements are to 'culturally redevelop' the English, western based material and translate it 'to support the learning of Maori learners'. Maori are still being controlled and contained within a western framework.

It appears that although policy making for Maori has had the appearance of shifting from Maori friendly to Maori centred in the past decade (Johnston, 1999, p. 80). Maori views and aspirations are "made to conform to an already defined structure and framework", which "in effect neutralise Maori involvement, participation and interests" (*ibid*, p. 84). We were involved in the development of a mathematics curriculum for Maori but it was not on our terms.

Maori parents have also shifted. They are not just focusing on language proficiency for their children. They also want cognitive understanding so their children can stand tall in both worlds. Although it is fair to say some of them are still anti pakeha as they are still suffering backlash from the effects of colonisation. Maori parents want to know what are the achievement gains of their children over time. They want to know what areas their child is finding difficult, and how they can help them improve over time.

Linguistically we have made some big gains, both our children and our teachers, from the teaching of mathematics in Maori. However our knowledge of authentic Maori knowledge is lagging behind. If we think of a rope and its various strands, we could say some of the strands are stronger than others. To have an effective rope, each of the strands needs to be as strong as each other and to support each other.

We have to strengthen our understanding of mathematics both from a western and a

Maori perspective. Currently we are largely getting a translation of western ideas. We could say we are essentially not any closer to Maori epistemology than we were 10 years ago while we continue to teach from a western based document. Our linguistic strand may be strong but cognitively we still have a long way to go.

Summary

We maintain that the promotion by the state of mathematics for Maori is still ethnocentric, and still techno-rational. The mathematics curriculum for Maori is more about controlling Maori teachers and preparing Maori for a job market.

Our contention is that we are seeing changes both at a linguistic level and at a conceptual level. There needs to be teacher development and ongoing research and discussion on where changes are occurring and what is underlying these changes. We need to ask if these changes are a usual part of a developmental continuum or are they the outcomes of colonisation. If these changes are a normal part of learning and developing another language then we're not worried. However we need to be clearer about the issues and separate out the two – colonisation and growing living languages.

As receivers of this curriculum document we are beginning to really understand the tensions and dangers. We need to disseminate these ideas and continue the ongoing debate so our practice is informed. We need to support such forums as NAMSAT for ongoing networking, debate and sharing. We need to ensure our practice, our research and our discussions inform the curriculum stock take. We need to fight for a Maori strand in organisations such as NZAMT and MERGA. Our debates and sharing should involve classroom teachers as well as 'academics' and in turn these need to inform our teaching and underpin resource making, policy, curriculum.

We believe the givers of the curriculum (MOE and politicians) have made some shifts. There is an awareness of our differences and some recent developments are less ethnocentric (eg the NEMP process, and the Early Numeracy Project). At times there is a feeling of partnership, but mostly there is a feeling of Maori Friendliness rather than Maori Centeredness.

We as Maori educators need to engage in ongoing critical examination and debate about the curriculum. As professionals we "must question and constantly reflect on the ideological bases and the theoretical underpinnings of curriculum" (Carpenter, 2001, p. 131). If we don't want to be assimilated by Trojans then we need to be continually alert to 'gifts' that are offered to us.

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Use of Graphics Calculators in School Tests and Examinations

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This paper reports on the outcomes of one component of a study carried out during 2001 to assess the impact of graphics calculator use on Year 12 school-based assessment in a situation where access to the calculator is assumed in the external tertiary entrance examination. The focus of the paper is on test and examination items that form part of the school-based assessment programs contributing to a student's final grade in the Western Australian Year 12 Tertiary Entrance Examination (TEE) subject *Applicable Mathematics*. Assessment items developed by participating schools during 2001 for use with *Applicable Mathematics* were collected and coded. While it was found that each of the participating schools was incorporating use of graphics calculators into questions to approximately the same extent, there was a wide variety of usage apparent within some curriculum components. This suggests there is scope for wider incorporation of the technology.

Background

Students in Western Australia sit for the external Tertiary Entrance Examinations (TEE) at the end of Year 12, and three mathematics subjects, *Applicable Mathematics*, *Calculus* and *Discrete Mathematics*, are examined. *Applicable Mathematics*, the focus of this paper, lies between *Calculus* (the most demanding) and *Discrete Mathematics* (the least demanding) in terms of difficulty. There are five components to *Applicable Mathematics:* Systems of Linear Equations and Matrices (25 hours of tuition time), Graphs and the Solution of Equations (18 hours), Descriptive Statistics (20 hours), Sets, Counting and Probability (18 hours) and Random Variables and their Distributions (24 hours). Each year, approximately 5000 students sit the TEE in *Applicable Mathematics*.

Marks obtained in the TEE contribute 50% towards the student's assessment in the subject, and the other 50% of the assessment is school-based. In *Applicable Mathematics*, the school-based assessment is to consist of 25 to 50% from extended pieces of work and 50 to 75% from other forms of assessment. Tests and examinations figure prominently in the latter but it may include checklists, homework assignments and oral presentations. In this paper, we focus on tests and examinations.

Since 1998, it has been assumed that students sitting for *Applicable Mathematics* and the other TEE mathematics subjects have access to graphics calculators. With the exception of the Hewlett-Packard HP-38G and HP-39G that have limited symbolic capabilities, calculators approved for use are non-symbolic. Given their inclusion in the TEE, it follows that students would make regular use of graphics calculators during mathematics lessons and for assessment tasks throughout Year 11 and 12, in order that

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they become familiar with their capabilities. The syllabus for *Applicable Mathematics* makes clear the expectation that students should "select and use appropriate technologies" and "appreciate the benefits of using technology in mathematics" (Curriculum Council, 2001, p. 37).

Literature Review

Internationally, the use of graphics calculators is permitted in public university entrance examinations in the USA and United Kingdom, amongst other countries. In Australia, Victoria and Western Australia were the first two states to allow use of the technology in external statewide examinations at the end of secondary schooling and this occurred in 1997 and 1998, respectively. South Australia has recently followed suit and the technology is allowed for the alternative entry paths in mathematics in Queensland. However, few empirical studies examine school-based assessment in the presence of graphics calculators at the secondary level. The study by Senk, Beckham, and Thompson (1997) into assessment in eight high schools in the USA is an exception. They identified that inclusion of the calculators and other computer technologies, even when readily available, was limited to approximately 3% of assessment items.

Amongst the literature, several articles have focused on the incorporation of graphics calculators in *Calculus* examinations, while relatively few deal with other areas of mathematics.

Forster and Mueller carried out a longitudinal study on the impact of the inclusion of graphics calculators in the *Calculus* TEE in Western Australia. This study encompassed tertiary entrance examinations from three years before and three years after the introduction of the graphics calculators. Aspects of the study included misuse and misconceptions and problems in students' use of the technology (Mueller and Forster, 1999, Forster and Mueller, 2000a). Their analysis of characteristics of questions revealed (Forster and Mueller, 2000a, 2000b) an increase of questions set in real-life contexts from 1996-1997 (pre graphics calculators) to 1998-1999 (post graphics calculators), a greater role for diagrams in the solution of problems and a change in the level of difficulty of examination questions in some curriculum components. The findings of Forster and Mueller are mirrored in the paper by Anderson, Bloom, Mueller, and Pedler (1999) who investigated the impact of graphics calculators on the assessment of calculus and modelling at the undergraduate level. They also found that the new skills needed to do mathematics in this environment necessitated a change in the tasks set to assess student achievement and envisaged questions of analysis and interpretation assuming greater importance.

Little work has been done in relation to the subjects covered in *Applicable Mathematics*. Bradley (1999) looked at the efficiency of graphics calculator use in the 1998 *Applicable Mathematics* TEE in Western Australia and Mueller, Pedler, Anderson, and Bloom (1998) evaluated students' uptake of graphics calculators in an undergraduate linear algebra unit with content that overlaps *Applicable Mathematics*.

Apart from those of Anderson et al. (1999) and Mueller et al. (1998), each of the studies reported above used as its focus data from Tertiary Entrance Examinations in Western Australia. The present study differs in that the data consist of items developed in schools as part of the school-based component of the student's assessment for Applicable Mathematics. Its focus is on how teachers are responding to the availability of graphics calculators when developing test and examination questions.

A coding schema developed by Senk et al. (1997) in their investigation of assessment and grading in high schools in three states in the USA was adapted (Table 1) to facilitate analysis of these items. The most significant change to their schema was in the coding characteristic of 'technology'. We narrowed 'technology' to 'graphics calculator' and expanded their categories of active, neutral and inactive to essential, advantageous, neutral, checking, and inactive. The categories 'essential' and 'advantageous' refer to uses over and above those possible on a scientific calculator.

Table 1
Item Characteristics and Categories

Characteristic Category	Description
Skill	
Yes	Solution requires the application of a well-known procedure. The item does not require translation between representations. (Word descriptions that aren't merely procedural, tables, graphs, algebraic.)
No	No algorithm is generally taught for answering such questions, or the item requires translation across representations.
Level	
Low	Using the most efficient method, a typical student in that course would use less than four steps for a solution.
Other	A typical student in that course would use four or more steps for a solution, or the content is new to the course.
Reasoning Required	
Yes	The item requires justification, explanation or proof.
No	No justification, explanation or proof is required. (By itself, "Show your work" is not considered reasoning.)
Role of Diagram	
Interpret	A graph or diagram is given and must be interpreted to answer the question.
Make	From some non-graphical representation (data, equation, verbal description) student must male a graph or diagram.
Assist	A diagram could assist the solution.
None	No graphical representation give is needed.
Graphics Calculator	
Essential	Use of the calculator is necessary to obtain a solution.
Advantageous	Use of the calculator greatly simplifies the work needed to get a solution.
Neutral	It is possible to use the calculator to obtain part or all of the solution; but the question could be reasonably be answered without the tool.
Checking	Calculator can only be used for checking and not for obtaining the solution.
Inactive	Use of the calculator is not possible, or can be done using a scientific calculator.

(Adapted from Senk, et al., 1997)

The purpose of the current study was twofold: to ascertain the extent to which the availability of the graphics calculator was reflected in teachers' assessment practices; and to develop a means of categorising assessment items that teachers might find of use when developing tasks for their students.

Data Collection

The authors sought the advice of the Western Australian Curriculum Council to determine schools where teachers were proactive in the use of graphics calculators. Of the eight schools nominated and approached, all agreed to participate in the study. A mathematics teacher involved in the teaching of *Applicable Mathematics* in each school became the contact person and accepted the responsibility of passing on to the authors each test and examination item used for the school's *Applicable Mathematics* program during 2001. All of the test and examination items collected were classified first in accordance with the role played by the graphics calculator in answering the question. Next, items other than those coded inactive for calculator use were coded further using the modified categories described above.

Each item was coded independently by two of the authors. In the first instance it was determined whether the use of the calculator was essential, advantageous, neutral, for checking only, or inactive. Questions other than those where the calculator was coded as inactive were further coded under the characteristics of 'skill', 'level', 'reasoning required', and 'role of diagram'. Where our coding varied, we negotiated agreement. The results of the coding were collated in order to produce an assessment profile for each of the participating schools. Examples that illustrate use of the schema are included in the results section that follows.

Results

Table 2 shows percentages of marks allocated to test and exam items used by the eight schools in the study during 2001 for which graphics calculator use was coded either essential or advantageous. These are displayed both for the five curriculum components of *Applicable Mathematics* and overall. Complete test and examination results from three schools were unavailable at the time of analysis and, because data from School C were limited, it has been excluded when calculating means.

It can be seen from Table 2 that curriculum components 1 and 5 have the highest average percentages of marks allocated to essential/advantageous graphics calculator use (25.3% and 29.7% respectively). The results indicate these are the curriculum areas in which teachers find it most appropriate and/or convenient to devise test and examination that exploit the capabilities of the instrument. Matrix functions on the graphics calculator lend themselves readily to component 1, with its focus on systems of equations and matrices. The capacity of the calculator to evaluate integrals and determine probabilities associated with the various probability distributions appear to make it a valuable tool in test and examination items for component 5. Ignoring School C under curriculum component 1, the percentage of total marks allocated to items requiring essential/advantageous use ranged from 41.6% for School F to 15.2% for School G. In curriculum component 5, percentages ranged from 45.4% for School H to 12.8% for School A.

For curriculum component 2, graphics calculator items are either reasonably well represented in tests and exams (from Schools B, D E and G with around 25% of marks allocated to items which included essential/advantageous calculator use), or modestly represented (ranging from 8.1 % to 13.4% from the remaining schools). This suggests teachers in the latter schools may not be as aware of how use of the graphics calculator can be incorporated in this component or are choosing not to include the technology.

Ignoring School A under curriculum component 3, it can be seen that School G stands out here with a 22.8% of marks allocated to items including essential/advantageous graphics calculator use. The percentage drops to 14.3% and 12.4% for Schools F and H respectively, with a considerable drop in percentages for the remaining schools. Most of the measures discussed in this curriculum component are available on scientific calculators, including fitting exponential curves to data. Where a graphics calculator offers additional benefit for the student is in enabling a quick visual assessment of the quality of the fit of a particular curve to data and in the analysis of time series. It is therefore not surprising that the percentages of questions coded as advantageous or essential in regard to graphics calculator use is not as high as in components 1 or 5.

Table 2

Percentages of Marks for which Use of the Graphics Calculator was Coded Essential or Advantageous in school-based tests and examinations

School			Curriculum Component			
	(1) Systems of linear equations and matrices	(2) Graphs and the solution of equations	(3) Descriptive statistics	(4) Sets, counting and probability	(5) Random variables and their distributions	Across all curriculum components
A[1]	31.2	9.2	0	0	12.8	13.5
B	23.7	28.9	7.3	0	38.4	19.9
C[2]	0	8.1	3.7	0	0	5.0
D	37.7	26.4	6.3	0.9	39.4	19.3
E[1]	37.5	24.8	6.2	0	19.0	13.9
F	41.6	8.6	14.3	0	13.6	16.1
G	15.2	22.6	22.8	0	39.5	19.9
H	27.8	13.4	12.4	0	45.4	14.8
Mean[3]	25.3	19.1	9.9	0.1	29.7	16.8

[1] These two schools used a commercially prepared final examination and this was not included in the analysis.

[2] This school only provided limited data.

[3] School C was excluded when calculating means.

Regarding curriculum component 4 (sets, counting and probability), an examination of test and exam items used by schools indicates that the activities students are called upon to complete in these topics do not lend themselves to simplification through use of the graphics calculator. The usage of a graphics calculator in, and hence its impact on, this curriculum area is negligible and this is reflected in the table.

The right hand column of the table gives the school percentages of total marks from test and examination items across all curriculum components in which graphics calculator usage was essential or advantageous. Again ignoring School C, it can be seen that the percentages from all the other schools are remarkably consistent around the average of 17%, ranging from a low of 13.5% to a high of 19.9%.

We now illustrate the use of the schema (Table 1) by coding three examples taken from those provided by the participating schools and indicating the reasons for our decisions on coding. Example 1 is from curriculum component 1, Example 2 from component 2, and Example 3 from component 5.

Example 1

A company produces three products X, Y and Z. Each of these products undergo three processes on two different floors in the factory. The time required for each process, in hours, is given by matrices ***A*** *(floor 1) and* ***B*** *(floor 2).*

Floor 1 *Floor 2*

$$\boldsymbol{A} = \begin{array}{rc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{r} \text{Cutting} \\ \text{Sanding} \\ \text{Zinc Coating} \end{array} & \begin{pmatrix} 3 & 1.5 & 1.1 \\ 6 & 2.8 & 1.2 \\ 1.3 & 1.5 & 1.1 \end{pmatrix} \end{array} \qquad \boldsymbol{B} = \begin{array}{rc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{r} \text{Assembly} \\ \text{Polishing} \\ \text{Packing} \end{array} & \begin{pmatrix} 0.7 & 1.8 & 2.1 \\ 1.2 & 1.3 & 2.3 \\ 0.3 & 0.2 & 1.5 \end{pmatrix} \end{array}$$

The cost for each process per hour is given by matrices ***C*** *and* ***D***.

$$\boldsymbol{C} = \begin{array}{ccc} \text{Cutting} & \text{Sanding} & \text{Zinc Coating} \\ (\$10.30 & \$5.57 & \$12.80) \end{array} \qquad \boldsymbol{D} = \begin{array}{ccc} \text{Assembly} & \text{Polishing} & \text{Packing} \\ (\$12.50 & \$4.20 & \$1.53) \end{array}$$

Determine whether the following expressions would have meaning in the context of the problem. If so, find the answer and explain what the resulting matrix represents.

(i) $\boldsymbol{A}+\boldsymbol{B}$ *(ii)* $\boldsymbol{AB}$ *(iii)* $\boldsymbol{AC}$ *(iv)* $\boldsymbol{AD}$

(v) $\boldsymbol{AC}^T$ *(vi)* $\boldsymbol{B}^T\boldsymbol{D}^T$

Note: $\boldsymbol{M}^T$ *refers to the transpose of matrix* ***M*** *(columns and rows interchanged).*

We coded use of the graphics calculator as being 'advantageous' for this question because, although multiplication of 3 x 3 matrices can be done without it, it does take away the tedium of doing so and helps reduce the incidence of error. The question was coded as 'no' for 'skill' since it did not rely solely on the application of algorithms, 'other' for 'level' because of the number of steps involved to answer the question as a whole; 'yes' to 'reasoning required' because explanation was needed; and 'none' for 'role of diagram'. We chose to present this question because of the graphics calculator use it entailed and thought that it was a good question because the student was referred back to the context of the question in order to provide a solution. The question is an example of how access to the graphics calculator can allow greater scope for setting questions in a real-life context.

Example 2

(a) *Solve the following inequality, correct to 5 decimal places.*

$$5-\frac{3}{x}<6-x^2$$

(b) *What integral values of k will give three solutions to the equation below?*

$$5 - \frac{3}{x} = k - x^2$$

We coded use of the graphics calculator 'essential' for both parts of this question. For the first part it is necessary to find the points of intersection of the two graphs. This can be done either via the equation solver of the graphics calculator or via graphs of the functions. Consideration of the graphs of the functions helps to ascertain that there is only one point of intersection of the two functions and assists in determining the lower bound of the interval and so might be more beneficial than using the equation solver. For part (b) consideration of the graphs in part (a) will alert students to the fact that translation of the parabola will lead to an increased number of points of intersection and so greatly helps in finding the least integer value of k required in part (b). Both parts were coded 'yes' for skill, since a procedure is taught; 'low' for level, as each required less than four steps to solve when using the graphics calculator; 'no' for reasoning required, as no explanation was sought; and 'assist' for role of diagram.

Example 3

Suppose that x is a continuous random variable with a probability density function given by:

$$f(x) = k(1 - x^2) \qquad \text{for } -1 \leq x \leq 1$$

(a) *Find k.*

(b) *Evaluate* $P(-1 < x < 0.5)$.

(c) *Find t (to two decimal places), if* $P(x > t) = 0.35$.

Use of the graphics calculator was coded 'advantageous' for each part of this question since utilisation of the solve and integration facilities could be used to automate the work. Each part was coded 'yes' for skill since a procedure is taught, 'low' for level as no more than three steps were required, 'no' for reasoning required as no explanation was sought, and 'none' for role of diagram.

Discussion

To measure the impact of graphics calculators on test and examination questions, a coding schema was adapted from one developed for a previous study (Senk, et al., 1997). Application of this schema revealed that around 17% of the marks allocated in tests and examinations were to questions for which use of the graphics calculator was either essential or advantageous.

Closer examination of the coding revealed a wide discrepancy between participating schools in their incorporation of the use of the graphics calculator in four of the five curriculum components of *Applicable Mathematics*. In the fifth of these, Sets, Counting and Probability, the content does not readily align to functions that are unique to the graphics calculator, but uses functions also on a scientific calculator. The wide range of incorporation in the other four components, for example 12.8% to 45.4% in Random Variables and their Distributions, suggests some teachers may be better informed about the capabilities of the calculator as they relate to the *Applicable Mathematics* curriculum or are choosing to retain traditional approaches. Analysis of the sample items collected indicates there is considerable scope for the calculator to have a wider impact on assessment practices in *Applicable Mathematics*.

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Aspects of Written Performance in Mathematics Learning

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This paper examines two early high school students' beliefs about mathematics, writing in mathematics and their writing audience. Students who do not value writing as being part of mathematics or are unable to conceive of the needs of an examiner as their audience are unlikely to perform in mathematics assessment tasks to their ability level. Teachers may need to increase students' awareness of the value of writing in their learning of mathematics and how different audiences will have different requirements for their writing.

With the introduction of the National Certificate of Educational Achievement (NCEA) in New Zealand, valuing of language in mathematics has become more explicit. In order to achieve the higher levels of merit and excellence in mathematics, students need to write accompanying notes to their calculations. For example, in the assessment schedule for a Measurement Activity (Ministry of Education, 2000, p. 6), to gain excellence, students are expected to answer one question in the following way:

> 4a) Answers will vary but students must produce evidence of an appropriate model selection. E.g. Soil on trailer may be seen as a cuboid topped by a pyramid, cone or curve of some sort or any of these shapes alone. They must use this model to calculate the height and show how they did so. Their results must be consistent with their model.
>
> 4b) 2 valid statements. Valid statements could include:
> - Soil slipping off the sides if too high on trailer
> - Model may not fit the soil exactly - must say why.

Morgan (1998, p. 25) suggested that writing in mathematics has been limited to ensuring that students used the correct symbols. If this is the case, then it may be that the students who achieve the merit or excellent levels will be ones who come from backgrounds where explaining and justifying are valued (Morgan, 1996, p. 2). However, being able to explain and justify answers has significant advantages in mathematics learning as it causes students to reflect on what they have done. MacGregor and Stacey (1995, p. 26) stated that 'students' ability to articulate ideas, to others or to themselves, is an essential support for their mathematical reasoning and further learning'. Reflection supports the learning process because it 'leads to synthesis, validation, integration and appropriation of knowledge' (Southwell, 1993, p. 228). By assessing students' writing, the education system (in this case the Ministry of Education) has reinforced the usefulness of this articulation and reflection. It is disappointing to find that it is only students attempting the higher levels who are expected to use writing in this way.

The Study

With the introduction of the NCEA, there was a need to better understand some of the aspects of the writing process and how they combine to support or hinder student writing. This report is part of a wider study which investigated spoken and written examples of mathematical communication and considers the relationships between students' writing

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and their beliefs about audience, mathematics and the place of writing within mathematics classrooms.

As NCEA texts are written by students to demonstrate their understanding, they share many attributes with oral performances such as attention to audience and the situation of the performance (see for example Finnegan, 1992, pp. 91-110, on discussion of the characteristics of performance). Morgan (1998, pp. 43-49) in considering the literature on writing across the curriculum stated that many studies suggested that 'one of the roots of students' difficulties and lack of motivation in their development as writers' was the fact that the teacher as examiner was the audience on most occasions. Students' previous experiences of mathematics classrooms also influence their expectations of what should occur (Alrø and Skovsmose, 1996, p. 7). It, therefore, seemed worthwhile to consider students' beliefs about mathematics and the role of writing within it and how these combined with their ideas about audience.

One way to examine links between these perceptions and student texts is through Functional Grammar. Functional Grammar connects certain grammatical features to the functions that they perform for the writers (Halliday and Hasan, 1985, p. 25). For example, in determining how students' conceptions of mathematics are reflected in their writing, Functional Grammar can be used to analyse transitivity and the labelling of mathematical ideas including symbolic mathematics. Halliday refers to these as *experiential meanings* as they reflect what is going on or the *field of discourse*. In looking for how audience is acknowledged in students' writing, Functional Grammar would examine the use of mood, modality and personal pronouns. *Interpersonal meanings* illustrate the *tenor of the discourse*. Lastly how new information is introduced and the cohesion of a text maintained are examined to determine the writer's beliefs about how language is used to explain or justify. Halliday uses the term *textual meanings* to describe the role assigned to language within the *mode of discourse*. Morgan (1996) used Functional Grammar to uncover the choices in these different kinds of meaning that students made when producing texts and a similar approach has been taken in this research.

The students were in one of my third form class in 2001 at Logan Park High School. They would do NCEA in 2003 and there was an expectation from the school, the parents and the children themselves that many of them would achieve good results. The mathematics program often involved students taking part in group work which led to the completion of a project or an assignment. Sometimes, this would be written up by the group and at other times it would be written up by individuals. At the end of each topic, all students in third form completed a test which included some items requiring students to justify and/or explain their working.

At the beginning of the year and at the end of third term, students completed a questionnaire in which they were asked to give their views on mathematics and the role of writing within it. Written work for assessment was collected from all students during the year and ten students were interviewed in September. During this interview, students described how they solved a problem and gave their opinions about their audiences and the language they used for writing in mathematics.

From the questionnaires at the beginning of the year, 15 students (10 boys and 5 girls) out of a total of 22 described mathematics as solving problems by calculating numbers. 3 students stated that mathematics was used everyday and the remaining students gave their feelings towards it or described aspects of it such as equations, graphs, diagrams. By the middle of September, these views of maths were still supported, with a total of 16 students out of 20 (11 boys and 5 girls) describing maths as problem solving by calculating numbers, with 1 girl considering it as having everyday application. In the first lesson of the year, 9 students (7 boys and 2 girls) thought that writing in maths would help students learn. 3 boys thought that it would have no value and 8 students (5 girls and 3 boys) felt that there was value for some students but not all in this approach. 1 boy and 1 girl expressed no opinion. At this point, many felt that writing in maths involved doing calculations and printing neatly so it is perhaps not so surprising that after doing several projects, their views on writing in mathematics had changed. By September, 2 boys and 2 girls felt that writing was useful, while 9 students (7 boys and 1 girl) believed that it had no value. 6 students (2 boys and 4 girls) thought that some students would benefit from this approach. In September, of the 10 students interviewed, 4 students believed that the audience for their writing was the teacher (2 boys and 2 girls) while another 4 students felt that the audience was themselves. 1 boy believed that the audience was both the teacher and himself while 1 girl felt that the audience was other students.

This paper gives examples of work of two boys (B1 and B2). Both students were middle achievers and native English speakers. B1 was chosen because although he had good writing skills in mathematics, his beliefs about what mathematics involved meant that he did not consider writing useful. From the questionnaire results, it can be seen that there were many others, especially boys, who shared similar beliefs. B2 was chosen because like others in the class his inability to provide the details that an examiner needed seemed to be gaining him poor results. One example of each student's work is examined using categories from Functional Grammar and extracts are provided from the questionnaires and interviews. All spelling and punctuation has been left, although the interview extracts have been edited for clarity.

Results

B1

B1 did not think that writing in mathematics was useful. At one point during our ongoing, year-long discussion about mathematics and how it should be taught, he presented me with the five exercise books which had been his previous year's work as examples of what mathematics should be like. All the books were filled with neatly completed exercises with red ticks to show that the work had been correct. There were occasional paragraphs which were obvious copied notes from the teacher but there were no examples of B1 composing his own sentences. Table 1 provides B1's answers to the questionnaire on his ideas about mathematics and writing in mathematics.

Although B1 had agreed to take part in the interviewing, he was disinclined to explain how he gained his solution to the problem. He was therefore released without being questioned about his views on audience or language. Figure 1 provides his written work from the last project done in the year. In this project, students had to design a bandanna which incorporated different transformational geometry operations and a packaging box and its net.

Table 1
B1's Questionnaire Comments on Mathematics and the Role of Writing.

Questions	1/2/2001	19/9/2001
1) What is maths?	1) In a way fun but sometimes challenging	1) Mathematics is when numbers are used to figure out certain problems or give a quantity to something
2) What writing do you do in mathematics?	2) Problem solving involves writing	2) You write when you do problem solving and writing out equations
3) It has been suggested that students who write about mathematics they are doing will learn more easily. Why do you think that is?	3) I don't see how any one could possibly learn more about something by writing about it, but I suppose the theory could be correct, depending who it came from	3) I think its stupid, and is probably said from a teachers point of view, one who likes to give the students more work.

B1 initially tried to hand in the project without any written descriptions of what he had done. He was then asked to re-read the requirements. At this point he wanted to verbally explain his actions but I did not accept this and insisted that he wrote it down. When he came to hand it in again, he had to be reminded to explain why his initial pattern was not symmetrical. After he added this, the project was accepted although he had still not fulfilled the requirements as he did not hand in a net of his three-dimensional shape.

> What I have done to get the red pattern to where it is, is first rotate the whole original pattern on the vertex "A" 90° then translated the vertex (A) to the vector: $\begin{pmatrix}0\\5\end{pmatrix}$
>
> If the light *rays were both either plain white or rainbow, then the mirror lines would be as shown.*

Figure 1. B1's geometry project from December, 2001.

The writing shows sophisticated use of language as it includes many of the features of the mathematics register (see Meaney, in press for a description of the features of the mathematics register). This suggests that B1 is capable of writing mathematical texts but is unwilling to do so because of his beliefs about mathematics and the role of writing within it as described in Table 1.

Experiential meanings: The first sentence has two parts which are related through the second "is". It contains several embedded clauses providing information about both "what I have done" and the actual actions. By beginning the first sentence with "What I have done", the emphasis is moved away from the student as the doer and onto the actions themselves. Morgan (1998, p. 35) identified the issue of agency, who has taken part in the actions, as one which teachers did not agree on in their conceptions of what "good mathematical writing" was.

> Making agency explicit may be valued as it can appear to present a clearer picture of what was done, particularly in a context in which students' individual work is to be assessed. This may however, conflict with traditional conventions of 'sophisticated use of nominalizations and the passive voice".

Although the NCEA model answers use passive language, the examples given of the marking of actual student work do not penalised the use of first or second person. However, as students move into the final year of high school and onto university, students may be at a disadvantage if they are not fluent users of the mathematics register which would include mastery of the passive voice. Nominalisations provides lexical cohesion in texts of mathematics and allows ideas to be combined and manipulated in a succinct way (Meaney, in press). In this example, the vector $\begin{pmatrix}0\\5\end{pmatrix}$ could be considered a nominalisation as it can be acted upon as though it was an object (noun) but contains an implicit action (movement of five units up). However by describing the action as translating the vertex "A" *to*, rather than *by* , B1 has shown that he may be thinking of a vector as being more like a Cartesian point and has not understood the action that it contains. From examining his writing, it was possible for me as the teacher to determine what work needed to be done with him about vectors. Language-wise, there would also be a need in the future to support his understanding about nominalisations and passive voice. As a teacher it would have been difficult to discover these points if the student was only prepared to participate orally in class.

Interpersonal meanings: Although "I" is the second word in the sentence, it is the actions of the mathematical objects which are described in detail. B1's use of declarative clauses in the first sentence indicates that he is confident in the knowledge that he is describing. He is thereby suggesting that the relationship between himself and his reader is one of equals. The use of 'would', a modal finite, in the second sentence reflects B1's acknowledgment that he had not conformed to the requirements of the project and change would be needed to conform. This is also the only indication that he has perceived (finally) that the audience, his teacher, had a specific purpose in reading it. The relationship of the writer and the audience is no longer one of equals but a situation in which he has to conform to the reader's requirements.

Textual meanings: B1 used conjunctions to join many of his clauses together: first; then; if; then. Mathematics educators have referred to these conjunctions as logical connectives (including 'or' in the second sentence) and they have a significant part in the mathematics register as they provide information about the relationship between ideas (Meaney, in press). Some students especially those coming from an ESL background often have great difficulty in mastering the use of logical connectives in mathematics language (Roberts, 1991 p). B1 has good control of this aspect of the mathematics register, reinforcing his ability to use sophisticated language.

B2

B2 showed considerable interest in mathematics by attending lectures by visiting mathematicians and reading books about mathematics. However, his results especially in the common tests were poor. His answers to the questions in the questionnaire (see Table 2), like those of B1, show little change in his ideas about mathematics and writing in maths. His answer to question three in September is confusing, suggesting that he has not understood the question.

Table 2:
B2's Questionnaire Comments on Mathematics and the Role of Writing.

Questions	1/2/2001	19/9/2001
1) What is maths?	1) Solving problems using numbers.	1) The study of things numerical
2) What writing do you do in mathematics?	2) Little outside of numbers. Any letters are used to describe the situation or outcome of the problem.	2) Notes and numbers
3) It has been suggested that students who write about mathematics they are doing will learn more easily. Why do you think that is?	3) This could be right, but I really don't know. I do think its a good idea though	3) No, Lots and lots of small tests (e.g. 10 long division questions, when everyone gets a good score, the class could move onto something else)

When he was asked about who he wrote for, B2 replied:

> Just mainly so I can look back over, sort of like notes, so if I can't remember how to get something, I can just look back over it. It's actually a lot more useful than people seem to think it is.

When further queried, he stated that he also wrote for the teacher. The only difference that he could identify between writing for himself and writing for the teacher was that when he wrote for himself he might not use *big words* in case they confused him. He gave the example of perpendicular as a word whose meaning he often forgot. His sample of work comes from an open-book test at the end of the first term. The question gave the growth rate of the fastest growing tree in the world. Students were then asked to calculate its height after ten years, showing all their working and then comment on whether the height that they obtained was realistic.

In this test, B2 was one of the few students who commented on the growth rate of trees. Most students said that the height must be correct because they had done all the calculations correctly. However, B2 left it up to the reader to make the connection that it was unlikely that the tree would continue to grow so fast for such a long period. As the growth rate was given as a fact, B2 needed to state that although this tree might have an initial growth rate (or at some point in its growing) this fast, it was unlikely that this rate would continue for ten years and/or 97.28 m is an enormous height which no tree could grow to (especially within 10 years). In not providing this information, an examiner is left uncertain whether B2 has understood this or just believes that the given fact was incorrect.

6) 10 years = 120 months
120 ÷ 15 = 8
10.66
X 8
85.28 m

85.28 m
+12.00 m
97.28 m

I do not think that a tree growing 97.28 m in 10 years is a likely thing to happen. That's growing at 9.728 m a year which is about 19 cm a week! No, I do not think this is possible.

Figure 2. B2's test answer from March 2001.

Experiential meaning: B2's belief that mathematics is 'solving problems using numbers' is illustrated in his answer as, apart from 10 years being equivalent to 120 months, none of the calculations are explained. Although the students were not explicitly asked to describe each of their steps, for B2 the numbers themselves seem to be sufficient to describe what is happening. He does not feel it necessary to state why he did certain calculations with certain numbers. Even in the second part of the answer where his description of his disbelief in the tree height begins and ends with clause complexes based around mental process "I do not think ..." with embedded clauses, it is the growth rate of the trees which is commented upon. The mathematics itself has a life of it own which is self-sustaining.

Interpersonal meaning: By beginning the first descriptive sentence with *I do not think*, B2 is lessening the impact of his belief that the tree growth is unrealistic. In this way, the ideas are not put forward as fact but as his opinion thereby reducing his disagreement with the authority of the examiner. When there is a power relationship as there is between a student and an examiner, it is common for the student to try to lessen anything which could be seen as a contradiction of the authority (in this case the test itself which gave the growth rate). This is counterbalanced by using declarative clauses to provide information and reiterating his belief in the last sentence.

Textual meaning: As in most sets of calculations, there is a repetition of numbers which provides the lexical cohesion which suggests that he is following a logical train of thought. This type of cohesion is repeated in the sentences. At the beginning of the second sentence he uses 'that' to refer to the tree. In the third sentence 'this' refers to the growth rate of the tree. If B2 had used logical connectives as did B1, then his ideas rather than being seen as individual ideas may have produced a more convincing argument. This would have been because he was forced to state what he perceived as being the problem and a solution to it. For example, he could have written 'as it is impossible for a tree to grow 13 cm every week for ten years, either the growth rate is not correct or the tree could only grow this fast for a short period of time. For the rest of the ten years it would grow more slowly'. Providing opportunities where B2 would need to use logical connectives was one teaching point I gained from this investigation.

Discussion

Although only two examples of students' work are provided, it can be seen that both students showed considerable ability in expressing their ideas mathematically. However, these examples raise several issues. One is that of considering writing as being a legitimate part of mathematics. B1 had very strong views about what constituted doing mathematics and as a consequence resisted writing. These views may have derived from his previous mathematics learning experiences which appeared to match his preferences for how he liked to learn (doing series of exercises which could be then given ticks to show that they were correct). By not considering writing to be an important part of his learning, he restricted his ability to reflect upon his learning and receive feedback about it. From the questionnaire results, other students especially boys, seemed to resent this need to write explanations and justifications in mathematics. These students are likely to gain poor results in assessments such as NCEA unless their beliefs can be changed.

As he valued writing in mathematics learning, B2 raised other issues. It is possible that his view that the writing was for his benefit meant that he was not always able to consider what another audience may need. As a result, the main point in his answer about the growth rate of trees was missing. As a teacher I value the fact that this students writes for himself rather than writing just to achieve marks. By continuing to write for himself, I believed that not only his writing but his mathematical understanding would improve. Yet this did not happen. He continued to find it difficult to imagine what an audience would want to know because he felt that much of what he did was obvious. It may have been that using a process writing approach would have been helpful where another person commented on a first draft. However, he did receive considerable comments about his work throughout the year and his writing continue to leave out important parts in his descriptions of what he had done. This not only meant that he achieved poor results but also that he was unable to use his writing of a way of checking what he had done which may have identified errors. Further research is needed to discover how students such as B2 can learn to understand and respond appropriately to the reading needs of his audience.

While conducting this study, it also became apparent that more work needs to be carried out to find out what constitutes good mathematical writing (and speaking) for students at different stages in their mathematics education. At present, it is difficult to say what is acceptable writing at a particular stage in students' mathematical careers.

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A Comparison of Novice and Expert Views of the Features of Quality Mathematics Teaching

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Results from part of an earlier survey of teachers and academics were compared with student teachers' responses to the same open question. The question asked respondents to detail features of quality mathematics lessons. The students identified many of the same components of effective teaching as did the experienced respondents, but there were some notable differences and omissions. The resulting information has implications for planning of future teacher education experiences. Most importantly, it raises questions about the relative emphases in pre-service mathematics education courses that need to be placed on general pedagogy as opposed to mathematical content and pedagogical content knowledge.

Introduction

Researchers in the Early Numeracy Research Project (Clarke, Sullivan, Cheeseman, & Clarke, 2000) found that about 75% of children entering school could "compare, order and match using the attribute of length" (Clarke et al., 2000). Yet teachers of these young children would spend much of their measurement curriculum time, in at least the children's first year of schooling, on comparing and ordering lengths of objects. Given the "crowded curriculum", focusing on what is already known is questionable.

At Deakin University, we spend a considerable amount of mathematics education time on the study of features of general pedagogy. This is not a separate topic, *per se*, but a component of each topic, particularly in the introductory units. One must ask whether this expenditure of time on development of pedagogical (as opposed to content or pedagogical content knowledge, to use the categories of Schulman, 1986) is warranted. What knowledge of these elements do student teachers bring to their mathematics education units? Do their views on effective teaching vary markedly from experienced teachers' views?

These are the questions that the research reported in this article aimed to answer. They are significant questions because time in any pre-service course is precious, and because (just as in any early learning experience) we do not wish to turn students off learning by focusing on what they already know well. Further, we need to know what knowledge our students have so that we can draw on it as appropriate and build on it in order to further their professional readiness.

Features of Quality Teaching

Sullivan and Mousley (1994, 1997) noted that previous to their project entitled *Features of Quality Teaching* there had been few research projects aimed at identifying aspects of effective mathematics teaching. Today presented a different picture, with most Australian states supporting projects with this aim. For example, (a) the *Numeracy Research in NSW Primary Schools Project* subtitled "What's making the difference in achieving outstanding primary school learning outcomes in numeracy?" (b) Queensland's

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 489-496). Sydney: MERGA. ©2002 MERGA Inc.

What elements of learning environments promote enhanced student numeracy outcomes? (c) South Australia's *Numeracy Research in Primary Schools* project exploring practices that have led to improvement results, and (d) Western Australia's *Numeracy Research Project* "unravelling" factors that make a difference to numeracy outcomes. These initiatives were preceded by the same focus of interest in the United Kingdom (e.g., Askew et al., 1997. Askew & William, 1995; OfSTED 1995, 1996, 1997, 1998) and the United States (e.g., Borich, 1996; NCTM, 2000; Wood & Sellers, 1997).

The Original Survey

Sullivan and Mousley (1994) found that that there was some consensus amongst their subjects about features of effective teaching. The prompt below was administered to 125 experienced teachers and mathematics education academics (called the teachers for the rest of this paper):

> Imagine a mathematics lesson, at any year level, where the students are learning to estimate the weights of various objects, or to add fractions, or to record given information as a graph. Please write down the most important characteristics that a *quality* mathematics lesson on any of these concepts/skills would usually have.

Details of the data-handling and sorting process are outlined in Mousley, Sullivan and Waywood (1998). The researchers identified major components of quality teaching, along with the sub-categories listed in Figure 1.

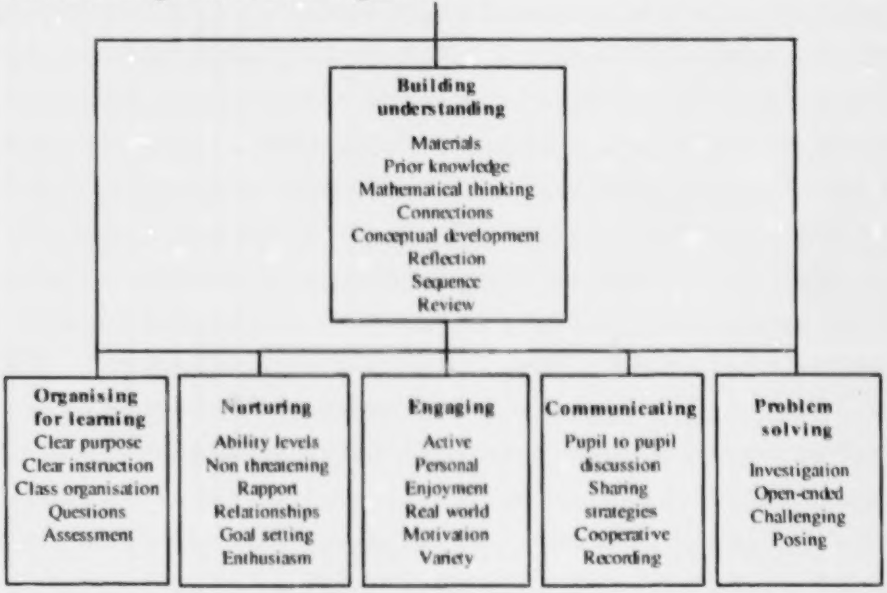
Building understanding
Materials
Prior knowledge
Mathematical thinking
Connections
Conceptual development
Reflection
Sequence
Review
Organising for learning
Clear purpose
Clear instruction
Class organisation
Questions
Assessment
Nurturing
Ability levels
Non threatening
Rapport
Relationships
Goal setting
Enthusiasm
Engaging
Active
Personal
Enjoyment
Real world
Motivation
Variety
Communicating
Pupil to pupil discussion
Sharing strategies
Cooperative
Recording
Problem solving
Investigation
Open-ended
Challenging
Posing

Figure 1: Features of quality teaching (from Sullivan & Mousley, 1996, p. 13)

A Comparison with Pre-service Teachers' Perceptions of Quality Teaching

It is generally assumed that student teachers bring to their courses limited and limiting ideas about the ways that mathematics should be taught. Schuck (1999) would have struck a chord with many of us with the title of her paper "Driving a mathematics education reform with unwilling passengers". Schuck found that many of her own students held beliefs about mathematics and pedagogy that constrain their access to alternatives. She

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realised that her task was to make students more aware of their beliefs as well as the implications of these for their future as teachers. When discussing the implications of this research for the ways that we might plan for and evaluate our own tertiary teaching, we decided to find out whether perceptions of quality mathematics teaching vary between experienced teachers and novices. Our research question became "Do our own students hold beliefs that are likely to constrain their access to effective teaching practices?" Given that we could gain access to the Mousley and Sullivan data as a basis for this comparison, we used the same open-ended instrument as above.

For this investigation, our subjects were 123 teacher Education students. About eighty were entering the second level of a four-year undergraduate degree, but had studied only two Education units because the first year of their course is heavily weighted with general arts and discipline area studies. This group had experienced one practicum round. The others were postgraduates who had just entered our pre-service two-year end-on course (called the students for the rest of this paper).

All of the students were new to mathematics education. They were in the first few weeks of their first mathematics education unit, with more units to follow in subsequent years. These were *our* entry students, so it was their knowledge that was relevant to our planning and teaching in order to help us with delivery of appropriate content and activities. We will repeat the exercise at the end of semester to gauge how their beliefs have changed as a result of their mathematics education experiences during the semester—including lectures, tutorials, group and individual assignments, reading tasks, preparation for the exam, and a 10–15 day practicum with some mathematics teaching.

The students were shown the open-ended prompt above and given class time (five minutes) to write their responses. These were typed and sorted into the same categories used by Sullivan and Mousley shown in Figure 1 above. The only change in method was that the students responded only to this prompt whereas the teachers had gone on to a more structured section of the questionnaire.

Results

This paper focuses both on comparing the numbers and proportions of responses in each category and sub-category as well as their nature. There are three points to note regarding the numbers reported below. First, as in Mousley and Sullivan's data sorting, some sentences or phrases were entered into several sub-categories. Second, although by coincidence the number of students (123) was very close to the number of teachers (125), the students wrote a lot more comments (529) than did the teachers (384); perhaps because they had been given a whole sheet of paper, and/or they did not have three pages of structured questions following this initial open one. Thus we have reported in the results below not only numbers of phrases but also these numbers expressed as percentages of responses from each sample. Third, the data were qualitative so we make no claims about statistical significance of differences noted, seeking only to give a sense of these. Also, while we know what teachers and students wrote, we do not know why they focused on those aspects of teaching. In some cases it was even difficult to infer what they meant. For example, does "Explain clearly" refer to conceptual or procedural explanations? We acknowledge the limitations of the data gathering and analysis processes in both studies in this respect.

Comparison of Categories

The number of comments allocated to each category by the teachers and the students is summarised in Table 1 below. Results and discussion for each sub-category follow, although space did not allow tabular presentation of these.

Table 1
Number of Entries Under Each Category

Sub-category	Teachers		Students	
	n[1]	%[2]	n[1]	%[2]
Building understanding	106	28	167	32
Communicating	75	20	66	12
Engaging children	74	19	174	33
Problem solving	53	14	11	2
Nurturing children	44	11	78	15
Organising for learning	32	8	33	6
	384	100	529	100

[1] Number of comments allocated to the relevant category.

[2] Percentage of all responses by the relevant sample, rounded to the nearest whole number.

The table shows clearly that there are similarities between numbers of comments made by teachers and students for some categories. *Building understanding*, *Nurturing* and *Organising for learning* showed similar patterns of response, although these mask differences in sub-categories (see below). Teachers showed more awareness of the importance of forms of *Communication*, and made many more comments about *Problem Solving*, a strategy that was becoming a popular at the time of their survey. However, the students gave the notion of *Engaging children* more emphasis—possibly in response to their own learning preferences.

Comparison of Sub-categories

Building understanding. This term is used in recognition of "particular understandings to be developed, and of strategies to achieve this end by building on existing knowledge, using materials to explain and clarify concepts, choosing appropriate sequences, helping students to make connections, forming relationships, and knowing the meaning of terms. There is a strong inference of teacher decision, teacher direction, teacher explanations and teacher control" (Sullivan & Mousley, 1994, p. 9).

For *Use of materials* T = 36 (9%), S = 35 (7%); *Use of prior knowledge* T = 21 (5%), S = 29 (5%); *Mathematical thinking* T = 15 (4%), S = 17 (3%); *Making connections* T = 14 (4%), S = 14 (3%); *Conceptual understanding* T = 10 (3%), S = 54 (10%); *Encouraging reflection* T = 6 (2%), S = 3 (1%); and *Use of review/closure* T = 4 (1%), S = 15 (3%).

These figures general demonstrate a relatively close alignment in the thinking of teachers and students. A surprising difference related to the development of conceptual understanding, with many more student comments than teacher comments about the importance of this. Some examples of students' comments follow:

> Spend one-on-one time with kids to try to make sure they understand the concepts—no point in moving on if they don't. Make sure they understand why as well as how they are to do it.

> Learn, not regurgitate.
>
> Assuming the context to be fractions, a lesson that fleshed out the key concepts for the student would be beneficial. For instance the use of visual clues and divisible objects would seem likely—basic ideas that lead the children into forming their own patterns around the ideas.

The number of student comments on the importance of review and lesson closure may have been the result of students learning about lesson planning in Education Studies.

Communicating. This category includes "statements related to opportunities for talking, explaining, describing, listening, asking, clarifying, sharing, writing, reporting, and recording" (Sullivan & Mousley, 1994, p. 12).

For *Discussion between pupils* T = 34 (8%), S = 5 (1%); *Sharing strategies* T = 20 (5%), S = 10 (2%); *Co-operative situations* T = 15 (4%), S = 45 (9%); and *Recording* T = 6 (2%), S = 6 (1%).

As would be expected, the teachers here valued discussion and the sharing of strategies—two points emphasised frequently in professional literature and development activities. It would be interesting to probe further how both the teachers and the students define *Co-operative situations* and to try to find out why relatively more student comments were made. These comments included:

> All lessons would have opportunities to work in pairs or groups, to work out answers, enhance understandings with each other through experimentation and discussion with the teacher.
>
> Team work (of children) to find answers—bouncing ideas off one another.
>
> Occasional group work could aid in the brainstorming of solutions.

Engaging Children. This category is generally about students' involvement in their own learning. Included in it are both physical and mental forms of engagement such as keeping children actively and productively involved in thought-provoking activities, as well as more affective factors such as motivation and interest.

For *Active involvement* T = 34 (8%), S = 31 (6%); *Personally relevant* T = 16 (4%), S = 30 (6%); *Enjoyment* T = 5 (2%), S = 41 (8%); *Real world* T = 7 (2%), S = 42 (8%); *Motivation* T = 6 (2%), S = 18 (3%); and *Variety* T = 6 (2%), S = 12 (2%).

Thus while there was general agreement between teachers and students in four of the sub-categories, notable differences were evident in relation to both *Enjoyment* and the use of *Real world* contexts. We were not surprised that students seemed aware of the need to make mathematics enjoyable as well as clearly useful. Their sentences included:

> Fun—the more interesting you make it the easier to learn. Turn it into an enjoyable activity rather than a mathematical chore.
>
> Ensure all children enjoy and understand the activities and see how they apply in everyday life. Get all of the class involved in meaningful activities. Not just pencil and paper.

Problem solving. The term problem solving has been used to mean many things in many situations. It is not surprising that some variation was evident in responses

For *Use of problem-solving tasks* T = 27 (7%), S = 7 (1%); *Use of open-ended tasks* T = 14 (4%), S = 1 (<1%); *Use of challenging activities* T = 6 (2%), S = 2 (<1%); and *Problem posing opportunities* T = 6 (2%), S = 1 (<1%).

Note that there were many more teachers' comments than students' comments about all aspects of problem solving. The problem solving movement was influential at the time that teachers completed the survey and there was a blossoming interest in open-ended tasks and investigations. Perhaps the students of those times now see such activity as a "given" in mathematics. Despite the small number of comments, those made were insightful. For example:

> Provide challenging problems to convey the message that all problems can be attempted, and discuss creating means of solving them.
>
> Have problem solving activities where kids have to think constructively / critically and also let the students create their own real life problems and solve them.

Nurturing Children. Most of the descriptors categorised under this heading were related to the teacher being sensitive to the needs of the "whole child" or "individuals".

For *Catering for levels of ability* T = 11 (3%), S = 35 (7%); *Non-threatening atmosphere* T = 10 (3%), S = 22 (4%); *Rapport with children* T = 9 (2%), S = 11 (2%); *Enthusiasm* T = 8 (2%), S = 2 (1%); *Personal relationships* T = 4 (1%), S = 4 (1%); and *Goal setting and negotiation* T = 2 (<1%), S = 4 (1%).

The exception to the pattern of similarity—the emphasis by students on catering for a range of abilities—could be accounted for partly by the emphasis on equity issues and inclusive curriculum in first year Education Studies as well as by more focus on this in the broader society in recent years. Some representative comments here are:

> The teacher must be patient and understand that each child learns at a different pace. The teacher should have some background knowledge of what each child's level of understanding is.
>
> Be patient—some children might not understand the task at hand so be prepared to spend time with those who are struggling.
>
> Make sure everyone knows they can ask questions if they don't understand. Try to ensure everyone understands what they have to do. Follow up at the end by asking random students what they did, what answer they got and what does it mean. Doing this rather than saying "Who knows the answer?" may help identify children with problems, rather than them hiding that they don't understand so that the smart children tell everyone the answers and the confused children don't get an explanation from someone who might also be confused.

Organising for Learning. This component refers to actions designed to keep students working towards achieving a lesson's goals, decisions made by the teacher about a specific focus and a commitment to pursuing that focus, and communicating the focus to the students.

For *Clear purpose* T = 14 (4%), S = 8 (2%); *Clear instruction* T = 7 (2%), S = 20 (4%); *Organisation* T = 4 (1%), S = 2 (<1%); *Assessment* T = 4 (1%), S = 2 (<1%); and *Questions* T = 3 (1%), S = 1 (<1%).

The noteworthy differences here lie in clarity of purpose (which the teachers recognised more), and of instructions (which the students emphasised more). Perhaps this reflects traditional roles and experiences. Students' comments in this category included:

> Clear simple directions so as not to confuse. Straight-forward, detailed explanation of the topic.
>
> A clear explanation of the class objective.
>
> Activities where students can see for themselves the purpose of the concept.

Emerging Points

An emphasis on time. The only factor that was not covered explicitly by Sullivan and Mousley but that the students mentioned a number of times, was the need to give children time for learning.

> Allow enough time for children to work out the maths problems—ensure there is not time pressure that may mean children don't get long enough to think creatively about the answers.

> Time—do not rush.

Perhaps teachers see time as more of an inevitable constraint rather than an aspect of quality teaching; and once again, the students' comments may reflect on their perceptions (as school students) of the pressure and limitations imposed by teachers' and curriculum developers' expectations.

The quality of responses. Several aspects of the students' responses pleasantly surprised us. First, some showed that they had been analytical and critical observers of classroom practices during their practicum periods.

> Teacher demonstrating the "how to" on the board followed by students doing either workbook or photocopied examples to gain an understanding of the subject matter. This is what I have seen on teaching rounds. However, for me it is more important that each student understands the concept and can also relate it to everyday life and hopefully make use of the concepts.

Further, many students seemed both insightful and articulate. They demonstrated maturity in professional knowledge as well as the language used to express it.

> The children are encouraged to be imaginative and innovative in approaches and techniques.
>
> Let them experience the ideas.
>
> Have two different ways (more if possible) to explain a concept in case some children interpret and understand the information differently than others, or differently from what we expected.

Some demonstrated a willingness to engage in reflections about heir own teaching, making comments about on-going learning as a reflective teacher:

> Discussion of difficulties, what was hard and why, to improve teachers' understandings of children's abilities and of how the teacher may understand her or his own presentation of the mathematical concepts.

A number of responses put a focus on deeper levels of understanding.

> For fractions and mass, incorrect examples can be shown so students can differentiate and hence understand the correct ideas better.
>
> Physical example of abstract theory.
>
> At primary age, learning mass may involve using various size containers. This enables the kids to explore mass/weights for themselves to discover various characteristics and hopefully come up with general rules to be used.

A final pleasing point was that many students emphasised learning and linked this with generalisations about appropriate teaching strategies.

> Learning is the foundation of all good lessons. There is no point having good teaching approaches and activities that do not lead to extra learning.
>
> Quality maths lessons are all about concepts and exploration. They need to do the thinking about the maths at school. Students can do the exercises at home because the repetition might help a child remember what they have learnt.

Conclusions and Implications

An exercise that we thought would be useful for our planning and evaluation of unit foci and content proved more fruitful than expected. It has given us insights into many of the students' understandings and perspectives, their abilities to articulate and share these, and also some needs (such as attention to uses of problem solving). It was also reassuring that their responses mentioned the full range of areas that experienced teachers and academics mentioned.

However, we realise that while many of the comments demonstrated significant knowledge and maturity in thinking, there are a number of areas that need our attention. Not all students demonstrated a strong grasp of aspects of effective teaching. We also need to consider that even if students have beliefs about the worth of particular aspects of classroom practice; they may not have a strong grasp of the underpinning theoretical bases for particular components of quality teaching.

It is clear that more research is needed to find out the extent to which students hold a good level of understanding, and that we need to consider how we can make use of this knowledge. Also raised are questions of how best to organise learning for those who do not enter the course with adequate understanding of general mathematics pedagogy while still catering for those who do.

We believe that the most important point arising from the findings is that in many areas the novices' comments were very much like those of the experts, and were made in similar proportions. This raises questions about the relative emphases that need to be placed on pedagogy as opposed to mathematical content knowledge and pedagogical content knowledge in teacher education courses—ours specifically but perhaps pre-service mathematics education courses more generally. The data suggest that we could focus less on quality teaching in general and more on ideas related to the learning and teaching of specific content.

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The Role of Structure in Children's Development of Multiplicative Reasoning

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Key features of developing multiplicative structure were analysed from case studies of twenty-four students representing extremes in mathematical ability. Data drawn from a longitudinal study from Years 2 through 5 of schooling indicated that low ability students represented multiplicative situations without structure and development progressed from the use of pictorial to ikonic representations. Absence of any underlying structures persisted through to the end of Year 5 for half of these students. From the outset in Year 2, high ability students used notational representations with well-developed structures, and dynamic imagery featured strongly in their responses.

Multiplicative reasoning is essential in the development of concepts and processes such as ratio and proportion, area and volume, probability and data analysis. It is clear that failure to develop multiplicative structures in the early years impedes the general mathematical development of students into the secondary school, for example, in using algebra, functions and graphs. It appears that difficulties faced by older students can be attributed, at least in part, to the lack of development of an equal-grouping structure in early concept formation (Mulligan & Mitchelmore, 1997). Often young students' own representations lack any recognisable cohesive structure and they are unable to use their representations flexibly. Using structure is also important in the organization and interpretation of multiplicative situations shown as models, diagrams, tables and graphs. It is still unclear how the development of underlying structures, whether they are essentially mathematical or related to spatial organisation, influences mathematical development. This paper reports one aspect of a 4-year longitudinal study on children's number concepts. It describes the development of a theoretical framework for analysing the role of structure in multiplicative reasoning supported by data from twenty-four case studies of students followed from Years 2 through 5 of schooling.

An Overview of Research

Children need to develop and recognise underlying mathematical structure in order to understand how the number system is organised and ordered by grouping in tens, and how equal groups form the basis of multiplication and division concepts (Boulton-Lewis, 1998; English, 1999; McClain & Bowers, 2000; Mulligan & Mitchelmore, 1997; Mulligan & Watson, 1998; Sullivan, Clarke, Cheesman, & Mulligan, 2001; Willis, 2000). Research in early number development has identified counting, subitizing, grouping, partitioning and sharing as essential elements of multiplicative structure (Mulligan, in press). Structure can be identified in a variety of ways, such as by finding patterns of five dots in an array of twenty-five items rather than seeing twenty-five individual items. Children's early multiplication and division knowledge results from cognitive reorganisations of their counting, addition and subtraction strategies, and builds on number word sequences, combining and partitioning. The development of multiplication and division knowledge is described in order of increasing sophistication from initial grouping and perceptual counting to abstract composite units and repeated addition and

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combining and partitioning. The development of multiplication and division knowledge is described in order of increasing sophistication from initial grouping and perceptual counting to abstract composite units and repeated addition and subtraction, and to multiplication and division as operations (Mulligan & Wright, 2000).

Studies investigating the role of imagery in children's counting and numeration have also identified structural development of the number system from elements such as grouping, regrouping, partitioning and patterning found within the recordings of the numbers 1 to 100 (Thomas & Mulligan, 1995; Thomas, Mulligan & Goldin, in press). Emerging structure is typified by numerals organised in a counting sequence, recorded continuously in a horizontal, vertical, curved or spiral formation. Mathematically gifted students' images have recognisable mathematical and spatial structure and use dynamic imagery (changing or moving images), whereas low achieving students show no signs of underlying structure. Students showing evidence of a more developed mathematical structure record accurate counting patterns using multiples (e.g., 3, 6, 9…), and marks or pictures showing a ten by ten (10 x 10) array structure. For example, Mellissa, aged 7 years (Fig 1) gave evidence of a high level of structure in her representation of 100 with her drawing of ten cubes. Robert, aged 9 years (Fig 2) drew a square and subdivided rows of separate squares, each square not being aligned to adjacent squares, and then recorded numerals for the numbers in squares, 1 to 17 being in the first row. It can be inferred that Robert has a much weaker understanding of the number system, as evidenced by his lack of structure, than Melissa.

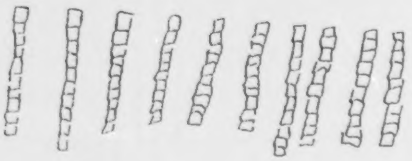

Figure 1. Melissa's image of 100.

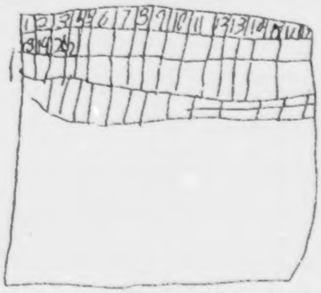

Figure 2. Robert's image of 1 to 100.

Mathematical structure has also been described in terms of students' spatial organisation of collections of two-dimensional and three-dimensional objects such as arrays of squares in rectangles and cubes in rectangular boxes (Battista, 1999a, 1999b; Battista, M. T., Clements, D. H., Arnoff, J. Battista, K. & Borrow, C., 1998). Other studies have investigated students' lack of structure in measurement of rectangular arrays, squares and other two-dimensional objects (Outhred & Mitchelmore, 2000; Reynolds & Wheatley, 1996). One of the difficulties is that students may not construct the row-by-column structure that identifies how squares fit together in a rectangle. Another problem is the underlying equal-groups structure required for counting rows and layers in multiples.

Background to the Study

The use of mathematical structure was highlighted in a longitudinal study of young children's intuitive models for multiplication and division. Mulligan and Mitchelmore (1997) found that the intuitive model employed to solve a particular problem did not

Observed Learning Outcomes model (SOLO) (Biggs & Collis, 1991). The analysis showed that the early formation of visual images in the prestructural and unistructural levels in the ikonic mode was critical to development of the equal-grouping structure of multiplication. Images initially lacking in structure at the prestructural level became more organised mathematical elements in the ikonic mode, i.e. random drawings of ikons began to show the structure of groups. Children initially giving prestructural responses became less reliant on physical models and idiosyncratic images and began to focus on numerical aspects of the problem in the ikonic mode. This development influenced their ability to interpret the semantic structure of the problem and their ability to represent equal-sized groups through physical or concrete models.

Children's Representations of Multiplicative Structures: A Longitudinal Study

Mulligan & Mitchelmore extended their research on intuitive models in a 3-year longitudinal study of second graders' representations of numerical situations involving counting, grouping, place value and multiplicative reasoning (Mulligan, Mitchelmore, Outhred & Russell, 1997). This project advances prior research through a longitudinal study integrating three key elements:

- a notion of mathematical structure commonly described as equal grouping, partitioning, or patterning inherent in multiplicative reasoning (Mulligan & Mitchelmore, 1997);
- the use of spatial structuring in mathematical representations which pertain to organisational features such as arrays and charts (Battista, 1999b); and
- the role of imagery in relation to developing both mathematical and spatial structure (Thomas et al., in press).

Analysis of children's drawings, symbols and explanations of their representations identified how children imposed structure, or lack thereof, on numerical situations. Low achievers were more likely to produce poorly organised, pictorial and ikonic representations that were lacking in structure. *These children lacked flexibility in their thinking; they were only able to replicate models of groups, arrays or patterns that had been produced by others.* Poor performance in multiplicative tasks was attributed not just to a lack of underlying equal-groups structure, but also to the primitive idea that unitary counting can be used to solve everything. Poor performance on simple ratio tasks was also linked to children's inability to visualise fractions.

Follow-up Case Studies

Further study focused on key features of developing mathematical structure, and monitored changes in children's multiplicative reasoning. This is shown through in-depth case study in the fourth year. Twenty-four Year 5 students (aged 10 to 11 years), representing extremes in mathematical ability (twelve high ability and twelve low ability), were drawn from the 3-year longitudinal study. Interview data were combined to form a more coherent picture of these student's key aspects of multiplicative reasoning over a 4-year period. Analysis of interviews was supplemented with work samples of students' representations and classroom assessment records. Interviews were videotaped to ensure the reliability of coding and for in-depth analysis.

Subjects were interviewed four times during the fourth year at regular intervals by a trained research assistant. At the time of final interview twenty one of the twenty-four subjects remained. Tasks initially developed to assess key aspects of multiplicative reasoning were modified for the low ability group of students (Mulligan et al., 1997). Task categories were extended in the fourth year to include fractions and decimals and area concepts. In some cases interview procedures were modified to allow students to further opportunities to demonstrate a variety of responses to the one task. The researcher asked further questions to evoke other available representations from the child.

Analysis

Data were analysed initially for performance across tasks and for individual patterns of strategy use across the four-year period. This paper reports further analysis of the interview data for the twenty-four subjects.

Table 1

Classification of Representations by Imagery, Structure and Nature of Image

CATEGORY	*DESCRIPTION*
Mode of Image	
Perceptual	Objects, figures or gestures used as models by the child which refer to quantities
Pictorial	Pictures drawn or descriptions of objects, events or situations given or described by the child which refer to quantities
Ikonic	Notations such as dots, dashes, tally marks or shapes that represent images of quantity
Notational	Conventional numerals (drawn) or descriptions of numerals represented individually or a series of numerals; signs and symbols e.g., × ÷
Type of Structure	
No structure	Objects, pictures or numerals which do not depict equal grouping structure
Partial structure	Objects, pictures or numerals in rows and columns but not a complete pattern of equal groups
Structure	Objects, pictures or numerals in equal grouping patterns, arrays, grids
Nature of the Image	
Static image	The representation is drawn and/or described as a fixed object or notation
Dynamic image	The representation is drawn or described as changing or moving

(Adapted from Thomas, Mulligan & Goldin, in press)

Their pictorial and notational recordings were coded according to three dimensions: (a) the type of representation identified by the perceptual/pictorial, ikonic and notational recordings; (b) the level of structural development and (c) evidence of a static or dynamic nature of the image. Table 1 describes the classifications for each of these dimensions by *mode of representation, type of structure* and *nature of image*.

Discussion of Results

Figures 3 to 6 compare high and low ability students' use of structure and type of representations across tasks and interview stages. Interview Years 1 to 4 refer to grade levels 2 to 5 in New South Wales schools. For the purpose of this summary, one category of 'structure' and one category of 'representation' were reassigned for each student for the first three years and at two interview stages in the fourth year. This was based on the most dominant characteristics shown in their responses and drawn representations across tasks.

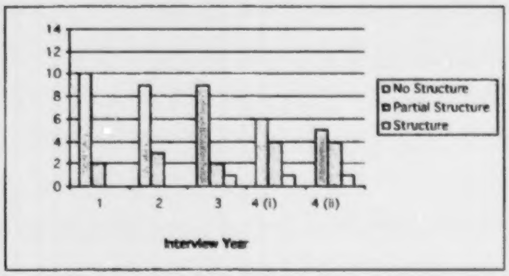

Figure 3. Low ability students' use of structure.

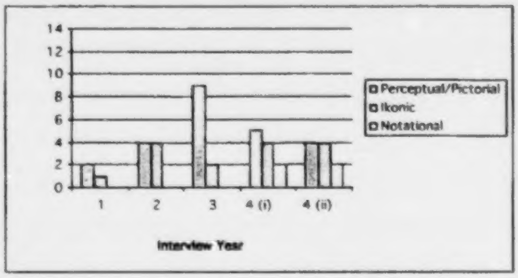

Figure 4. Low ability students' representations.

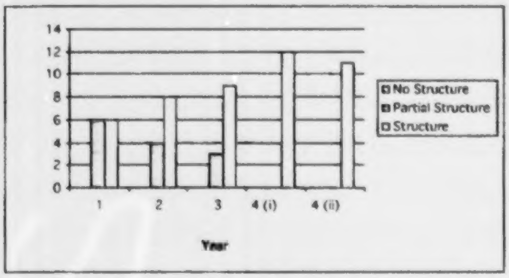

Figure 5. High ability students' use of structure.

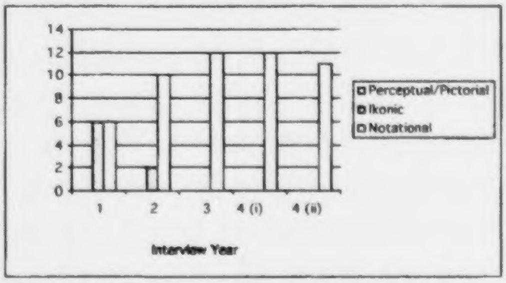

Figure 6. High ability students' representations.

Figures 3 and 4 indicate that the lower ability subjects differed significantly from the higher ability subjects in both their use of structure and their representations. In all but one case their perceptual/pictorial and ikonic representations lacked structural cohesion and mathematical structure depicting equal groups. Another problem detected was an underlying difficulty with basic fractions concepts and appropriate representations of 'part/whole' and 'equal parts'. There was, however, some evidence of 'partial structure' reflected in recordings of equal groups combined with single objects and unitary counting. The absence of underlying structures persisted through to the end of Year 5 for five of these students. Notational representations of multiplicative situations were not forthcoming because these students could not use multiplication and division as operations and could not recall number facts.

Figures 5 and 6 show that the high ability group used notational representations with recognisable mathematical structure. For example, arrays, diagrams, webs or grids were often used in conjunction with number patterns or number sentences, for example 3, 6, 9, 12,... drawn on an array. Most of these representations were reported as having dynamic

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properties (moving or changing). This was consistent with patterns shown in earlier studies of gifted students (Thomas & Mulligan, 1995). This dynamism showed evidence of underlying mathematical creativity in four of the cases where multiplicative problems were solved using a variety of non-standard, mathematically sophisticated methods.

The use of perceptual/pictorial and ikonic representations for low ability students is not surprising and supports other findings (Mulligan et al., 1997; Thomas et al., in press). The level of representation alone, however, does not explain the inability to initiate and use structure in multiplicative situations. Even when low ability students are given 'models' to promote structure, for example arrays, they are unable to depict spatial organisation or mathematical grouping. In contrast, high ability students indicated existing structures from early Grade 2 (Year 1 of the study), and partial structures for some students progressed to more sophisticated structures by Year 3. This raises the question of why high ability students develop structure even before Grade 2 and whether this reflects high-level visualisation skills or innate mathematical ability.

Implications

Data drawn from twenty-four cases does not permit immediate generalisation. Further longitudinal study needs to focus more explicitly on the origins of both spatial and mathematical structure. It is critical to identify and prioritise those processes that promote the development of structure across a range of mathematical concepts. Part of this identification will involve investigating how young students establish, or fail to establish, relationships between one aspect of mathematical structure and another, such as using a 2 x 10 grid to identify two groups of ten at the same time as recognising the perimeter as 24 units.

The analysis of students' interviews and representations over a four-year period showed some developmental trends that can contribute to a more coherent framework for assessing multiplicative reasoning. It appears that current curriculum contributes to the underlying problems because it lacks emphasis on early development of fractions and multiplicative concepts. Further, can teachers become more aware of these apparent differences and develop strategies for promoting 'structure' across mathematics curricula? In 2002 a new study entitled "Young children's construction and application of mathematical pattern and structure in early numeracy" is examining first graders structural relationships between early number and space concepts while also examining their mathematical representations.

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Fostering Authentic, Sustained and Progressive Mathematical Knowledge-Building Activity in CSCL Communities

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In this paper, we report on a study where students engaging in model-eliciting mathematical problems with collective discourse mediated by Computer Supported Collaborative Learning (CSCL) software achieved the kind of authentic, sustained, and progressive knowledge-building activity that until now had not been achieved in CSCL software-mediated mathematics communities.

Background to the Research

A major pursuit of most mathematicians working in the living discipline of mathematics is the production and improvement of mathematical conceptual artefacts such as mathematical models. In order to do this, mathematicians often form research communities who invest their resources in the collective pursuit of understanding by engaging in:

- knowledge-building discourse where problems are formulated and posed, assumptions are identified and evaluated, conjectures are made, and findings are shared and critiqued; and
- the investigation of authentic mathematical problems which advances the corpus of knowledge within the discipline of mathematics by: producing new conceptual artefacts and/or by improving existing conceptual artefacts.

This contrasts with the sort of mathematical activity most children engage in within their classrooms. Most classroom mathematical activity tends to be a severely atrophied version of the mathematical activity engaged in by mathematicians (Bereiter, in press; Lesh & Doerr, in press).

According to Bereiter (in press), the major focus of students in most mathematics classrooms is the completion of tasks (e.g., the completion of worksheet and textbook exercises). He, therefore, advocates that mathematics classrooms be reformulated as knowledge-building communities where students are engaged in authentic mathematical activity and produce mathematical conceptual artefacts that can be discussed, tested, compared, and hypothetically modified. In these communities, students see their main job as producing and improving such artefacts, not simply the completion of school tasks.

Information technology has been recognised as an important tool for supporting knowledge-building communities in classrooms (Scardamalia & Bereiter, 1996). However, because most existing information technology-based educational materials (e.g., CAI, ITS, educational games, simulations etc.) tend to foster neither collaborative learning nor knowledge-building (Bereiter, in press), Computer Supported Collaborative Learning (CSCL) software systems such as *Knowledge Forum®* (Scardamalia & Bereiter, 1998) have been designed. *Knowledge Forum®* is a communal multimedia database into which students may enter various kinds of text or graphic notes. When off-line, students plan

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knowledge-building projects, seek information from a variety of sources, and engage in small group and whole class discussions of questions, ideas and findings. When on-line, students describe plans, enter new information through text and graphic notes, and participate in more pointed discourse on questions, ideas and findings. Constructive commenting on other students' notes is proactively encouraged and has been shown to be effective in facilitating the construction of conceptual artefacts (Woodruff & Brett, 1993).

However, establishing and maintaining knowledge-building communities of practice with *Knowledge Forum®* (and its earlier version called *CSILE*) in the domain of mathematics has been found to be a rather intractable problem (De Corte, Verschaffel, Lowyck, Dhert, & Vanderput, 1999; Nason, Brett & Woodruff, 1996; Scardamalia & Bereiter, 1996). This contrasts with the experiences in other discipline areas such as science (Brett & Woodruff, 1993; Scardamalia & Bereiter, 1996).

One possible reason for why researchers have had more success in establishing and maintaining CSCL-mediated knowledge-building communities in science classrooms could be due to qualitative differences in the nature of the problems being investigated in the mathematics and science classrooms.

Credence for this viewpoint is provided by recent research conducted by Lesh and his colleagues into the design of "Real Life" Model-Eliciting Activities (Lesh, Hoover, Hole, Kelly, & Post, 1999). In Model-Eliciting activities, the products that students produce include more than answers to questions; they involve producing models or other conceptual artefacts for constructing, describing, explaining, manipulating, predicting and controlling complex systems. Based on their analysis of activities which were most effective in eliciting high levels of symbolizing, communicating and mathematizing, Lesh et al. (1999) developed the following six principles to inform the design of good model-eliciting and thought-revealing activities:

1. *The Personal Meaningfulness Principle* (sometimes called the reality principle)
2. *The Model Construction Principle:* Does the task create the need for a model to be constructed, or modified, extended or refined? Is attention focused on underlying patterns and regularities?
3. *The Self-Evaluation Principle*: Will students be able to judge for themselves when their responses are good enough? For what purposes are the results needed? by whom? when?
4. *The Model-Documentation Principle:* Will the response require students to explicitly reveal how they are thinking about situation? What kind of system (mathematics objects, patterns, regularities) are they thinking about?
5. *The Simple Prototype Principle:* Will the solution provide a useful prototype (or metaphor) for interpreting a variety of other structurally similar situations?
6. *The Model Generalisation Principle:* Can the model generated be applied to a broader range of situations? Is the model reusable, sharable and modifiable?

A close analysis of the science problems used in the studies where CSCL software systems have been very successful in establishing and maintaining science knowledge-building communities (e.g., Woodruff & Meyer, 1995; Woodruff and Meyer, 1997) reveals that the science problems more than adequately meet Lesh et al.'s six principles. These problems were effective in eliciting high levels of authentic science activity such as hypothesising, symbolizing, communicating and "scientifizing". In contrast however, most of the problems used to promote software-supported mathematical knowledge-building communities (e.g., De Corte et al., 1999; Nason et al., 1996) do not adequately meet this set of six principles. Thus, in hindsight, this may explain why the studies were unlikely to be effective in eliciting high levels of symbolizing, communicating and mathematizing.

Therefore, in the study being reported in this paper, the major aim was to investigate whether having students engage in model-eliciting mathematical problems with collective discourse mediated by Knowledge Forum would achieve the kind of authentic, sustained, and progressive knowledge-building activity that has been achieved in more content-rich discipline areas such as science.

Method

Participants

The twenty-one participants in this study were students from a Grade 6 class at a private urban Canadian school for girls. The twenty-one students in the class were assigned to seven mixed-ability groups of three students.

Data Sources

The investigation involved detailed analysis from four data sources: (1) Observation notes from the lessons; (2) Unstructured on-the-run interviews in which students were asked to explain their models, how they had created their models, and what they had learned, (3) Students' spreadsheet models; and (4) Knowledge Forum notes.

Procedure

Initially, the teacher/researcher presented the students with an article "Best City in the North-East: Providence RI" (Money, 1998) that explained why Providence had won "Money" magazine's award for being the best city in the north-east of the United States. A whole class discussion based around a set of four focus questions followed the reading of the article. The purpose of these questions was to help the students "read with a mathematical eye" while also familiarizing them with the context of the model-eliciting activity – so that their solutions would be based on extensions of the students "real life" knowledge and experiences (Lesh, Cramer, Doerr, Post, & Zawoiewski, in press). After the discussion based around the set of four questions, the students were presented with the following problem:

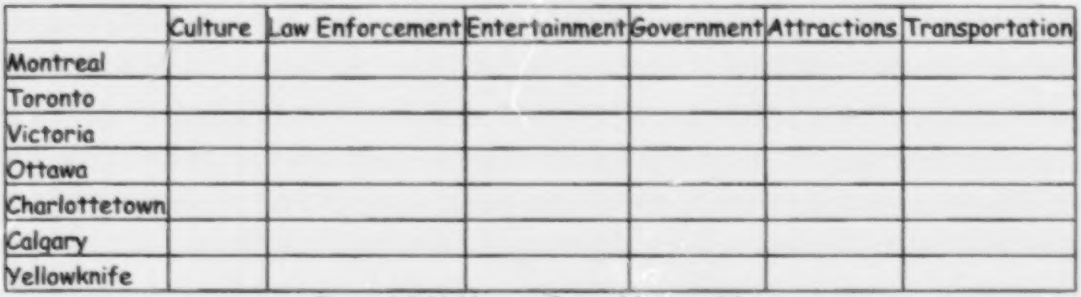
> Ms Estrina has recently emigrated from Russia. Although she has been living in Toronto for about a year, she has not yet made up her mind about where she would like to live in Canada. In order to help her decide, she wants to create a table ranking Canadian cities in terms of quality of life. Your task is to create a model that will enable her to create this table.

The production and the revision of the systems proceeded in four phases.

Phase 1: Initial model-eliciting activity. After the problem had been introduced, the whole class engaged in a "brainstorming" session in which they identified and briefly discussed some of the criteria that would need to be considered when one was ranking cities (e.g., safety, health services, educational services).

Following the brainstorming session, the class broke up into their seven groups. Each group then set about producing their initial model. They did this by first listing all the criteria that could be used for ranking the cities. Because their lists contained more than ten criteria, each group of students then decided to organise their list of criteria into a set of five or six categories. For example, Kath's group categorised criteria such as multiculturalism, variety of ethnic foods, cultural celebrations under the rubric of "culture". Anne's group categorised criteria such as safety, education, multiculturalism and leisure under the rubric of "quality of life".

Once they had decided on their set of categories, each group of students then entered their set of categories into the top row of an Excel spreadsheet. After they had entered their categories into the spreadsheet, each group of students listed 6-8 Canadian cities in the left hand column of the spreadsheet (see Figure 1 below).



	Culture	Law Enforcement	Entertainment	Government	Attractions	Transportation
Montreal						
Toronto						
Victoria						
Ottawa						
Charlottetown						
Calgary						
Yellowknife						

Figure 1. Initial layout of spreadsheet models.

After they had entered the categories and cities into the spreadsheet, each group then used books from the library and internet databases to gather information with respect to each of categories for each of the cities. Based on this information, they then awarded each city a score out of ten for each category. Following this, they used the spreadsheet to generate a total score for each city and then to rank the cities in descending order according to their total scores.

Phase 2: Sharing of initial models. During this phase, each group of students presented their initial model to other members of the class. The other members of the class were invited to make comments and/or ask questions about the model.

Phase 3: Revision of models. Following the sharing of the initial models, each group then proceeded to further explore and modify its spreadsheet model. During the sharing of the initial models, there was much debate about whether each category should be given a score out of ten. As a consequence, all of the groups then decided that the categories needed to be prioritised. Some groups did this by allocating more points for some categories than others (e.g., scoring culture out of five and transportation out of ten). Other groups did this by building a weighting into the formula used to generate the total score (e.g., multiplying the score out of ten for culture by five and the score out of ten for transportation by eight).

As this was occurring, it was noted that each group of students made sure they were aware of what ideas the other groups were investigating by intermittently "scouting" (Meyer & Woodruff, 1997) around the room looking at and talking to other groups about their computer spreadsheet models. This informal, "scouting" form of discourse played a very important role in facilitating the progress in each group's model. For example, some groups changed their weightings after their scouting forays.

Phase 4: Dissemination and revision of models in Knowledge Forum. This phase began with each group posting their revised model on the *Knowledge Forum®* shared database.

Then, during the next three sessions, each group of students read and provided constructive feedback (such as comments, questions, propositions) to other groups about their models. Based on the feedback from other groups and the teacher/researcher, each group also iteratively revised their own model. In order to facilitate the process of constructive feedback, each session began with a brief whole-class discussion about how to provide constructive feedback. The classroom teachers further reinforced this instruction during other classroom activities.

Results and Discussion

Three important elements of knowledge-building activity that have been noted in mathematical research communities were observed during the course of the study: (1) the posing and exploration of conjectures, (2) the collective pursuit of understanding of key mathematical concepts, and (3) the incremental improvement of the mathematical models.

Table 1

Knowledge-Building Activity of Students

Elements of knowledge-building	Exemplars
Conjecturing	• Changes to weightings and definitions
Collective pursuit of understanding	• Investigation of meanings of numbers
Incremental improvements of models	• Weighting of scores
	• Explanation of how and why scores were generated

Conjecturing

During mathematical investigations, mathematicians often engage in the process of conjecturing. For example, they often investigate "what-if" some of the assumptions underlying the solutions to the problems they are investigating are changed (Association of Symbolic Logic Association, 1998). This often can lead to the alternative means for solving the problems being identified and to the development of better mathematical models.

When the students generated their initial models in Phase 1, they awarded a score out of ten for each of the categories. They assumed that this was the "only correct way" by which to generate valid scores for each city. During the sharing of the initial models in Phase 2, a few of the students challenged this assumption and argued that because some categories were more important than other categories, different weightings should be apportioned to each of the categories. The whole of the class accepted this argument. Therefore, during Phase 3, each group of students experimented with different weightings for each of the categories and investigated how this influenced the rankings of the cities.

As they engaged in this "what-iffing" activity, the students often opened up new vistas for investigating the problem. For example, some of the groups relooked at their definitions of each of the categories and generated new definitions. This not only enabled them to better justify the sets of weightings they wished to adopt for their models, but also it gave them the option of modifying their models by revising definitions of categories rather than by revision of weightings. Whilst engaged in this type of conjecturing activity, the students gained deeper insights into the nature of ranking tables; many came to realise the subjective nature of ranking tables which in turn led them to question the validity of some ranking tables they had found in the mass media.

Collective Pursuit of Understanding

As the students engaged in the iterative revisions of their models, it was noted that each of the groups continuously delved deeper into the meaning of the scores their models generated for each category. This was reflected in the qualitative changes made to the explanations of how the models generated the scores for each of the categories.

During the earlier iterations of their models, each group of students usually just indicated the total number of points to be allocated for each category and what factors they had considered when generating the scores for each city. For example, Ayn's group initially provided the following explanation of their model's scoring schema for "Quality of Life":

> Quality of living is how good life is in the city: safety, education, multiculturism, leisure (recreation) etc. Score out of 10.

Like all of the other initial explanations provided by the groups, this explanation did not provide details about: 1) how the score for each city had been generated, or 2) what a score such as $^6/_{10}$ meant (e.g., is it a poor, average or quite a good score).

As they continuously received comments, propositions and questions from other groups of students and their teachers during Phases 3-4, the explanations were elaborated on and when necessary clarified so that the readers were provided with more details about how the models generated the scores and what the scores meant. For example, by its final iteration, Ayn's group had modified the explanation of their model's scoring schema for "Quality of Life" to:

> *Quality of living is how good life is in the city*: safety, education, multiculturism, leisure (recreation) etc.
> *How to Determine the Score*: If the city was in the top six of the twenty-four cities for its crime rates, then we gave it ten points. If the city was between seven and twelve for crime rates, we gave it seven and a half points. If the city was between thirteen and eighteen for crime rates we gave it a score of five. If it was between nineteen and twenty-four we awarded it two and a half points.
> We added on three points for good entertainment; two points for fairly good entertainment, and one point for poor entertainment. Next, we gave the city a score out of three for education based on the population. We divided the cities into categories based on the population: millions, hundred thousands, and ten thousands. Points were awarded, giving the cities with the highest number of number of secondary school graduates highest number of points in each of the categories. Three points were added for good multiculturalism, two for fairly good, and one for poor.
> The score is out of nineteen. /19

Another qualitative change that occurred to the explanations was that many of the groups utilised If-Then conditional statements in the later versions of their models to indicate that a score generated by their model for any category would be based not only on objective information (e.g., average temperatures etc.) but also on the personal preferences of the person using their system. A good example of this is provided by Mary's group when they explain how Ms Estrin could use their model to generate scores for "Environment:

> Environment means the climate, biome, and landforms. The climate should meet what Ms. Estrin likes. If she likes warm climates, maybe she could move to the prairies; if she likes cold climates, maybe she will like Iqaluit. The biome will depend on which part of the country you are. If Ms. Estrin likes a temperate biome, Toronto would be a nice place…

Incremental Improvement of Models

During the course of the study, the models developed by the groups of students were incrementally improved and became mathematically more complex and sophisticated. This was exemplified in the final versions of the models by the:

1. different methods utilised in the models to weight the scores for each category
2. complex schemes developed for scoring each of the categories
3. detailed explanations of the meaning of a score
4. inclusion of If-Then conditional statements in the explanations that enabled users of many of the models to utilise both objective and subjective factors when generating a city's score for a category.

This realisation by most of the groups that subjective factors play very important roles in ranking table models such as "Which is the Best City in Canada" was probably one of the major factors that led to the improvements made to the models.

Conclusions and Future Directions

The students in this study engaged in authentic, sustained, and progressive knowledge-building activity very similar to that which has been achieved in more content-rich discipline areas such as science. Indeed, the quality of their symbolizing, communicating, mathematizing and collective pursuit of understanding had much in common with that observed in many mathematical research communities.

Much of the success we had in establishing and maintaining the mathematics knowledge building community in this study can be attributed to the rich context for mathematizing provided by the problem. However, when analysing why this might be, we realized we needed to modify Lesh et al.'s (1999) six principles. Specifically, both Principle 1 (*personal meaningfulness principle)* and Principle 3 (the *self-evaluation principle*) need to be modified to address the collective goals of the community. Therefore we postulate that the personal aspects of principles 1 and 3 be modified as follows:

Collectively meaningful principle: Will members of the community be able to *collectively* make sense of the situation?

Negotiated collective-evaluation principle: Will all members of the community be able to judge for themselves when the responses are good enough? For what purposes are the results needed? by whom? when?

Further, the Knowledge Building community provided contexts and scaffolds for both intra- and inter-group discourse during the construction and iterative revisions of the models. It enabled the best ideas to surface, be explored and evaluated and then modified/improved. It also enabled the students to identify limitations in their models. For example, Kate and her group presented a model in which safety and environment received the highest weightings. They felt that they had created a final version of their model until it was pointed out to them that their model would rate cities like Flin Flon, in northern Manitoba (a mining community of approximately 7000 people at 54 degrees north latitude) higher than Toronto and Vancouver. Flin Flon has little crime and automobile pollution; however, it also has no movie theatre complex, no public transportation and no shopping malls! After receiving this feedback, Kate and her group rapidly and extensively modified their model to take cognisance of factors such as these. The community also provided the opportunity to see other groups demonstrating varying degrees of success. This peer-modelling promoted the thinking of all, provided examples of success and moved the class ahead--often overcoming impasses that can stall the efforts of groups working alone.

In the near future, students in an Australian school will be posed a similar problem where they will be required to generate a model for ranking Australian cities. Then via the means of *Knowledge Forum*®, the Canadian and Australian students will comment on one another's models and iteratively revise their models based on these comments. This will provide us with opportunities to engage in further analysis of inter-group data--with a condition that looks at cross-country/cultural configurations as a means of assisting us in refining the pedagogical approach (Woodruff, 2001) and also with opportunities to evaluate and verify/modify the altered set of principles for the design of model-eliciting mathematics problems that can provide the context for establishing and maintaining mathematics knowledge-building communities.

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Rethinking Curriculum: An Ethical Perspective

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It is argued, using Levinasian ethical philosophy, that dominant approaches in curriculum design and implementation are more than just an ineffectual system forced on teachers, they are also injurious to those who work in schools because they manipulate and erode a primary ethical relation between people. Although these approaches are used in several subject areas besides mathematics, mathematics is the key subject area for a more general critique of these approaches because it is the "paradigm case" subject area.

This paper follows from a recently completed research study into the application of ethical philosophy to mathematics education (Neyland, 2001), and applies the work of the ethical philosopher Emmanuel Levinas to the mathematics curriculum. Levinas is regarded by some commentators as the pre-eminent ethical philosopher of the 20th century (see, for instance, Nemo's introduction to Levinas, 1985). Levinas comprehensively analysed the ethical dimension of human interaction and published his findings in, among other places, *Totality and Infinity* (1969), *Time and the Other* (1987), and *Otherwise than Being* (1998). His ideas have been developed and applied by scholars of note such as Paul Ricoeur in *Oneself as Another* (1992), Richard Sennett in *The Corrosion of Character: the personal consequences of work in the new capitalism* (1998), and Zygmunt Bauman in his essay *Social Manipulation of Morality: moralizing actors, adiophorizing action* (1989a).

Dominant Approaches

In 1989 a newly published mathematics curriculum for England and Wales reformed the teaching of mathematics for these countries (Department of Education and Science, 1989). This curriculum statement divided school mathematics into 296 statements of attainment, organised into 14 parallel hierarchies, most having ten levels. Similar large-scale education reform of the whole school curriculum has recently occurred in a number of other industrialised countries (Levin, 1997; Smyth, 1995; Smyth & Dow, 1998). The reform programme in each of these countries has been similar because each has drawn on the same well of ideas (Levin, 1997). New Zealand, England and Wales have arguably been the recipients of the most thoroughgoing reform. The American researchers, Fiske and Ladd (2000, p. 3), for instance, write: "In 1989 New Zealand embarked on what is arguably the most thorough and dramatic transformation of a state system of compulsory education ever undertaken by an industrialized country". Thrupp, Harold, Mansell, and Hawksworth (2000), a New Zealand research team, however, believe that England and Wales received a more severe restructuring than New Zealand.

The driving thrust behind these education reforms is the determination by state and national governments to manage education scientifically with the aid of legislation. This began in 1969 with the United States Federal Government's National Assessment of Educational Progress (NAEP) project. Wise (1979) outlines how NAEP spawned a raft of state legislation based on the principles and concepts of Scientific Management Theory (SMT). The following examples are illustrative of how SMT influenced legislation in a number of states during this time. The 1969 California Education Improvement Act mandated that project grants were to focus upon basic skills and be evaluated in terms of

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student improvement and cost effectiveness. In 1971 the same state appointed an Educational Management and Evaluation Commission. Nine public members were to be appointed, three to represent the field of economics, three to represent the management sciences, and three to represent the learning sciences. In 1971 Colorado passed the Educational Accountability Act. This act introduced the term "accountability" into policy discourse in education, as a generic term, and referred to the legislative tools of scientific management in education. In 1973 Oklahoma passed an act which imposed systems analysis, originally developed to manage defence expenditures, upon school districts. It also introduced the term "needs assessment". In 1976 Florida's Educational Accountability Act mandated the specification of a system which guarantees that each student will attain a specified standard. It also specified the contingencies should a student fail to attain the minimum performance standards. In the teaching of mathematics, such policy developments resulted in outcomes approaches to mathematics teaching—the ones that behaviourally oriented curriculum theorists had earlier been experimenting with—becoming mandatory. These developments in mathematics also gave rise to attempts to resist the reform programme in mathematics. The National Council of Teachers of Mathematics' *An Agenda for Action* (1980) is a striking example.

Scientific management approaches in education were later adopted elsewhere. In the United Kingdom, for example, Skilbeck (1976) observed that a new educational model, the "technocratic-bureaucratic ideology", was gaining in influence in education. In New Zealand, in the New Zealand Treasury's (1987) briefing paper on education, education is discussed in terms of a "production function" with "inputs" and "outputs", and the various curriculum statements that resulted from the subsequent education reforms were expected to fit with this conception. The first subject to be reformed was mathematics, in *Mathematics in the New Zealand Curriculum* (Ministry of Education, 1992).

There are four requirements that go hand-in-hand with the decision to legislate within education according to the principles of SMT. The legislator requires: (i) a statement of unambiguous *outcomes* (to clearly specify what teachers are required to produce), (ii) a theory of *compliance* (this gives the legislator confidence that teachers will act as directed), (iii) a system of *auditing* (for ongoing monitoring), and (iv) the provision for *instrumentally-oriented research* (to supply precise data to aid rationalistic decision making). The outcomes-based curriculum is intended to provide the first. Contemporary variants of contract theory provide the second (Boston, 1991). In New Zealand, the Education Review Office is the statutory auditing agency. Dependable information necessary for the application of SMT to education is understood to be supplied by educational consultants, researchers under contract to the ministry, and officials who monitor national assessment data.

Mathematics as the Paradigm Case

It has been noted that, in New Zealand, mathematics was the first subject to be restructured. It was written even before the umbrella document, *The New Zealand Curriculum Framework* was published (Ministry of Education, 1993). It is probable that one of the main reasons for this is the fact that mathematics is the subject most easily revised as a hierarchy of outcomes, and therefore, in the minds of some, the logical subject with which to begin. The fact that mathematics is the curriculum subject most amenable to this form of presentation, however, is not to be taken to imply that mathematics should be used as an exemplar for how other subjects can be restructured as a hierarchy of outcomes. The opposite is the case. Mathematics is in fact what I call the "paradigm case" subject because, while it is accepted that mathematics is the subject most suited for presentation hierarchically, this cannot be done without unacceptable distortion of its disciplinary character. Thus, mathematics is the pre-eminent subject that can be used to argue a general case against the scientific management of the curriculum. Accordingly, mathematics education research refuting the outcomes-based curriculum in mathematics is of crucial importance for all subjects in the curriculum, and mathematics education researchers have this unique role to play on behalf of other subjects.

Macro-Level Critique

Features of these educational reforms may be usefully categorised as being either macro-level or micro-level. The former term refers to both the processes used in setting up the "reformed" education system, and the consequences for curriculum design, pedagogical style, and student achievement. The task of analysing the macro consequences has been made relatively straightforward because the reforms have a common source. The latter term refers to the post-reform consequences, for those who work in education, at the level of human experiences and social interactions. Identifying these micro consequences is fraught with difficulties for two reasons. First, because restructuring policy vastly underestimates the complexity of schooling (Thrupp *et al.*, 2000), and because mainstream education research misconstrues the nature of teaching—since it is dominated by cognitive and behavioural conceptions (Nias 1998; Gewirtz, 1997)—research into its impact is susceptible to a tendency to mimic this weakness. Second, the reform process results in the reformees experiencing difficulty accounting for, and articulating, its consequences for them personally. Menter, Muschamp, Nicholls, and Ozga (1997), for instance, argue that the reform process manufactures consent to the reforms, and that this leads to an under-representation of such things as experiences of alienation and deprofessionalisation.

At the macro-level the reform process has been criticised for (i) the hostile nature of the discourse that accompanied it (Ball, 1990; Ellerton & Clements, 1994), (ii) the anti-democratic processes that characterised its introduction (Lauder, 1991), and (iii) the assumptions that underlie the epistemological, ontological and social theories used to justify these approaches (Boston, Haig, and Lauder, 1988). The reforms have been criticised for (i) the over-extensive use of hierarchies to organise and teach mathematics (Ernest, 1991; Freudenthal, 1978, 1979), (ii) the ineffectiveness of outcomes approaches for teaching mathematics (Kuchemann, 1990; Kulik, Kulik, & Bangert-Drowns, 1990; Erlwanger, 1973), and (iii) the idea that standards of achievement can be precisely defined (Neyland, 1998).

Micro-Level Consequences and the Hidden Injuries

An investigation of the micro-level consequences of the reforms involves a recognition that teaching is, in part, a *felt*, and not just a *technique-oriented*, occupation. Nias (1998) is one who emphasises this dimension of teaching. She observes that teaching has an emotional component that includes "the inherently moral nature of teaching and the *sense of responsibility* that springs from this; [and] allegiance to the *ethics of care and service*" (p. 424, emphasis added).

Recently, some empirical researchers have made a first attempt at identifying the micro-level consequences of scientific management in education. Ball (1997), Gewirtz (1997), Helsby (1999), Menter *et al.*, (1997), Reay (1998), Thrupp *et al.*, (2000), Woods, Jeffrey, Troman, and Boyle (1997), and Wylie (1999), have turned up evidence that suggests that in the post-reform period, social bonds are being weakened and people are being damaged. These empirical studies report that in the aftermath of recent educational restructuring, the following have occurred: (i) "the nature of the tasks that are absorbing increased quantities of teacher time and emotional labour and the climate of surveillance within which those tasks have to be carried out" has changed for the worst (Gewirtz, 1997, p. 224); (ii) professionalism has been replaced by accountability; and, collegiality by competition and surveillance (Ball, 1997); (iii) initiative, creativity and teacher-led innovation have been constrained (Woods *et al.*, 1997); (iv) teaching has become technicised, and learning experiences impoverished (Woods *et al.*, 1997); (v) the workforce has become overworked, stressed, anxious, shell-shocked, distrustful, cynical, demotivated and alienated (Thrupp *et al.*, 2000); and (vi) teachers pragmatically respond to the "excessive demands of reform" by neglecting what they believe to be the more important components of their work, and doing "what is required of them by the managerial imperatives of the central agencies or the market imperative of local competition" (Thrupp *et al.*, 2000, p. 208).

Helsby (1999) reports that there is now a tension between the ethical professionalism of the pre-reform period, and the technocratic managerialism of the reformed period. In her view this is a result of a "transformation of teachers' work and a mutation of their role from semi-autonomous professionals to managed expendable employees" (p. 172). In Reay's (1998) view it is a result of the "discursive working of the new managerialism [that] constructs all teachers as managers" (p. 188). Woods *et al.* (1997) found an overall erosion of teacher professionalism caused by the managerialist imperative. They also note that while many do resist the downgrading of their role as professionals and its replacement by that of the technician, this resistance leads to teacher burnout, especially among creative teachers.

Gewirtz (1997) reports that the reforms have changed vertical relationships between management and staff, horizontal relations among staff, and relationships between teachers and students. In general there is "pressure for intimate and complicated relationships to give way to the "social relations of the production line"" (p. 226). Reay (1998) saw evidence of the senior management team being isolated from the rest of the staff who see it as "insular" and "exclusive" and "autocratic". She writes:

> Relationships between junior and senior staff are pared down and perfunctory within the new managerialism. Communication between staff is increasingly dominated by top-down, vertical interactions in which junior staff are informed about decisions rather than being part of them and cursory consultation rather than negotiation is increasingly the order of the day. (p. 192)

It is further reported that teachers are experiencing less collegiality with their colleagues at the same school (Wylie, 1999), a decline in the sociability of school life, meeting times being taken up working to agendas set by senior managers, less opportunity for collaboration with colleagues, and less informal socialising amongst staff (Gewirtz, 1997). Teachers also claim that they have less time to give attention to individual students, and are not getting a better picture of individual students' learning needs despite doing more assessment (Wylie, 1999).

Gewirtz (1997) observed that teachers often talk of such things as the "manic grind" and "frenetic pace" of work, and of being "squeezed dry".

> [Most] teachers resent having to spend so much of their time recording and monitoring students' work in the formal way now required of them. Such work is viewed as a distraction from what they see as the real work of teaching. [They] resent having to expend energy working to objectives, targets and requirements set elsewhere, particularly where those objectives conflict with their own. (pp. 224-225)

Further, they "are having to expend enormous quantities of energy [protecting] their students and what they consider their real work from work which is imposed on them" (Gewirtz, 1997, p. 225). Ball (1997) writes that he saw evidence of one teacher experiencing a sense of personal meaninglessness in the context of teaching as a result of scientific management approaches creating an *"existential separation"* (p. 327) from the moral resources necessary for a satisfying existence. "[Teachers are] exhausted by additional work and, in some cases, *alienated from their selves and their colleagues*" (p. 334, emphasis added).

It is evident, then, that some people in schools are experiencing significant negative consequences of the reforms. There is even Ball's tentative suggestion that people may be being damaged as selves. As we will see, by drawing on the ethical philosophy of Levinas, there is cause to conclude that there is good reason to believe that Ball's tentative suggestion may have real substance.

Towards an Ethical Alternative

Scientific management approaches provide a means of control over the process of education, and, for this reason, they typically maintain an appeal to us. They allow us to act decisively in situations that otherwise may appear beyond the range of our intervention strategies. For example, in order to try to improve the teaching of mathematics we specify norms and professional standards with the aim of bringing about improvements (e.g.: *Professional Standards for School Mathematics*, 1991). However, when we make this step, we tacitly accept that it is desirable for the actions of teachers to be *mediated* by such protocols. Unfortunately, there are grave risks associated with this because, as is evident in the empirical research just reported, the mediation of action inevitably involves the suppression of direct or *unmediated* action by teachers in response to the immediacy of the needs of students. A major problem with the scientific management of education, then, is the fact that it is designed to both (i) shape teacher actions according to a blueprint, and (ii) curtail direct action.

In order to construct an alternative conception of the primary motivation for teacher action it is useful to identify a fundamental assumption upon which scientific management approaches are based. These approaches are based on the assumption that moral and socially sanctioned human action is *caused* through the taming of pre-social and animal-like instinct by social institutions, rules, norms, and educative processes; including the professional and curricular standards used in mathematics teaching. Bauman (1989b), following Levinas, argues that this assumption is a "myth" that ought to be replaced. This myth, he shows, is deeply embedded in the modern sensibility and sustained by a broad coalition of "respectable" opinion which includes the view that the modern state has a "gardening" function; that is, that "the society it rules [ought to be viewed] as an object of designing, cultivating and weed-poisoning" (pp. 12-13). Failures within the education system are, according to the gardening notion of the state, and the central myth upon which it is based, the result of a breakdown in the specification of rules, standards and protocols. Scientific management approaches are used to deal with such failures.

Levinas proposed an alternative to the fundamental assumption above about the causes of human action. For him, social institutions and norms do not *cause* ethical behaviour, they *manipulate* and sometimes *erode* a primordial ethical relation between people. What is this fundamental ethical relation? It is a relation of responsibility for the other person. A trace of this relation is evident, to use one of his favourite examples, when I communicate directly with another person. When I speak to another person I am first *acknowledging* him or her as another person. Thus, as he puts it, before every "said" there is first of all a "saying". When I acknowledge another person, when I focus on his or her "face", I do more than just gaze, I actually *encounter* him or her. This encounter, Levinas argues, is, at its deepest level, an awareness of the other as one who in some way needs me. This, according to Levinas, is the source of the social bond. He emphasises that there is no compulsion involved. I am not obliged to respond to the other. I can choose to break the encounter. But if I do, I weaken the social bond. Further, because my selfhood—my self-concept and self-identity—depends on my responding to the need I recognise in another, when I break the social bond, I impair my selfhood.

So, from Levinas' ethical perspective, teacher action ought to be, at source, an overflow of this primary ethical relation. It ought to be, in the first instance, a direct response to the need I see in another. Priority, then, should be given to this relation of responsibility as a determiner of action. Priority should not be given to society's gardening function. In education, therefore, priority should not be given to the mediation of action. When the priority of the ethical relation over a prescriptive and moralising function is abandoned, as it is with the scientific management of education, the ethical force of the direct interpersonal relation is, at best, neutralised, and at worst, eroded. It is not unreasonable to infer, based on the empirical research just outlined, that we are in the midst of the latter scenario.

Social institutions and professional protocols do have a function. We cannot do without such social agreements. Levinas emphasises that it is not a question of one or the other—direct action or mediated action—it is a question of which ought to be accorded priority. For Levinas, protocols should *complement*, but not *precede*, the force of the direct ethical relation. The crucial question is, does a given protocol enhance the primary relation of responsibility, and support the direct action that flows from it? If this can be answered in the affirmative, then the ethical and the societal reside in an appropriate complementary relation. If not, we risk human injury.

The Erosion of Proximity

What causes the erosion of the primordial ethical relation and hence injury? The interpersonal relation of responsibility is maintained by what Levinas calls ethical *proximity*. He outlines what he means by this in the third chapter of *Otherwise than Being* (Levinas, 1998). Proximity is not a spatial, cognitive or emotional closeness. It exists insofar as I feel, or am, responsible for the other person. Ethical proximity can be undermined, and, as will be indicated shortly, the mechanisms that are central to SMT do this. As has been noted, scientific management approaches are built upon a belief that education is a raw material for social engineering; an object to be shaped and administered. The scientific management of education is geared towards rationalised processes of change, the meeting of targets, standardisation, the specification of outcomes, and the disciplined observance of protocols.

How is proximity eroded by the scientific management of mathematics education? Kelman (1973) argues that ethical relations are weakened when three conditions are met:

(i) particular procedures are *authorised*, (ii) actions are *routinised*, and (iii) people are *dehumanised*. These conditions have increasingly been met in contemporary scientifically managed mathematics education. There is, for instance, increasing standardisation and measurement of both the work of mathematics teachers and the achievements of their students. In addition, students and teachers are increasingly dissembled into aggregates of isolated traits in such a way that it is difficult to re-assemble the face of the human individual. Students, for instance, are categorised as either possessing, or not possessing, certain capacities and attributes, and identified on complex hierarchies of levels of mathematical outcomes. Teachers are viewed, according to SMT, as contractual agents performing a carefully specified and monitored service within education, conceived as a production function.

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Year Seven Students' Representation of Numerical Data: The Influence of Sample Size

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Ten children in Year 7 of primary school were given two numerical data sets to represent graphically – one with 10 pieces of data, and one with 30. Increasing the size of the data set had little effect on the children's organisation of the data. In general, the children were no more likely to represent the larger data set in an organised form than the smaller data set. The majority of the children represented the smaller data set without any reorganisation. Similarly, the majority of the children required prompting to organise the larger data set. Mathematical ability did have some effect in that some of the more mathematically able children found it easier to organise the data than their less able counterparts. Possible explanations for the results are explored and the implications for teaching and the curriculum are discussed.

This paper concerns the ability of upper-primary children to represent numerical data in a graph, and explores factors which assist children to organise the data before drawing the graph. The term numerical data refers to counts or measures (e.g., number of CDs bought), in contrast to categorical data which refers to categories (e.g., eye colour).

The ability to draw an organised graph is one in a suite of skills expected of all students according to recent curriculum documents. The Australian Numeracy Benchmarks (Curriculum Corporation, 2000) include the ability of primary school students to organise, summarise, and display information in graphs. Similarly, the National Council of Teachers of Mathematics Standards (NCTM, 2000) has highlighted the need for students at all levels to be able to organise and represent data.

Research into children's ability to draw graphs has included the development of a framework for statistical thinking by Jones, Thornton, Langrall, Perry, & Putt (2000) (Framework). The third construct in the Framework - Representing Data - is the main issue under consideration in this study, and incorporates constructing representations that exhibit different organisations of the data. As with the other constructs, four levels of thinking have been proposed for this construct. The levels are defined by statements describing students' data displays in terms of the validity of a display when asked to complete a partial graph, and the degree of reorganisation of data when asked to produce a display. The evidence obtained in this study relates to the latter – the degree of reorganisation of data shown in the display.

According to the Framework, at Level 1 the student produces an idiosyncratic display that does not represent the data set. At Level 2, the student produces a display that represents the data but does not attempt to reorganise the data. At Level 3, the student not only produces a display that represents the data but also shows some attempt to reorganise the data. At Level 4, the student produces multiple valid displays, some of which reorganise the data.

Research in the area of students' representation of data is not extensive, however a small number of studies provide some background for this study. Lehrer and Schauble

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B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 520-527). Sydney: MERGA. ©2002 MERGA Inc.

(2000) investigated the process of data organisation with elementary school children in grades 1, 2, 4 and 5. They examined how these children developed and justified models to categorise (by grade level) drawings made by children in the same grade levels as themselves. Their results suggest that, at higher grades, children use more sophisticated strategies for organising data.

Nisbet (1999) examined the representations of categorical data generated by teacher-education students. The majority (99%) drew representations of the data showing some reorganisation of the data. However, the data was categorical, not numerical. Nisbet, Jones, Langrall and Mooney (submitted) analysed primary-school children's representations of categorical and numerical data. The study revealed that numerical data was significantly harder for children to organise and represent than categorical data. Children beyond Grade 1 can make connections between organizing and representing data when the data are categorical but generally not when the data are numerical. Whereas 60% exhibited Level 3 thinking with categorical data by reorganising the data, only 20% exhibited Level 3 thinking with numerical data. Two of the three Level-3 thinkers produced a tally table while the third drew a pictograph.

Another study (Nisbet, 2001) found that teacher-education students had similar difficulties with organising numerical data. All could produce an organised graph from categorical data, but only 19% could produce an organised graph from numerical data. For the latter, the majority of students merely drew separate bars for each individual piece of data without organising the data into numerical categories.

Why do more students find it difficult to represent numerical data in an organised way, compared to categorical data? It could be that the way to organise categorical data is obvious, but less obvious for numerical data. Maybe the need for organisation is not perceived to be great when there are only 10 items in the data set. Perhaps, if the data set was made larger, then the students would be more likely to see the need to organise the data, and subsequently draw an organised graph based on numerical categories. To test this hypothesis Nisbet (2002) presented students in Years 9 and 11 with two data sets, one small (10 pieces of data) and another larger (30 pieces of data) and asked them to represent the data graphically. With the small data set most students drew graphs showing no organisation of the data. However, increasing the size of the data set to 30, lead more students to organise the data and draw a representation based on number of times a value was recorded rather than individual values. There was an effect of mathematics ability with the Year 11 students. Most of these students in the higher ability group were able to organise and represent the data from the larger data set in an organised way without any prompting. However, only one student in lower ability group drew an organised graph without any prompting. There was no similar ability effect for the students in Year 9.

The current study was therefore designed to test the same hypothesis with upper-primary children. Hence in this study, children in Year 7 were given the task of drawing two graphs – one graph to represent a small data set (n = 10), and second graph for a larger set (n = 30). The Framework was used to evaluate the graphs produced by the children's organisations and representations of the data, and an interview protocol was employed to ascertain the extent of prompting required before children realised that the larger set needed to be organised before it could be represented graphically.

Method

Participants

The target for this study was the upper-primary level, because this age group had not been included in previous studies on this topic by the researcher. A sample of 10 children at a suburban primary school was drawn from one of the Year 7 classes; their mathematics abilities varied across the range of ability, high, medium and low.

Tasks

Participants were given two tasks. The first required them to draw a graph representing the information in the following scenario.

> Ten students were asked about the number of novels they read during the term. These are their answers.
>
> **NUMBER OF NOVELS: 5, 4, 1, 7, 5, 0, 3, 4, 5, 6**
>
> Draw a graph which represents the data.

The numbers were presented in a vertical list.

After the children had drawn their graphs, they were asked the following questions:

> What sort of graph did you draw? and Why did you draw it that way?

The second task required the children to draw a graph representing this information.

> Thirty students were asked about the number of CDs they bought during the year. These are their answers.
>
> **NUMBER OF CDs: 2, 4, 2, 7, 5, 0, 3, 4, 5, 1, 5, 4, 1, 7, 5, 0, 3, 4, 5, 6, 3, 4, 8, 6, 3, 2, 3, 4, 5, 6**
>
> Draw a graph which represents the data.

The numbers of CDs were presented in a vertical list. [Note that the data did not include names or any identifying information of the people who bought the CDs. This was done so that the children would see any loss of such data in the reorganisation process.]

If a student had drawn an organised graph successfully, he/she was asked:

> What sort of graph did you draw? and Why did you draw it that way?

Table 1

Prompt No. 2 – Blank table of values (No. of CDs)

No. of CDs									
No. of people									

If the student was experiencing difficulty in working out what to do, a protocol comprising a series of prompts was available, and the extent of prompting necessary for the student to embark on the correct course of action was noted.

1. How many students bought no CDs? How many bought 1? Does that help with your graph?
2. Could you fill in a table of values like this?(Show blank table of values as in Table 1)
3. Could you draw a graph with this table of values? (Show completed table of values.)

If the first three prompts were not sufficient to assist the child to draw an organised graph, the child would be shown a graph of the data (Figure 1) and asked the following question.

4. Here's a graph drawn by someone else. What does it tell you?

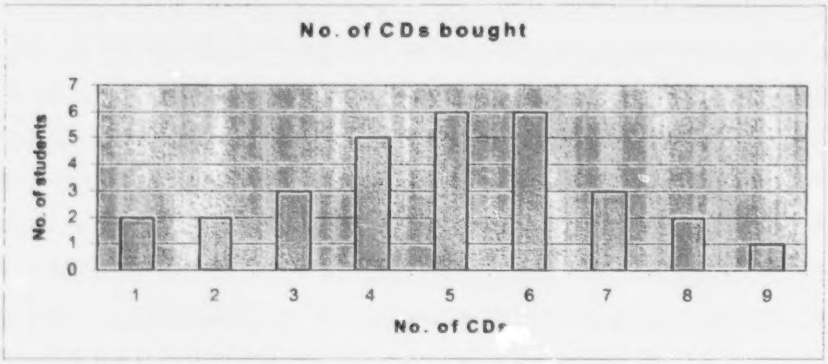

Figure 1. Graph shown in Prompt 4.

Each child was interviewed individually in a separate room away from any distractions of the class. All materials including sheets of graph paper, rulers, pencils and erasers were supplied by the researcher. The researcher kept notes of the answers given to the questions in the protocol, the amount of prompting required, and other comments made by the children in the interviews. All graphs drawn by the children were collected by the researcher for analysis.

Results

Overview of Results

The effect of size of data set. Increasing the size of the data set seemed to have little effect on the children's propensity to re-organise the data. With the smaller data set, the majority of the children drew a bar graph with separate bars for each piece of data. With the larger data set, the majority of the children needed prompting to group the data.

The effect of mathematics ability. There was a limited effect of mathematics ability shown and it was more evident with the larger data set. The children in the high mathematics ability group required fewer prompts to reorganise the data.

The effect of order of tasks. There appears to be no effect of order of tasks on the likelihood of the children to organise the data.

Results in Detail

The effect of size of data set. Increasing the size of the data set seemed to have little effect on the children's propensity to re-organise the data while drawing a graphical representation. Only a small proportion of the children was able to reorganise the data in both examples. One child out of the ten reorganised the smaller data set, and two out of ten reorganised the larger data set without prompting.

With the smaller data set concerning number of books read by students (n = 10), nine out of ten children drew bar graphs with individual bars – one bar for each student (Figure 2). These nine graphs were assessed to be at Level 2 according to the Framework because they did not show any reorganisation of the data. The other child drew a graph of grouped data without hesitation, showing how many students read zero books, how many read one book, two books, etc. (Figure 3). This graph was assessed to be at Level 3 because of the reorganisation of the data.

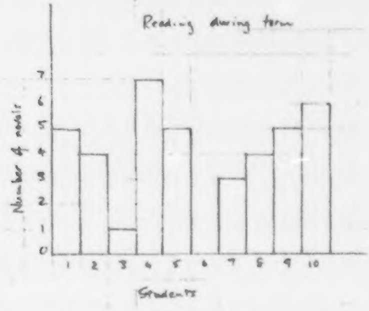

Figure 2. Typical Level 2 graph of smaller data set (number of novels read).

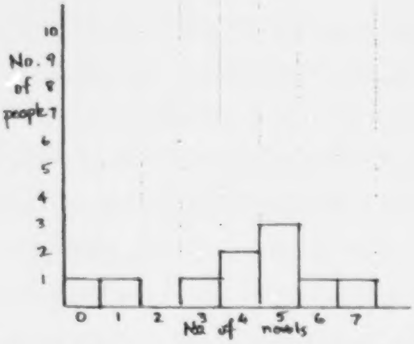

Figure 3. Level 3 graph of smaller data set (number of novels read).

[Note that the graphs shown in this paper have been redrawn by the researcher, as the originals by the children were drawn in pencil and did not scan well enough for reproduction.]

With the larger data set concerning the number of CDs bought by students (n = 30), eight out of ten children needed prompting (as described in the method section) before they thought of grouping the data according to how many CDs were bought. Seven of the graphs were valid and were assessed at Level 3 because they showed a reorganisation of the data The eighth graph was assessed at Level 2 because it was not a valid display (Figure 4). The other two children were able to draw an organised graph without any prompting at all. These two graphs were assessed to be at Level 3, but the difference with these two children is that they produced the organised graphs without any prompting (Figure 5).

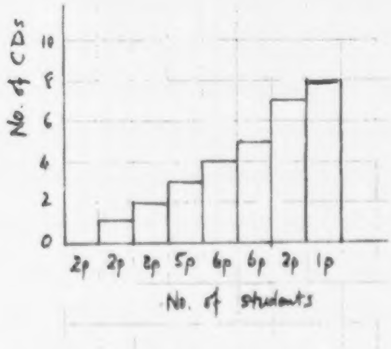

Figure 4. Invalid graph of larger data set

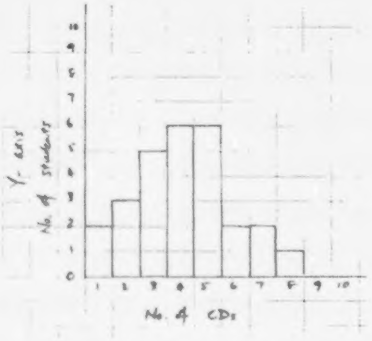

Figure 5. Level 3 graph of larger data set

The effect of mathematics ability. The effect of mathematics ability was demonstrated only to a limited extent. With the smaller data set, the only child to draw an organised graph was a child of high mathematics ability. However, the two other high mathematics ability children drew graphs without reorganisation, similar to the children of low and medium mathematics ability (Table 2).

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Table 2
Number of Children and Level of Thinking with the Smaller Data Set by Ability Group.

Mathematics ability group	Level of thinking (representing data) 1 (Idiosyncratic)	2 (Transitional)	3 (Quantitative)	Total Children
Low	0	3	0	3
Medium	0	4	0	4
High	0	2	1	3
Total children	0	9	1	10

The effect of mathematics ability. This was more evident with the larger data set. Although the number of children in each ability group is small, the average number of prompts for each group gives an indication of the general trend: the more mathematically able children required fewer prompts to reorganise the data than their less able counterparts. The average number of prompts was 0.33 for the high ability group, 1.75 for the medium ability group, and 2.67 for the low ability group (See Table 3). The only children to draw an organised graph without any prompting were high ability children.

Table 3
Number of Children Requiring Prompts for Task 2 by Ability Group.

Mathematics ability group	Number of prompts to organise the data 1	2	3	4	Average no. of prompts
Low	1	0	1	1	2.67
Medium	2	1	1	0	1.75
High	1	0	0	0	0.33
Total children	5	2	2	1	1.9 (overall)

The effect of order of tasks. There appears to be no effect of order of tasks on the likelihood of the children to organise the data. Towards the end of the interviews with the children, the researcher thought about the possible effect of order of tasks. Perhaps the smaller data set and no apparent need to reorganise was influencing the children to maintain a 'no reorganisation' stance. So the final two children were given the larger data set before the smaller set. However both of these children required prompting to organise the larger data set. Then, having completed the organised graph for the smaller data set, they reverted to an unorganised graph (bars for individual people) when presented with the smaller data set! This effect needs further investigation.

Other Points Noted During the Investigation

The mechanics of drawing graphs. All the children appeared to be very familiar with the mechanics and conventions of drawing simple graphs. When drawing the graph for the smaller data set, there was little hesitation in drawing the axes and deciding that the number of novels went on the vertical axis (referred to confidently by one child as the Y-axis). One child did label the vertical axis as the 'number of students' but then quickly changed it to 'number of novels'. Two children omitted zero from the scale of number of novels, but inserted it when they realised from the data that it was necessary.

Alternative ways of reorganising the data. Although most children were able to reorganizing the larger data set and draw a graph of number of people (vertical axis) against number of CDs (horizontal axis), albeit with prompting, three children came up with alternative ways of reorganizing the data. One child made 15 pairs from the data (n = 30), and drew a bar graph with 15 bars – one for each pair of students. Another combined the numbers of CDs for each number bought; he drew a bar of height 6 for the two students who bought three CDs, a bar of height 24 for the four students who bought six CDs, and so on (See Figure 6). A third child attempted to group together the students who bought the same number of CDs. However, the graph that he drew was not a valid display (Figure 4). It was merely a sequence of bars showing the number of CDs bought without any representation of the number who bought each number of CDs.

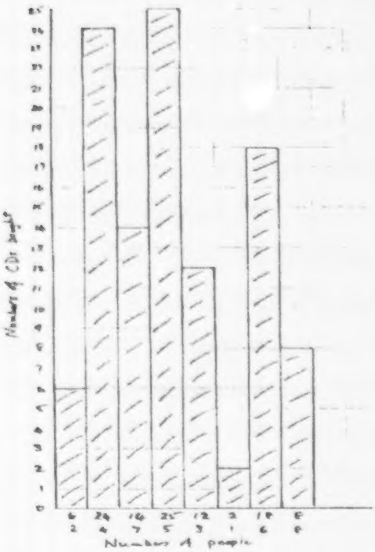

Figure 6. An invalid display.

No table of values. All of the children who required only one prompt to organise the larger data set omitted the drawing of a table of values in the process of generating the organised graph. Instead, for each value of numbers of CDs, they looked down the list of numbers to count how many people bought that many CDs. I assume that these primary children did know about tables of values, because most of those who were shown the table of values in prompt 2 recognised it. However, that step was omitted, leaving the children open to a possible source of error.

Language. It is possible that there exists an ambiguity in the phrase 'number of students', or 'number of children'. For instance, one of the children who drew an unorganised graph for the smaller data set (novels read), wrote 'no. of children' along the horizontal axis, along with the digits '1' to '10' when these digits actually referred to an implied student number, i.e. 1st student, 2nd student, etc.

Discussion

The minimal effect of size of data set on Year 7 children's propensity to reorganise the data raises some questions about (a) the formal primary school curriculum in data handling, and (b) the data handling activities that are given to children throughout the primary grades. It is hoped that the current rewriting of the mathematics syllabus in Queensland will place greater emphasis on data organisation, and its role in the early steps of preparing data for representation. Reference to the Framework by Jones et al (2001) in particular to the constructs 'Organising and reducing data' and 'Representing data' would highlight the importance of having children group data into classes and articulate the basis of the groupings.

Regarding the data-handling activities employed in primary classes, one can only speculate that a mechanistic approach (Ernest, 1989) involving the use of rules and formulae and data found in a textbook may be prevalent in schools. A more beneficial approach may be the dynamic approach (Russell & Friel, 1989) in which students investigate an issue of interest to them and collect their own data, then analyse it, organise it and represent it. Students need experience in organising their own data and representing it, rather than representing data already collected and organised by a textbook author. It would seem worthwhile for students to organise their data by making rank orders, drawing line plots and stem-and-leaf plots, and making tables of grouped data. Further research, in the form of teaching experiments, needs to be carried out to ascertain the impact of such learning activities on children's ability to draw an organised graph.

The fact that the only children to reorganise the data without prompting were high mathematics ability children seems to imply that a component of mathematics ability is the tendency to organise information. However, not all children identified by the school as high mathematics ability did reorganise the data. This issue needs further investigation with respect to both sides of this issue – the defining of high mathematics ability and the ability of teachers to correctly identify children with such ability.

Another point worth considering is the fact that the equivalent study conducted with secondary students (Nisbet, 2002) revealed a greater effect of increasing the size of the data set especially with high mathematics ability students. Further studies could investigate whether this is a curriculum issue or a developmental issue.

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The Role of Models and Representations in the Development of Multiplicative Thinking

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An important aspect of concept development in primary school students is the development and use of models, symbols, and representation. This paper deals with conceptual development of notions of multiplication and ways primary school students use representations such as equal groups, arrays, region models and stories to extend primitive or simple conceptions of multiplication to more complex, sophisticated ideas leading to more effective transfer to examples involving decimal fractions.

Research about multiplication has addressed comprehensively the development of multiplication concepts in early primary school students (Davydov, 1992; Anghileri, 1989; Mulligan & Mitchelmore, 1997; Clark & Kamii, 1996), including the way concrete materials are used as models in this development. Other studies have focused on some misconceptions (Tirosh & Graeber, 1989) that have been attributed in part to the influence of primitive understandings and models generalised from whole numbers to decimals. Greer's (1992) classification of multiplicative situations, referred to in many other studies (e.g., Verschaffel & De Corte, 1996) is accompanied by pictorial illustrations of external representations for the operations, showing a variety of possible models. This paper draws on these studies and on other research that expands on the use of models and representations in the development of more sophisticated understandings of multiplication (Steffe & Cobb, 1994).

The aim of this paper is the exploration of children's understanding of multiplication of whole numbers, how models can be useful in building understandings of the operation, and how this understanding helps or hinders an extension of multiplication to deal with decimals. It is part of a larger study which was set up in response to suggestions by Greer (1992) "to improve teaching of multiplication concepts" (p. 292). We need first to know how children understand multiplication of whole numbers, how this would be useful in extending understandings for use with decimals, and what intervention would be successful in helping students to move forward in this area. Graeber and Tirosh (1990) had earlier investigated the concepts of multiplication and division held by a group of primary students in Israel and the United States of America. They also investigated how these concepts affected development in the area of decimal fractions. Their study showed that the commonly held beliefs that "multiplication always makes bigger, division always makes smaller" (p. 565) were evident "as early as fourth and fifth grades" (p. 582).

A teaching program was devised by the author of this paper to allow students to develop a more sophisticated understanding of multiplication and so move more comfortably into the area of multiplication with decimals. This program was used as the basis for a larger study by the author. Some of the tasks in the Graeber and Tirosh study (1990) related to models for teaching and interpreting multiplication. Through the teaching sessions and the follow-up interviews, it was hoped that the role these models played in developing understanding of multiplication would be better understood. This aspect of the study, related to models and representations, will be the focus of this paper. The following

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questions will be addressed:

1. To what extent can upper primary school students' understanding be classified in terms of documented concepts of multiplication, that is, groups of, arrays, region, area, factor/factor/product?
2. To what extent can students' primitive or naïve views of multiplication be impacted by a specifically designed intervention program?
3. What models of multiplication do the students hold that may be enabling or disabling as they attempt to make sense of multiplication involving decimals?

Method

Participants and Procedure

The interview sample comprised fifteen Year 5 students, aged 10-12, five boys and 10 girls. All students attended the same inner Melbourne primary school. They were from three different classes. Three students had a language background where English was not the main language spoken at home. The students had been instructed in basic multiplication from Year 2 and in decimals and fractions from Year 4.

The study was conducted as a teaching experiment. The main data was collected from interviews. The interview protocol was a section from Graeber and Tirosh (1990). It was used to establish understandings and conceptions that students already had about multiplication and decimals. Interviews, using the same questions and format, were conducted after the teaching sessions as well. At each interview the student was asked a series of questions relating to the four major areas to be covered. This paper is concerned with only two of these, models in the domain of whole numbers (Tasks 2, 3 and 4) and facility with decimals (Task 7) (see Figure 1). The students were interviewed before the intervention program. Information from the initial interview was analysed, and an intervention program of six lessons (to be delivered over six weeks) was designed and developed from data collected. The follow-up interview was conducted immediately after the last of these lessons. A further interview was conducted four months later.

Each interview was approximately 20 to 25 minutes. Students used blocks, pens and plain and grid paper. Responses were recorded, including attitudes and behaviour, and students' written responses were kept. Each question from the multiplication protocol was presented in both written and oral form. If any student experienced difficulty with the tasks, the interview was stopped.

Results were analysed according to the students' understandings of multiplication, based on the Graeber and Tirosh (1990) categories and other categorisation according to types of multiplication situations as outlined by Greer (1992). The categories for analysis were as follows:

Category A: Confusion with task requirement.
Category B: Equal groups, without showing commutativity.
Category C: Equal groups, showing understanding of commutativity.
Category D: Array presentation, without showing commutativity of model.
Category E: Array or region representation showing understanding of commutativity.
Category F: Models reflecting multiplicative comparison /Cartesian product/rate.

Tasks 2:	(Students given cubes or blocks all of one colour and shown the sentence *7x2=14)* *Can you use these cubes to show me 7x2=14?* *Can you show me another way?* *Another way? (looking for sets and arrays)*
Task 3	*(Students given a worksheet with cm square grid and shown the phrase 3x4.)* *What is 3x4?* *Can you use this (the grid) to show me 3x4?* *Can you show it another way?*
Task 4	*(Students shown the number sentence 2+4=6.* *The written story problem, and the number sentence 5x6=30.)* *If I showed you 2+4=6 and asked you to tell me a story problem about it you might say: "Joe had 2 biscuits in the afternoon and 4 biscuits after dinner. How many did he have altogether?' This (point) is an addition sentence and an addition story. Six (point) is the answer to the question. We have been working on multiplication. This (point) is a multiplication sentence.* *Can you tell me a multiplication story problem for 5x6=30? You can write it or just tell me, whatever is easier. Take your time.*
Task 7a	*15x.6=?* *Can you tell me what the hidden number is?* *Can you tell me anything about it? Why do you think so?*
Task 7b	*1x10=?* *Can you tell me what the hidden number is?* *Can you tell me anything about it? Why do you think so?*

Figure 1. Tasks for the interviews.

The teaching program was designed particularly to introduce and allow exploration of (a) array and area or region models of multiplication; (b) discussion of the multiplication symbol and verbalisation; (c) models of multiplication situations where one factor is a whole number and one a decimal, and where both factors are decimals; (d) word problem representations (whole numbers by whole numbers and whole numbers by decimals); and (e) understanding of factor and product.

Results

Table 1 summarises the responses of all 15 students to Tasks 2, 3 and 4 at the three interviews, in terms of categories A to F. This first section reports the responses to these tasks separately. The second section reports the responses to Tasks 7a and 7b, and then links these to the responses to the models tasks.

Task 2

Category A responses indicated confusion with the task and use of an addition approach. Typically, seven blocks were laid out and then a further two blocks, with a response such as, "*Well, we need these two blocks because there is 'times 2' in the sum*". It seems that for these students the 7 and the 2 were indicators of the number of blocks needed. In Category B the responses were multiplicative, but each student arranged the blocks only one way – 2 groups with 7 blocks in each.

Table 1

Summary of Responses by Student to Tasks 2, 3 and 4 at Interviews 1, 2 and 3

Student	Task	Int. 1	Int. 2	Int. 3	Student	Task	Int. 1	Int. 2	Int. 3	Student	Task	Int. 1	Int. 2	Int. 3
Tt	2	A	E	A	O	2	C	E	C	D	2	C	D	E
	3	A	E	E		3	E	E	E		3	E	E	E
	4	A	D	D		4	B	B	F		4	B	B	D
Q	2	A	absent	C	Ki	2	C	E	E	Ta	2	C	E	E
	3	C	absent	E		3	C	E	E		3	C	E	E
	4	B	absent	B		4	B	B	E		4	A	B	E
Ka	2	B	D	E	Mc	2	C	E	E	P	2	C	C	E
	3	C	E	E		3	C	E	E		3	C	E	E
	4	B	B	D		4	A	B	E		4	B	F	D
F	2	A	D	E	E	2	C	E	E	B	2	C	E	D
	3	C	E	E		3	E	E	E		3	C	E	E
	4	B	B	D		4	B	B	E		4	B	B	D
Tc	2	C	E	E	Dh	2	B	E	E	L	2	C	E	E
	3	B	E	E		3	C	C	E		3	E	E	E
	4	B	A	E		4	B	B	E		4	F	B	F

Category C responses were those where the students made 7 groups with 2 in each group, and in response to, "Can you show me another way?" made 2 groups with 7 in each. When asked the question again, the students typically replied, *"There is no other way"*, and *"Don't think there is another way"*. Category D responses were those where the students arranged the blocks in 2 groups of 7 (or 7 groups of 2). The students showed another way by rearranging the blocks in an array as 2 rows of 7 (or 7 rows of 2). Others arranged the blocks in an array initially, but rearranged them into a second array to show the 'other way'. The students showed an understanding of commutativity, but did not see the array as showing both 2 by 7 and 7 by 2. In Category E, the students used the blocks to form an array which was explained as 2 by 7 and 7 by 2. Some students arranged the blocks so they were touching, representing a region model. In Category F, the representations reflected either a multiplicative comparison situation, a Cartesian product or a rate situation. No students used the blocks in this way.

The responses at the first interview differed markedly from those at the second and third interviews as is shown in the table. In the first interview almost all of the students used the "groups of" representation and no other, but in the second and third interviews the majority of students arranged the blocks in the rows and columns of an array and were able to read these as both 7 by 2 and 2 by 7. Overall, there was a definite shift from the representation of 7 x 2 = 14 in the "groups of" arrangement (seen predominantly in the first interview) to representing the expression in arrays and region models. This was a little different when the students moved to the next task, where concrete materials were not used in their representations.

Task 3

For this second task the students were asked to use grid paper to show a representation of 3 x 4. The students were familiar with grid paper, but not for a task such as this. At first the majority of students used the grid paper to show "neat" groups, but for other students the grid paper seemed to act as a cue. The same six categories, used for Task 2, were used again to group these responses.

Category A responses were confused. The grid paper was used as a recording sheet on which was written the equation and the answer, or an algorithm and the answer. Category B contained the responses where 3 x 4 was represented as 3 groups of 4 (or 4 groups of 3). In this category, the students shaded in squares on the grid paper to show the groups of 3 or 4, with no second representation. In Category C, the students shaded in squares in rows or columns on the grid paper to show 3 groups of 4 (or 4 groups of three) and a second representation of groups. Also in this category were the students who modified their existing representation to show the other way. If the first representation showed 3 rows with 4 squares shaded in each row, the student crossed off the fourth square in each of the three rows and then shaded another row of 3 making it 4 rows with three in each.

In Category D, the responses were those where the rows or columns of squares were shaded as groups of 3 or 4, but to show another way, an array presentation of 3 x 4 was made, but this was not seen as representing both 3 by 4 and 4 by 3. In the same category were the students who first shaded the squares to show an array and then had shaded a further array. The students indicated an understanding of commutativity, but not in their models. Category E responses were region and array representations of 3 x 4, which were readily explained as showing both 3 by 4 and 4 by 3. Category F responses represented models of multiplicative comparison and Cartesian products. No responses with the grid paper were categorised in this way.

The responses to this task differed somewhat from the responses to the task using the blocks. The categories were the same, but the responses in the first interviews were different, in that no student at the first interview used blocks to show an array, but five students did show an array on the grid paper at the first interview. This is where it appeared that the grid paper acted as a cue for the student to represent 3 x 4 in an ordered array, rather than in the "groups of" form. In the second interview, all except one student showed a region or area representation, and in the third interview all students shaded in an area or region representation, using only one model to show both 3 by 4 and 4 by 3. Once again this differed from the response to the task with the blocks, where in the final interview a group of students had used the blocks to show a "groups of" representation.

Overall, the shift from the "groups of" representation to the region model was more definite with this task, with no regression at the final interview, as seen in the blocks task.

Task 4

Task 4 was a story problem. Responses were again grouped as they were for Task 2. Category A responses were an incomplete or inappropriate multiplication story. A typical response was "*There were five apples and six children.*" In the subsequent categories, the students correctly gave multiplication stories. In Category B, the responses reflected an equal groups situation. The majority of students responded in this way in the first and second interviews. One example was "*During the day I ate 6 chocolates, 5 times, how many altogether?*" By the third interview, stories were representing arrays or area situations. Category C, a story to reflect equal groups with a sense of commutativity, was not possible. Category D had real life stories using the array model, but in this group the students did not express the idea of commutativity. Category E contained stories using an array or region model with the sense of the model representing either 5 by 6 or 6 by 5. All stories in this category used a region representation. An example was, "*There was a cake of chocolate with 5 squares one way and 6 the other, how many squares altogether?*" Category F contained those stories representing multiplicative comparison, rate or Cartesian products. One student at interview one, another at interview two and yet another at interview three responded with stories reflecting rate situations. An example was, "*There were five cats and each cat had six kittens. How many kittens?*"

In this task all except three students were able to give multiplication stories, but in the first and second interviews these were predominantly equal groups situations. There was a shift in the third interview when 12 students gave region or array stories of 5 by 6, and were able to precede this, or follow it, with a model on grid paper.

Task 7

Tasks 7a and 7b were used to measure any effect the new representations had on the students' ability to understand and work through examples involving decimals. The tasks in this group required the students to solve multiplication equations, each with a whole number and a decimal fraction as the factors. The results were allotted a score of 0, 1 or 2. The students who were confused with decimals and unable to continue, and those students who attempted to use an algorithm to solve this problem, but were uncertain about procedure and gave an incorrect response were scored as zero. The students who gave a correct response, but with no explanation, or by using rules to determine where to place the decimal point in the answer were scored as one. The students who gave a correct response with a clear and meaningful explanation of the product being smaller than one of the factors were scored as two.

Although not in the initial interview protocol, at interview two and three the students were asked to respond to either .1x10 or 15x.6 with a pictorial representation or a story.

Generally, in Task 7a there was a change in the responses from the first to the final interview. Initially, almost 50% of the responses were incorrect and a further 50% gave a rule as an explanation. There was a shift in the second and third interviews where 70% gave a correct response with clear explanations of the effect of multiplying by a decimal fraction. The responses to Task 7b were quite different where 40% gave a clear explanation at the first and second interviews, but at the third interview the majority of students gave a correct solution with no explanation.

Links Between Models of Multiplication and Responses to Tasks 7a and 7b

The results of the tasks involving decimals were compared with those of the models tasks to examine links between the development of sophisticated models and the students' ability to solve multiplication equations involving decimals. These results are displayed in Table 2. Categories A to F were rated with a score from 1 to 6, and these were tallied to find a total score for the models. Tasks 7a and 7b were scored as stated above and the total was entered as a combined score for the two tasks. Table 2 shows that generally the individual student's score rose for both the models and decimals from the first to the third interview. Of interest is the drop in the score at the third interview in Task 7b. The same pattern is observed in the means.

The two students who began with a low score on the models and finished low, were unable to do the decimals. The third student with a low score on the models was unsure at the first interview, but continued. At the second and third interview she drew models to assist understanding and check her explanations and algorithms. At the second and third interviews she was able to explain the product of 15 x .6 = 9 in terms of the second factor being a fraction therefore the product would be lower than the first factor. Of the four students who began with a low score (about 7) on the models, but finished with a high score on the models (about 15), three were unsuccessful on the decimal tasks of 15 x .6 and

.1 x 10 at the first interview, but correct on these tasks at interviews 2 and 3. Each of the four was able to give a story or draw a model to accompany one of the examples, and each explained the product being lower than one of the factors because the other factor was a fraction below one.

Table 2
Scores of Models' Tasks and Tasks with Decimals

	Models			Decimals		
Student	Interview 1	Interview 2	Interview 3	Interview 1	Interview 2	Interview 3
Tt	2	14	10	0	0	1
Q	6	absent	10	0	absent	1
Ka	7	11	14	1	1	3
F	6	11	14	1	3	2
Te	7	11	15	1	3	3
Ta	7	12	15	1	2	3
P	8	14	14	1	3	2
B	8	12	13	3	3	3
O	10	12	14	3	4	3
Ki	8	12	15	3	4	3
Mc	7	12	15	1	4	3
E	10	12	15	4	4	3
Dh	7	10	15	3	4	3
L	14	12	16	3	4	3
D	10	11	14	1	3	3
Mean	7.8	11.9	13.9	1.7	3.0	2.6

Overall, 12 of the students with correct responses were able to draw pictorial representations or generate a real world problem to model either 15 x.6 = 9 or .1 x 10 = 1, although some problems were without questions.

Discussion and Implications

The Graeber and Tirosh protocol (1990) was useful for understanding how students think about multiplication, allowing for explanations and discussion, in an accessible context. Some weaknesses were apparent, however. Examples were directed towards the equal groups, arrays, and region models. It would have been beneficial to include tasks related to multiplicative comparison, rate, and Cartesian products. No questions used examples where decimal fractions were both factors, so it was difficult to fully assess the impact of the region model on the understanding of decimals.

Results showed that students' understandings of multiplication, as gauged by the classification of the models they used, were still attached to the simple idea of equal groups at the first interview. While all students were displaying multiplicative thinking (i.e., recognising a multiplier, a multiplicand and coordinating these to find a product, Mulligan & Mitchelmore, 1997), the models they relied on were primitive and unable to be used with decimals. Initially, students only used algorithms to tackle the equations involving decimals, but after the intervention sessions, they were able to make sense of the decimal tasks, relating them to real world stories and representations. Arrays and regions had become the preferred models being used in situations involving decimal fractions. The fact that 12 of the 15 students were able to generate a story or draw a representation about the equations showed progression due to the intervention. Experience with a variety of models and representations is an important aspect in the development of more sophisticated notions of multiplication. Students use and draw what they know and, initially in this study, this was only the "equal groups" model.

Sullivan, Cheeseman, Clarke and Mulligan (2001) report on the problem of moving students from the early stages of multiplicative thinking to the "abstracting (stage) characterised by students moving beyond a need to create physical models as a prerequisite to solving multiplicative problems." (p. 233). The author acknowledges the problem, but the proposal to "explicitly remove physical prompts or supports" (p. 234) is only part of the solution after considering the findings of this study. Verschaffel and De Corte (1996) discuss the inhibiting effect of rushing into formal operations which prevents students from developing "a rich and deep understanding" (p. 111) of operations. The tendency to move students away from supporting materials to algorithms, as soon as they are seen to have developed multiplicative thinking, but before they have worked with a wide range of examples and larger numbers, may be setting up a culture that impedes the development of more sophisticated representations of multiplication.

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The Transition From Concrete to Abstract Decimal Fractions: Taking Stock at the Beginning of 6th Grade in German Schools

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When starting German 6th grade, students already have broad prior experiences with decimal fractions as numbers for measuring quantities. Diagnostic written tests and individual interviews were used to examine how far this experience promotes the transition to abstract decimal fractions shortly before they are taught systematically. Results showed that this transition is still only just beginning in many students, and that the necessary extension of the place-value system is still hampered by incorrect transfers from natural numbers.

German schools teach fractions and decimal fractions systematically in the 6th grade (approximate age 11 years). Before this, students have already gathered initial experiences with simple *fractions* at primary school (Grades 3 and 4) as well as in daily life. At this stage, they have far more experience with decimal fractions in the form of measures of quantity ("concrete decimal fractions"). These are addressed in primary school lessons (Grades 3 and 4) and are also reexamined--to a different extent--in Grade 5. However, they also play an important role in daily life--particularly because, unlike some other countries, the metric system has a long tradition in Germany and is fully integrated into daily life.

How far does this comprehensive experience with concrete decimal fractions lead to the formation of an abstract concept of decimal fractions (i.e., one that does not refer to concrete quantities) by the beginning of 6th grade? Has sufficient basic knowledge accumulated up to this time to provide a rapid and problem-free transition to decimal fractions as abstract arithmetic numbers when they are taught in the coming year? Is this transition even something that occurs almost automatically? Up to now, no empirical findings on these issues are available in German-speaking countries (see, for a related study on fractions, Padberg, 2002). Some interesting studies in Australasia, however, have been stimulating for my work, although they may not address precisely these topics. Particular mention should be given to the work of Baturo and Cooper (2000), Irwin (1999), Moloney and Stacey (1997), Stacey and Steinle (1999), Steinle and Stacey (1998), and Thomas and Mulligan (1999).

The Present Study

My empirical study is based on a specially developed diagnostic test followed by videotaped individual interviews. The diagnostic test (lasting for one 45-min lesson) was given to six complete classes from three different technical secondary schools (in German: *Realschule*) containing a total of 165 students who had just started 6th grade. It addressed the following topics: reading decimal fractions, illustrating decimal fractions, everyday experiences, decimal fractions and scales (lengths, volumes), the decimal place-value system, the local versus global perspective on decimal fractions, ordering decimals, and

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simple decimal arithmetic operations (adding, subtracting, multiplying with natural numbers, measurement and partitive division). For reasons of space, only a small part of this study is reported here.

The videotaped individual interviews were carried out with 13 of these students 2-3 hours after completing the diagnostic test. They took about 10-15 min for each student. Two students were selected from each class (apart from one class in which three students were selected) according to the following criteria: interesting solution approaches in the test, unclear solution strategies, or characteristic error strategies. Because not all parents gave permission for their child to be interviewed, a further selection criterion was parental permission.

The present study is the first, just completed part of a longitudinal research program planned to run for 2 years.

Results

Reading Decimal Fractions

Item 1(item numbers do not correspond to their sequence of presentation in the written text) "How would you read the following number: 3.25? Please circle only *one* answer". Participants could choose between the following answers: "Don't know", "Three hundred and twenty-five", "Three with twenty-five left over", "Three and twenty-five", "Three point twenty-five", "Three point two five", and "Three and one-twenty-fifth". Forty-seven percent gave the answer "Three point two five", and 44% reported "Three point twenty-five". Two percent gave no reply, and 8% gave one of the other replies.

It was reassuring to find that almost one-half of the students circled the correct reading before systematic instruction in decimal fractions, although their only prior experience with them had been in the context of quantities--which would point particularly to the other formulation (e.g., €3.25 = three Euro twenty-five cents). Nonetheless, it is problematic to see that almost all the others simply transferred the way of reading quantities to decimal fractions (a common practice in daily German language). This leads to typical errors such as: (a) 0.4 is smaller than 0.35 because 4 is smaller than 35; (b) 0.27 plus 0.7 = 0.34 because 27 plus 7 = 34; and (c) 0.97 minus 0.6 = 0.91 because 97 minus 6 = 91.

The strong trend toward this problematic reading has to be monitored carefully and countered in the subsequent systematic treatment of decimal fractions, because--as the interviews confirmed--there is a strong degree of uncertainty regarding how to read fractions correctly: one-half of these interviewed students named a different way of reading fractions to that they had given in the written test.

Illustrations of the Decimal Fraction 0.5

Item 2. "Draw a picture illustrating the decimal fraction 0.5 (using a line, a rectangle, a circle, or ...)". This item also presented the option of "I don't understand the task". Results revealed the following strategies:

Strategy 1 (14% of participants). Here, 0.5 was recognizably depicted as one-half of a line or a two-dimensional figure.

Strategy 2 (5%). 0.5 cm was marked out on a longer line with further subdivisions. These students are probably already on the path toward using 0.5 as a marker on a number line.

Strategy 3 (23%). These students drew a 0.5-cm-long line with a ruler (or, partly because of the item description, a square with 0.5-cm sides) and (still) identified 0.5 with 0.5 cm.

Strategy 4 (9%). These students drew only a line, a square, a rectangle, or a circle, sometimes with a length or diameter of 5 cm (avoidant reaction).

Various stages on the path toward a general concept of decimal fractions could be recognized in these four strategies. However, this item indicated that the majority of students still seemed to associate decimal fractions completely with concrete quantities. This is also indicated by the large proportion of no replies (42%). A final 7% percent of students gave other answers.

Decimal Fractions and Scales

To save space, I shall present only two items here with linear scales.

Item 3. "Use an arrow (↑) to mark 4.7 m on the following scale". The item then presented a scale calibrated in tenths from 4 m to 5 m.

This item was answered correctly by 87% of participants. A further 4% marked the scale one place too far to the left or right. Seven percent produced another error, and 2% made no reply.

Item 4. "What measure does the arrow (↑) mark on the following scale". The item then gave participants a scale calibrated in tenths from 3 m to 4 m with an arrow marking 3.4 m.

Ninety-two percent completed this item correctly, 4% gave a wrong answer, and a further 4% gave no reply.

The proportions of correct solutions to these items differed markedly from those for Item 2. Hence, at the beginning of 6th grade, almost all students associated visible representations reporting lengths with simple decimal fractions as measures (although a clear decline in performance was observed when decimal fractions had two places following the decimal point).

Decimal Place-Value Notation

Is the extension of the place-value concept from natural to rational numbers easy for students and thereby almost bound to succeed? The amount of prior work on decimal fractions as concrete quantities would seem to support this assumption. I shall report only two items from the large amount of data gathered in the present study.

Item 5. "How many tenths make up one whole"? Thirty-eight percent answered "10". Nineteen percent gave some other answer, but 43% gave no answer at all. These results speak for themselves (see, also, Baturo & Cooper, 2000). Another item in the test asked how many tens make up one thousand. Only just over 50% of the students managed to give the correct answer to this question on the place-value system for natural numbers--a devastating finding.

Item 6. "Use a cross or a circle to mark the *hundredths* in 7.654". Twelve percent marked the "5"; 28%, the "6"; and 9%, the "4"; whereas 10% gave other answers. A total of 41% gave no answer at all. Hence, despite a higher probability of guessing correctly, only 12% got it right (the proportion of correct solutions in another item asking them to mark the tenths was even lower at 10%).

The answer "6" may be traced back to an incorrect transfer from natural numbers in which the third place from the right (after the decimal point) is viewed as a hundredth--as in natural numbers in which the third place is occupied by the hundreds. This strategy may well also be supported by the fact that the number in this item has exactly three decimal places after the point, and this may encourage participants even more strongly to view the decimals as hundredths, tenths, and, perhaps, "oneths". However, the incorrect way of reading this number as "seven point six hundred and fifty-four" also suggests this error, along with a "point-separates" strategy that I have often observed after systematically teaching decimal fractions (Padberg, 2002). Students applying this strategy view the decimal point as simply separating two natural numbers.

There is a strong probability that the answer "4" is due to another incorrect transfer from natural numbers in which, for reasons of symmetry, the first place after the point is interpreted as a oneth (analogue to the units) followed by tenths and hundredths. The proportion of correct answers is very low on this item and the number of no replies is high, indicating that the extension of the place-value system needs to be dealt with thoroughly at the beginning of the systematic teaching of decimal fractions. The degree of prior knowledge is very low, and it is in no way something that emerges of its own accord (see, also, Baturo & Cooper, 2000). It is also important to contrast decimal fractions with natural numbers, because most of the many incorrect solutions were the outcome of erroneous transfers from natural numbers. This (incorrect) recourse to familiar natural numbers when learning decimal fractions is a frequent problem (see, also, Moloney & Stacey, 1997). However, great care is needed to prevent this misconception from obstinately consolidating (see, also, Moloney & Stacey, 1997; Padberg, 2002; Steinle & Stacey, 1998).

Local Versus Global Perspective on Decimal Fractions

Item 7. "Laura and Alexandra both want to explain the decimal number 0.75 to Sarah. Laura tells her: '0.75 is seventy-five hundredths'. Alexandra says: '0.75 is seven tenths plus five hundredths'. Which explanation is correct? Draw a circle round the name of the girl who has given the right explanation". Possible answers were "Laura", "Alexandra", "Both girls", "Neither girl", or "Don't know". Afterwards, the item asked: "If you have drawn a circle round neither girl, please use a few words to tell us why".

Nineteen percent said that both girls were correct; 22%, Laura; and 28%, Alexandra. Another 19% said that neither explanation was correct, and 12% said that they didn't know. Although answers were distributed fairly equally, the local perspective (Alexandra) seemed to be slightly more dominant than the global one (Laura). Both this equal distribution and the responses in the interviews also indicate that many students gave rather spontaneous decisions that they were unable to justify. The justifications for the answer "Neither" are particularly interesting, although only one-half of these students completed this part of their item. The most frequent argument was that 0.75 is composed of seven tenths and five units (oneths). This, like the answer, "0.75 is 75 *tenths*", reveals two different incorrect transfers from the domain of natural numbers (see the previous section on place-value notation). The answer 0.75 = 75 cm confirms that these students (still) identified 0.75 with 0.75 m.

Decimal Fractions "Between" Decimal Fractions

From the broad range of questions on ordering decimals, I shall report just one test item.

Item 8. "Can you name a decimal number that is larger than 1.5 and smaller than 1.6"? (This open question also provided the response option "No".)

Correct answers were given by 37% of the students; 6% produced other answers (see below); 16% produced incorrect answers; and 42% replied "No". The most frequent correct solution was the intermediate number: 1.55. The "Other" category contained rather creative replies that did not use standard notation (the most frequent was 1.5 1/2 followed by 1.5.5 and 1.5.1). Some of the justifications for "No" in the interviews were that there is no decimal number between the two, or that, although such numbers exist, the student did not know how to name them. There was a wide range of incorrect answers.

Some additional remarks. (a) A comparable item using *lengths* ("Mary wants to buy a new book shelf. It needs to be longer than 2.4 m and shorter than 2.5 m. Give an example of how long the book shelf should be".) led--contrary to my expectations--to only a slight increase in the proportion of correct solutions (from 37% to 42%), an equal proportion of nonstandard replies, and a markedly lower number of students who were unable to report an in-between number (from 42% to 32%), nonetheless, at the expense of a slight increase in the proportion of incorrect answers (from 16% to 20%). (b) Compared with a corresponding item using simple fractions (Padberg, 2002), far more students at the beginning of 6th grade associated more or less intuitive ideas with decimal fractions. In the study of fractions, only 5% of a comparable sample were able to report a correct fraction between one quarter and one half. Alongside the greater amount of prior preparation, this may also be due, in part, to the decimal notation being more familiar than that of (simple) fractions.

Discussion

Regarding their knowledge of decimal fractions, students at the beginning of 6th grade are going through a transitional phase between concrete decimal fractions as measures of quantity, which they have known for quite a while in both daily life and school, and the abstract decimal fractions as abstract computational numbers, which will be taught systematically during the course of this school year.

The study provides the following picture of their progress along this path: A high percentage of students are able to read decimal fractions (see Item 1), although almost one half of them use a common form of reading in daily life that leads to a variety of characteristic errors. They also have a confident command--at least in simple cases--of the notation of decimal fractions as measurements of quantity in scales (see Items 3 and 4). However, things are completely different when students are asked about the pictures they associate with simple decimal fractions such as 0.5. Although perhaps 20% of the students in this sample are already approaching the abstract concept of the decimal fraction, many still identify decimal fractions with specific quantities (in this case, frequently 0.5 cm), whereas many others are uncertain and give no answer (see Item 2). A good indicator of progress on the path toward an abstract concept of the decimal fraction is also the item on decimal fractions between two given decimal fractions, an item that is probably easier than illustrating decimal fractions (see Item 8). Almost 40% can report at least one decimal fraction here (a percentage that hardly increases in a parallel item with quantities). In contrast, the students still have hardly any experience with the two *perspectives* on decimal fractions (local vs. global, see Item 7). Their decision is reached more impulsively rather than on the basis of any plausible justification. The knowledge on extending the decimal place-value system (see Items 5 and 6), which is so necessary for a comprehension of decimal fractions, is still extremely overlaid by--what are here incorrectly transferred--prior experiences on the place-value notation of natural numbers. Hence, on two items, only about 10% of students mark the tenths or hundredths correctly. A very dominant misconception here is that the point in decimal fractions separates two natural numbers. As a result, it is not the point but the last place to the right that forms the reference basis in decimals as well (moreover, this misconception is supported additionally by the problematic way of reading decimal fractions mentioned above in Item 1). Furthermore, one can frequently observe the idea that decimal fractions following the point are constructed symmetrically to the whole numbers before the point (oneths). A high degree of uncertainty is also documented once again by the fact that approximately 40% of students are unable to solve this item. Fractions--at least in the form of fractions with powers of 10 as denominators--form a further important component of the concept of the decimal fraction that I am unable to discuss here because of lack of space (see Padberg, 2002, in press).

The study confirms very clearly that a confident mastery of concrete decimal fractions should not be equated in any way with a good knowledge of abstract decimal fractions. It is far more the case that the corresponding transfer needs to be carried out carefully and gradually--and not just quickly in one lesson. Great care is necessary when extending the place-value system. For example, the relations between the place values need to be discussed not just in the one direction but in both. The relations between non-neighboring place values and between place values to the left and the right of the decimal point need to be considered, to name just a few perspectives (see, also, Baturo & Cooper, 2000). The commonalities with natural numbers should not be overemphasized; it is specifically the *differences* that have to be worked out clearly. If this is not done with great care, it is easy for students to acquire misconceptions regarding decimal fractions that may well persist for a long time (see Padberg, 2002; Stacey & Steinle, 1999; Steinle & Stacey, 1998). However, a good and careful foundation in decimal fractions has a very high priority in our society with its widespread use of pocket calculators and computers as well as its daily use of the metric measurement system. When summarising the outcome of their wide-ranging own studies, Wearne and Hiebert (1988, p. 223) rightfully confirm that

> Many difficulties can be traced to an incomplete or nonexistent understanding of the meaning of the written symbol ... Without semantic meanings for the symbol, students have little choice but to memorize syntactic rules that prescribe how to manipulate symbols.

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A Systemic Program for Students Who Are Experiencing Difficulty with Mathematics as They Transition from Elementary to High School in Australia

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Counting On is a systemic mathematics program that targets low achieving students in the final year of government elementary and the first year of government secondary schools through the professional development of their teachers. During 2000, *Counting On* was implemented in 40 secondary schools, involving more than 600 students, 120 school teachers and 40 district mathematics consultants. In 2001, the implementation was extended to approximately 80 secondary schools and 80 final year elementary classes which 'feed' these secondary schools, involving some 1400 students, 320 teachers and 40 district mathematics consultants. This paper reports on evaluations of the program in 2000 and 2001, with particular reference to students' movement across the levels of conceptual development in place value contained in the numeracy framework which underpins the program.

The New South Wales Department of Education and Training (DET) is the government authority responsible for school education in the state of New South Wales, Australia. As such, it runs some 2000 elementary schools (students aged approximately 5 to 12 years) and 400 secondary schools (students aged approximately 12 to 18 years) as well as a small number of preschools and many technical and further education institutions. *Counting On* is a DET mathematics program that targets low achieving students in the first year of secondary (Year 7) and the last year of elementary schooling (Year 6). Hence, the students involved range from 11 to 13 years of age. In both 2000 and 2001, the authors of this paper undertook evaluations of *Counting On*. It is from these evaluations that the data for this paper are drawn. Earlier papers (Perry & Howard; 2001a, b) have explored the 2000 data on its own but this is the first paper to consider both years.

Background

The rationale and structure for the *Counting On* program are outlined in the following extracts from the program documentation:

> Too many students enter secondary school with calculating methods that consist solely of schemes of counting by one. While these methods will often result in the correct answer, they take so much effort that there is little chance of learning new material. The learning of these students in mathematics has reached a plateau. (NSW Department of Education and Training, 2002, p. 5)
>
> The intent of *Counting On* is to build the professional knowledge of the teachers involved in the project. Furthermore, it aims to assist teachers in addressing the learning needs of students who are not demonstrating progress … (NSW Department of Education and Training, 2002, p. 7)

The link between professional development and student learning outcomes is well established (Cook & Fine, 1997; Dockett, Perry, & Parker, 1998). The aim of teacher professional development is to produce change in teacher knowledge and teacher practice and, through these, change in student outcomes. Effective professional development

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B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 543-550). Sydney: MERGA. ©2002 MERGA Inc.

"establishes new expectations for students, teachers, and school communities" (Cook & Fine, 1997, p.1).

Counting On operates through a team of teachers from each secondary school and its feeder elementary schools. Each team consists of:

- the head teacher, mathematics; Year 7 classroom mathematics teacher(s) and the support teacher learning difficulties in the secondary school;
- Year 6 classroom teacher(s) from the elementary school(s); and
- the district mathematics consultant.

The theoretical foundation for *Counting On* is the *Learning Framework* (Mulligan & Mitchelmore, 1996; NSW Department of Education and Training, 2002; Thomas, 1999; Wright, Martland, & Stafford, 2000). This framework provides a structure of levels of conceptual development in place value and in multiplication and division. The place value framework relevant to this paper is provided in Table 1, along with a brief explanation of each level.

Table 1
Learning Framework Levels of Conceptual Development in Place Value

Level	Descriptor	Explanation
0	Ten as count	Student can count on but uses single units.
1	Ten as unit	Student treats ten as a single unit, made up of ten ones.
2	Tens and ones	Student solves addition and subtraction without representations of tens and ones being available. Ten is treated either as an iterable or abstract collectible unit.
3	Hundreds, tens, & ones	Student can use hundreds, tens and ones for standard decomposition of three-digit numbers and can mentally add and subtract combinations of numbers to 1000.
4	Decimal place value	Student uses tenths and hundredths with understanding of positional value.
5	System place value	Student knows that place value system can be extended in two directions.

Methodology

The evaluation of the *Counting On* program employed intensive case studies on four of the sites, pre- and post-test implementation of the student assessment schedule and team surveys. This paper concentrates on the data arising from the pre- and post-test student assessment data and uses some explanatory comments from the case studies.

A key feature of the *Counting On* program is an intensive assessment interview which is undertaken individually by each student before the teaching program begins and shortly after the teaching program has been completed. All assessment interviews were conducted by a member of the *Counting On* team on each site and were video-recorded for later analysis by the whole team. In 2000, the assessment schedule consisted of 19 questions covering place value, addition, subtraction, multiplication and division tasks. In 2001, the number of questions was reduced to 17 with the same coverage of topics. Many, but not all, of the 2001 questions were also part of the 2000 assessment schedule. A copy of the assessment record for each student completing the interview was made available to the authors for the two assessment implementations in both 2001 and 2002.

The assessment record for each student provided data on whether or not the student made a valid attempt at each question, whether the answer given was correct or incorrect and which strategies were used to answer the question. As well, the overall levels on both place value and multiplication and division at which the *Counting On* team deemed the student to be working—following a group analysis of the videotaped recording—were recorded.

In 2000, results from 671 Year 7 students were received for the first student assessment (2000,T1) with the results from 544 of this Year 7 cohort being received for the second assessment (2000,T2). The gender breakdown of these two cohorts was 2000,T1: 63.2% male, 36.8% female and 2000,T2: 62.5% male, 37.5% female.

In 2001, at least partial data were received from 1416 students from both Year 6 (30.3%) and Year 7 (69.7%). Of the total number of these students, 56.9% were male and 43.1% female. The gender mix for the Year 6 students was 53.8% male and 46.2% female while, for the Year 7 students, it was 58.3% male and 41.7% female. Almost all of the 1416 students for whom there are partial data (1404) undertook the first assessment (2001, T1) but only 1176 undertook the second assessment (2001, T2).

The decrease in numbers between the two assessments in both 2000 and 2001 can be ascribed to students leaving the schools, selecting not to continue with the program, being unwilling to be videotaped a second time and being absent from school on the second assessment day, along with the fact that the second assessment results were not received from a few of the schools.

Results

For this paper, only the data concerned with the overall place value levels for each student will be considered. In each of 2000 and 2001, each student was assigned a place value level (from 0 to 3 only as only whole number place value was assessed) from each of the pre- and post-assessment interviews. Tables 2 - 4 show the distribution of these levels for each year.

Table 2

Percentages of Year 7 Students in Each Place Value Level—2000, T1 and 2000, T2

Level 0		Level 1		Level 2		Level 3	
T1	T2	T1	T2	T1	T2	T1	T2
31	8	36	36	24	30	10	26

Table 3

Percentages of Year 7 Students in Each Place Value Level—2001, T1 and 2001, T2

Level 0		Level 1		Level 2		Level 3	
T1	T2	T1	T2	T1	T2	T1	T2
35	13	35	31	23	40	7	16

Table 4
Percentages of Year 6 Students in Each Place Value Level—2001, T1 and 2001, T2

Level 0		Level 1		Level 2		Level 3	
T1	T2	T1	T2	T1	T2	T1	T2
42	15	37	30	20	43	1	12

Discussion

These data show that, in both 2000 and 2001, there have been overall increases in place value levels for all groups of students as they have moved through the *Counting On* program. It is not possible to ascribe these increases to the program without a control group with which to compare. Such a control group was not available to the researchers. However, comparisons can be made between the two Year 7 groups from 2000 and 2001 and between the Year 6 and Year 7 groups in 2001. Figures 1 – 3 show clear increases in the place value levels for all three of the cohorts described above.

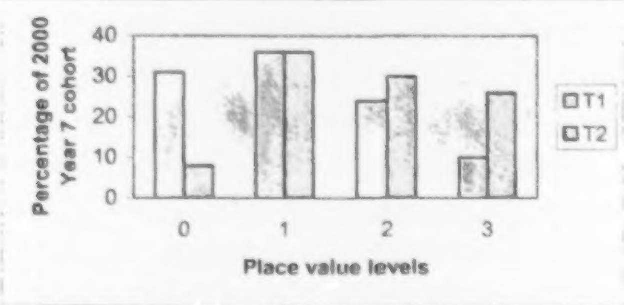

Figure 1. Percentage of 2000 Year 7 students in each place value level for 2000, T1 and 2000, T2.

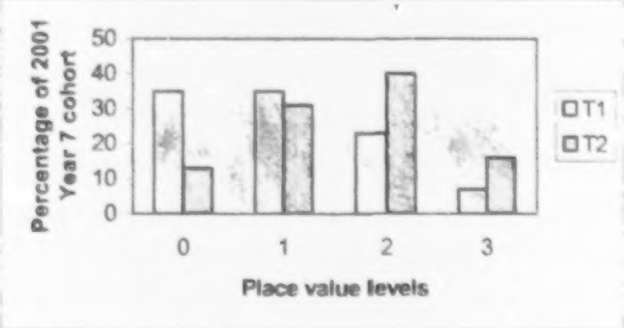

Figure 2. Percentage of 2001 Year 7 students in each place value level for 2001, T1 and 2001, T2

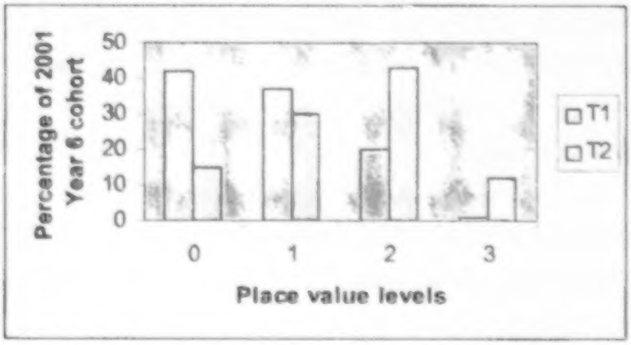

Figure 3. Percentage of 2001 Year 6 students in each place value level for 2001, T1 and 2001, T2

Two-way contingency table analyses showed that the changes in individuals' place value levels from T1 to T2 were highly significant for Year 7 in 2000 (χ^2=173.51, p<0.001) and each of the Year 6 and Year 7 cohorts in 2001 (Year 6: χ^2=190.96, p<0.001; Year 7: χ^2=590.86, p<0.001). In every case the differences were significant for both male and female students.

For the 2001 cohorts, Mann-Whitney U tests were conducted to evaluate the hypothesis that Year 6 and Year 7 students would score equally, on the average, on place value levels in T1 and T2. The results of the test were significant for T1, z=-4.11, p<0.001, with Year 7 students, on average, scoring higher than Year 6 students, but were not significant for T2. These results indicate that *Counting On* does seem to have had a differential effect on students in Year 6 and Year 7, with Year 6 substantially 'catching up' to the Year 7 cohort.

Further analysis can track individual growth across the place value levels achieved in each of T1 and T2. Table 5 and Figure 4 show the results of this tracking for each of the cohorts: 2000, Year 7; 2001, Year 7 and 2001, Year 6. In each case, small percentages of students have been recorded at a lower final place value level. This could be due to an incomplete understanding of the assessment process by the *Counting On* teams or a result of lack of attendance or illness on the part of the students or even transcription errors when recording the data. It is also possible that these students did not understand the task or the process. However, the overriding feature is the large percentage of students in all the cohorts who have maintained their level or have increased their place value assessment by at least one level, with more than 14% of the 2000, Year 7 cohort, almost 11% of the 2001, Year 7 cohort and more than 12% of the 2001, Year 6 cohort increasing by at least 2 levels. The graph in Figure 4 clearly shows that a greater proportion of the 2001, Year 6 students than either the 2000, Year 7 or the 2001, Year 7 students has increased one or two levels in their place value performance while greater proportions of each Year 7 cohort than the Year 6 group have decreased by one or two levels in their place value performance.

Table 5

Percentages of each assessment cohort and their growth in place value levels (missing data disregarded)

Growth in place value levels	-3	-2	-1	0	1	2	3
2000, Year 7	0.2	1.2	7.5	39.6	37.4	11.5	2.6
2001, Year 7	0.0	0.1	3.5	48.7	36.9	10.2	0.6
2001, Year 6	0.0	0.0	1.6	40.6	45.1	12.2	0.5

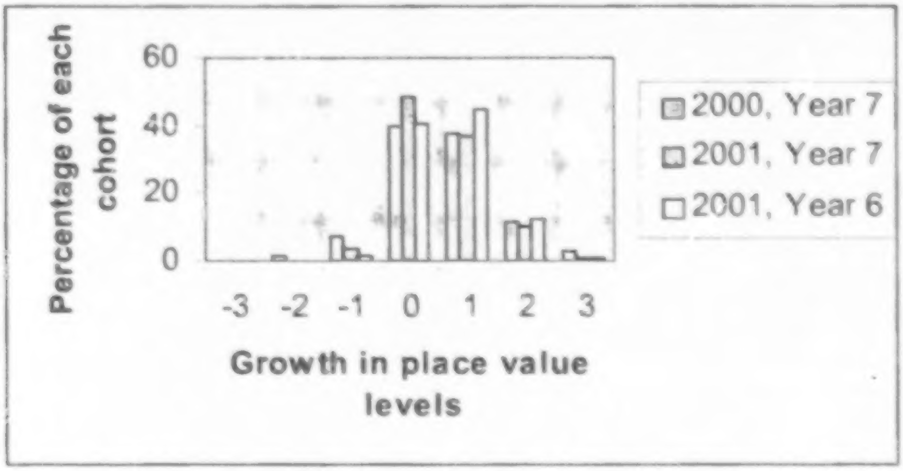

Figure 4. Growth in place value levels from T1 to T2 for 2000, Year 7; 2001, Year 7 and 2001, Year 6 cohorts.

The analysis shows that there are significant increases in the place value levels shown by the students in each of the cohorts as they move from T1 to T2. That is, the students are moving upwards through the learning framework. For all of the cohorts, there are statistically significant gender differences in the performances on each of the two assessment interviews but there are no statistically significant gender differences in the growth in place value levels in any of the three cohorts. That is, both boys and girls start at different points, finish at different points but their growth across the program is similar. This suggests that the *Counting On* program does not act differentially on either group.

Another interesting finding from this analysis is the 'catching up' phenomenon where the Year 6 students in 2001 seemed—as a group—to make up, by the time of the implementation of T2, a lot of the difference between themselves and the 2001, Year 7 group which was shown in T1. This suggests that the *Counting On* program is suitable for application in Year 6 and that, in fact, Year 6 may be a more suitable time than Year 7 for initial implementation of the program. Further investigation of this proposition needs to be undertaken.

Conclusion

> "The *Counting On* project concentrates on those aspects of the [learning] framework necessary to assist the movement of students from unitary to composite-based mental strategies." (NSW Department of Education and Training, 2002, p. 15)

Given that higher levels of the learning framework reflect more composite-based strategies for place value, the *Counting On* program in 2000 and 2001 has achieved this aim. With a relatively short period of intervention and focussed teaching activities, the students have generally advanced on the learning framework, as assessed in the pre- and post-tests. The increases in levels are statistically significant and pedagogically important.

Some idea of the importance of *Counting On* to the development of mathematical learning and teaching in the primary and secondary schools of NSW can be gleaned from comments of teachers in the four case studies undertaken by the authors as part of the overall evaluation.

In one of primary schools, the classroom teacher believed that he could isolate one feature of *Counting On* which stimulated substantial learning of mathematics. He illustrated it through a description of the progress of his most outstanding *Counting On* student.

> I was quite amazed by one of the kid's grades. I think that the thing that really suited him was the very explicit nature of the teaching and the fact that it built core numeracy skills which, for whatever reason, he hadn't obtained in his previous schooling. … There is a real pattern kind of thing built into *Counting On*. Kids see patterns. That really helped him. It helped all of them.

This impression was reinforced by the one of the secondary classroom teachers:

> Traditionally you would teach kids maths in a certain way to make sure that the answer is correct and there was always a standard procedure and now I do something in a classroom like counting up from 7 in 10s and we do it differently. I just find it a newer teaching and I think a lot of kids like it and will benefit from that. I think that most people know that is the thing to do. We have never been taught how to do it.

Similarly, another primary classroom teacher emphasised the importance of teachers and students thinking about their mathematics teaching and learning.

> It re-focuses teaching to the extent of … thinking to yourself, 'Hang on, this also has to do with place value and if they don't understand the place value then they are not going to.. They might remember to put the zero in but you are guaranteed to come unstuck'. It really makes you think what the purpose of a lesson is and gets the kids tuned in and focused. You think what your next step is—building in reminders for them.

The importance of being able to interact with students in a small group all focussed on one activity and with one aim was emphasised by a secondary school support teacher.

> The nature of the success is in the interaction. Some kids have a lot of trouble but when they have interaction they have success. … We can bring them into an environment where they are just a small group and they have access to the board and they get lots of teacher attention. In a small group of six kids the number of interactions that a teacher can have with a child is multiplied.

The quantitative results on changes in the learning framework levels and the anecdotal comments of students and teachers in the case study schools suggest strongly that *Counting On* has been very successful in its aim to help the targeted students to improve their application of mathematical thinking strategies in the area of place value. The expansion of the program into Year 6 classes in 2001 seems to have been successful, both in terms of student assessment results and teacher interactions. Primary and secondary mathematics teachers get very little opportunity to interact professionally. *Counting On* has the potential to encourage that to happen through the opportunity provided by the program for collegial discussions and professional discourse.

One of the interesting opportunities provided by the *Counting On* program as it is currently being implemented is to provide a real link in the mathematics learning of students as they make the transition from primary to secondary schools. The results discussed in this paper show that *Counting On* is an appropriate program for both Year 6 and Year 7 students. It could provide a familiar base for those students who have struggled with mathematics as they commence their high school careers.

The success of *Counting On* highlights the importance of:

- the use of individual video-taped interviews for the assessment of students' mathematical knowledge;
- an emphasis on student strategies as well as outcomes as a way of improving these outcomes;
- small group work in mathematics;
- collegial support amongst teachers, particularly as the students move between schools; and,
- using teaching strategies to change student and teacher beliefs about the learning and teaching of mathematics.

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Beliefs of Primary Teachers about Mathematics and its Teaching and Learning: Views from Singapore, Philippines, Mainland China, Hong Kong, Taiwan and Australia

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The authors have been working, separately as well as collaboratively, on the area of teachers' beliefs about mathematics and its learning and teaching for many years. In this paper, they report on the use of one instrument to ascertain measures of the beliefs of a total of 1254 primary school teachers in a number of different cultural contexts. The stimulus for this combined, cross-cultural study has been the variations in results achieved by students in international studies such as the TIMSS and TIMSS-R and the strong evidence that teachers' beliefs about mathematics and its learning and teaching play a critical role in determining how teachers facilitate their students' learning. Hence, differences in teacher beliefs may be one reason for the measured differences in student achievement. The results of the study clearly show that teachers' beliefs in all of the samples can be adequately described in terms of the statements in the survey and that there are there are marked differences in response to these statements across the different groups of teachers.

Introduction

This paper has grown from the long term commitment of the authors to the impact of teachers' beliefs on their teaching and their students' learning of mathematics. Previous studies have suggested that the cultural aspects and contexts of these beliefs are critical in their implications for student learning (Conroy & Perry, 1997; Howard, Perry, & Fong, 2000; Perry & Howard, 1999; Perry, Howard, & Conroy, 1996; Perry, Tracey, & Howard, 1998; Wong, Lam, & Wong, 1998). This paper extends the earlier work, introduces data from further contexts, and builds on the link between student achievements on international, cross-cultural assessments and the beliefs of the teachers involved in the classrooms.

Background

Results of the Third International Mathematics and Science Study (TIMSS) (TIMMS, 1998) and the Third International Mathematics and Science Study – Repeat (TIMSS – R) (Martin, Gregory, & Stemler, 2000; Wang, 2001) have highlighted the approaches of mathematics teachers and the findings of mathematics education researchers in many countries. On the other hand, both studies have also motivated mathematics educators from various parts of the world to engage in smaller-scale comparative studies in mathematics achievement—USA-Japan (for example, Stigler & Hiebert, 1999) and USA-China (for example, Ma, 1999). The objectives of such studies are usually twofold—to confirm results

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from the TIMSS studies and to gather more in-depth information than TIMSS has been able to collect.

With Singapore topping the mathematics achievement tests for 7th and 8th graders and for 6th and 7th graders (Population II) in TIMSS and TIMSS - R, respectively, it is only natural that some neighbouring countries would want to compare their results. One aspect of comparison is the teacher and, in particular, the beliefs of the teacher. It is well known that these beliefs are critical to what happens in the teacher's mathematics classroom (Barnett & Sather, 1992; Ma, 1999; Pajares, 1992; van Zoest, Jones, & Thornton, 1994) and that they guide the teacher in deciding what teaching and learning strategies are to be valued in that classroom (Baroody, 1987). Hence, in trying to understand fully the results of the international achievement studies, the affective aspects of teaching and learning mathematics, particularly teachers' beliefs, are likely to be important, even though some studies of the TIMSS data itself suggest that this may not be the case (Fullarton & Lamb, 2000). With this in mind, the authors of this paper embarked on their international, cross-cultural study of teacher beliefs about mathematics, and its learning and teaching.

Numerous studies on mathematics teachers' beliefs, attitudes, and conceptions have been reported (Garofalo, 1989; Sosniak, Ethington, & Varelas, 1991; Thompson, 1992) and many have offered implications for classroom reform (Battista, 1994; McLeod, 1992) and mathematics teacher education research (Thompson, 1992). Wong, Lam, & Wong (1998), in their study of students' and teachers' beliefs and views about mathematics learning, found that almost all teachers viewed mathematics as closely related to abstract concepts that required thinking and that may not necessarily be handled merely by routine manipulations. Some disparity was discovered between teachers' and students' beliefs. Nevertheless, the study provided important information on what Hong Kong teachers' believe about mathematics learning. Iida and Yamaguchi (1998) analysed novice and expert elementary teachers' views on the teaching and learning of mathematics in Japan. Both novice and expert teachers were classified as constructivist and non-constructivist. The study showed that constructivist teachers allow for situations that encourage discussion of informal ideas and that provide ease in learning. Moreover, the study showed that expert teachers were more open to arithmetic teaching based on constructivism than novice teachers. At the secondary level in Australia, this result has been supported by Perry, Howard, and Tracey (1999).

Perry and Howard (1999) and Howard, et al. (2000) reported significant differences in the beliefs among elementary teachers from Australia, Singapore and Indonesia concerning mathematics and its teaching and learning. First, they found that Singaporean and Indonesian teachers emphasised mathematics learning as a cognitive concern more than their Australian counterparts. Second, they found that the role of memorisation was more important to Singaporeans compared with Australians and Indonesians. These studies also showed that affect was a major concern in mathematics teaching and learning among teachers in these three countries. Howard, et al. (2000) also established the validity of the instrument devised by the Australian authors of the present paper as appropriate for use to measure the beliefs of teachers in a range of social and cultural contexts in other countries.

It is obviously important to investigate teachers' beliefs about mathematics teaching and learning if we are to gain a fuller understanding of differences in mathematics achievement across a variety of cultural contexts. Teacher beliefs have important consequences for mathematics teaching and learning and provide a glimpse of the teachers' philosophies of mathematics. Hence, they may help explain some of the differences in achievement noted by the international studies.

Method

Data for this study were gathered using a questionnaire containing 18 items dealing with teacher beliefs about mathematics and its learning and teaching. The questionnaire has been used in earlier studies (see Perry, et al., 1999 for a summary of this background) including cross-cultural studies in Australia, Indonesia, the Philippines and Singapore (Howard, et al., 2000; Perry & Howard, 1999).

For this study, data were gathered from a total of 1254 primary teachers in Singapore (S) (n=162), Philippines (P) (n=189), Changchun, China (CC) (n=111), Hong Kong, China (HK) (n=379), Taiwan (T) (n=161) and New South Wales, Australia (NSW) (n=252). In Singapore, Philippines and New South Wales, the questionnaire was presented in English and in Changchun, Hong Kong, and Taiwan, it was translated into the appropriate Chinese language. These translations were checked by one of the authors using the back-translation technique. Respondents completed the questionnaire by indicating on a three point Likert scale—Disagree, Undecided, Agree—to what extent they agreed with each statement.

Results

The 18 statements which made up the questionnaire are shown below.

Table 1
Belief Statements in Questionnaire

1.Mathematics is computation

2.Mathematics problems given to students should be quickly solvable in a few steps

3.Mathematics is the dynamic searching for order and pattern in the learner's environment

4.Mathematics is a beautiful, creative and useful human endeavour that is both a way of knowing and a way of thinking

5.Right answers are much more important in mathematics than the ways in which you get them

6.Mathematics knowledge is the result of the learner interpreting and organising the information gained from experiences

7.Students are rational decision makers capable of determining for themselves what is right and wrong

8.Mathematics learning is being able to get the right answers quickly

9.Periods of uncertainty, conflict, confusion, surprise are a significant part of the mathematics learning process

10.Young students are capable of much higher levels of mathematical thought than has been suggested traditionally

11.Being able to memorise facts is critical in mathematics learning

12.Mathematics learning is enhanced by activities which build upon and respect students' experiences

13. Mathematics learning is enhanced by challenge within a supportive environment
14. Teachers should provide instructional activities which result in problematic situations for learners
15. Teachers or the textbook – not the student - are the authorities for what is right or wrong
16. The role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received this knowledge
17. Teachers should recognise that what seem like errors and confusions from an adult point of view are students' expressions of their current understanding
18. Teachers should negotiate social norms with the students in order to develop a co-operative learning environment in which students can construct their knowledge

For each of the 18 items in the questionnaire, a two-way contingency table analysis was conducted to evaluate whether response to the statement was related to the respondent being a teacher in a particular geographical/cultural context. Significant relationships between response and context ($p<0.001$) were found for all of the statements. Follow up pairwise comparisons were conducted to evaluate the differences in these relationships among the different teacher groups. For every item of the questionnaire, significant differences at the 0.001 level were found between some groups, although not necessarily between all possible pairs. For the purposes of this paper, we have chosen to discuss just two items of particular interest— item 11 representing transmission approaches and item 12 representing child-centred approaches to mathematics learning and teaching (see, for example, Howard et al., 2000; Perry et al., 1998; 1999 for analyses in terms of these two factors). The distribution of responses for each of these questions across the teacher groups are shown in Table 2 and 3. The pattern of differences among the groups on each of these items are shown in Tables 4 and 5.

Table 2

Percentages of Each Teacher Group Responding to Item 11: "Being Able to Memorise Facts is Critical in Mathematics Learning"

Group	Disagree	Undecided	Agree
S	59.3	13.6	27.2
P	36.5	9.0	54.5
CC	24.3	19.8	55.9
HK	1.6	31.1	67.3
T	23.0	24.2	52.8
NSW	28.2	24.2	47.6

Table 3

Percentages of Each Teacher Group Responding to Item 12: "Mathematics Learning is Enhanced by Activities Which Build Upon and Respect Students' Experiences"

Group	Disagree	Undecided	Agree
S	1.2	3.7	95.1
P	5.8	5.8	88.4
CC	5.4	6.3	88.3
HK	23.0	54.6	22.4
T	1.9	1.9	96.3
NSW	0.4	2.8	96.8

Table 4

Differences Among Groups on Item 11: "Being Able to Memorise Facts is Critical in Mathematics Learning" (indicates statistical significance at the p<0.001 level)*

Groups	S	P	CC	HK	T
P	*				
CC	*	NS			
HK	*	*	*		
T	*	*	NS	*	
NSW	*	*	NS	*	NS

Table 5

Differences Among Groups on Item 12: "Mathematics Learning is Enhanced by Activities Which Build Upon and Respect Students' Experiences" (indicates statistical significance at the p<0.001 level)*

Groups	S	P	CC	HK	T
P	NS				
CC	NS	NS			
HK	*	*	*		
T	NS	NS	NS	*	
NSW	NS	*	*	*	NS

Analysis and Discussion

Items 11 and 12 are representative elements of the two major factors which have been identified in earlier studies using the beliefs instrument implemented in this cross-cultural study (Howard, et al., 2000; Perry & Howard, 1999; Perry, et al., 1996; 1998). These two factors—'child-centredness' and 'transmission'—have provided the basis for studying differences between the beliefs of groups of teachers and for linking these to curriculum reform and teacher education movements.

Three key findings appear immediately from an analysis of Tables 2 – 4 above. Firstly, Hong Kong teachers have made significantly different responses to all other groups on each of these statements. Secondly, there are many more significant differences between groups on the 'transmission' statement than on the 'child-centred' statement. Thirdly, both Singaporean and Philippines teachers have made significantly different responses not only to each other but to all other groups on the 'transmission' statement. Possible explanations for these findings are made below, using the experience of the authors in each of the contexts.

The relative timing of curriculum reform in the various countries and the consequent emphasis on understanding over memorisation in curricula may be a feature. Singapore's curriculum has emphasised the understanding of mathematical concepts, problem solving, thinking skills and creativity since its reform in 1997 (Fong, 1999; Kaur, 1999; Menon, 2000). The Singaporean cohort of teachers in this study may be reflecting this change. For Hong Kong, curriculum reform in mathematics came after the implementation of the beliefs survey reported here so responses would not reflect this reform, suggesting that teachers may not yet have problematised the role of memorisation in mathematics learning. Curriculum reform in Australia established constructivist approaches to learning from about 1990, while Taiwan has had two such reforms during the 1990s. The result from the Philippines suggests that the influence from the USA, Canada and Australia through the mathematics education literature coming from these countries and through various collaborations with colleagues may have contributed to the differences in the Filipino teachers' responses. Memorisation is very important to Filipinos because the examination systems encourage it. Curriculum reform, however, is not the whole story, as can be seen in Changchun where new curriculum standards were released only in 2001.

Teacher education and the quality of the teachers as mathematicians may also be a telling difference between Hong Kong and Singapore. All Singaporean teachers undertake their teacher education in one university which is closely linked to the Ministry of Education. There is a coherence of purpose which results from both the quality of the mathematical experiences which teachers have experienced during their own schooling (Menon, 2000) and their teacher education. In Hong Kong, many elementary teachers don't have a degree and of those who do, very few have majored in mathematics. As well, the level of support from professional associations of teachers of mathematics in Singapore is much higher than in Hong Kong. These features may combine to make the Singaporean teachers more open to reform and more comfortable in their own abilities to undertake such reform.

On Item 12, the high averages recorded by the Singaporean, Taiwanese and Australian teachers (almost total agreement in the three cases) suggests that the rhetoric of reform has been heard. This may also be the case for the Philippines teachers where influences from the field of psychology are very strong among elementary teachers. Thus, belief in child-centred methods may be strongly held but not concretely translated into action. On the other hand, Changchun and Taiwan teachers seem to reflect the respect they have for their students and their experiences (see, for example, Ma, 1999) without having this reinforced by recent reform documents. The response by the Hong Kong teachers is consistent with their response to Item 11 and further emphasises the importance of traditional transmission modes of learning and teaching to this cohort of teachers. As one Hong Kong teacher suggested:

> Transmission is direct, fast and objective. The process of discovery is a long process in which students might discover something that is outside of teachers' expectations. Teachers also have difficulty knowing whether or not what students have discovered is correct. On the other hand, transmission is more direct. Teachers only have to check the numerical answer to see if students are correct.

Conclusion

In this paper, we have been able to only just scratch the surface of the data which has been gathered in this study. Nevertheless, there are many clear distinctions which can be seen in terms of the differences in educational practices and policies of the different groups of teachers. Whether we can explain some of the differences that have been measured on international achievement tests in terms of the beliefs pattern measured is a moot point. However, we do know that the beliefs of teachers do make a difference in what happens in the classroom and that what happens in the classroom is an important factor in students' learning of mathematics.

The collaboration which is evident in this study, along with much higher profile studies such as those by Ma and by Stigler and his colleagues, augurs well for future work in the area of cross-cultural studies of mathematics learning and teaching. Our contribution is to suggest that the area of teacher beliefs is one which is critical to this work.

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Real-world Problem Solving in Small Groups: Interaction Patterns of Third and Fourth Graders

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The study reported in this paper is concerned with the investigation of primary children's real-world problem solving where particular research interest relates to the dynamics of group problem solving in an authentic classroom setting. Initial interpretative analyses of the video-taped group interactions revealed a kind of "interweaving" of the individuals' lines of thought. The application of an interaction model from social psychology lead to the identification of four basic types of interaction sequences in the group work episodes.

The 4-year study which provides the scientific background of this paper was initiated because of the concern that despite a vast amount of research literature on (real-world) problem solving in the last two decades, still little is known about primary pupils' real-world problem strategies as related to:

- their original mathematical modeling processes, and
- the dynamics of group problem solving and peer interaction during group work in an authentic classroom setting.

This paper focuses on the second major strand of the study—the analysis of group work episodes from authentic third and fourth grade classrooms with the aim of identifying interaction patterns which help to elucidate the dynamics of social mathematical learning processes in problem solving. Although we are reasonably aware of the individual capabilities of single students, the complexity of the 'normal' classroom in terms of the mixed abilities, the impact of the variety of socio-cultural experiences of children, and their quantitative and qualitative participation in classroom interaction have been almost completely neglected.

Learning in Small Groups: Interaction, Communication, Cooperation

Current research literature on learning mathematics suggests that the investigation of this phenomenon should not be limited to the analysis of internal mental processes of knowledge acquisition of individuals. Learning is also a social process that frequently occurs in the interaction of individuals (see e.g., Cobb & Bauersfeld, 1995; Bruner, 1996; Steffe, Nesher, Cobb, Goldin & Greer, 1996; Boaler, 2000). Interactive processes are integral elements of learning; that is, learning is socially constituted (Krummheuer, 1997). Interdisciplinary approaches in this context frequently include the concept of communication, since interaction is dependent on communication. An individual who forms a relationship with another person and in doing so conveys information. Communication is however, also dependent on interaction. A person who informs another person about something at the same time influences this other person. Therefore, the content of interpersonal behavior can be described as communication, while interaction is concerned with the observable actions of individuals (Crott, 1979). Graumann (1972)

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distinguishes these two concepts in relation to respective research approaches. Thus, communication theories aim at the method of information transfer, while the scientific interest of interactionist theories is concerned with why and how individuals interact with one another.

Curriculum documents and teacher handbooks that take into consideration the implications of interaction and communication with peers for individual as well as social learning (see also Australian Education Council, 1991; Johnson & Johnson, 1991, Davidson, 1990) emphasise the importance of group work:

> Education research offers compelling evidence that students learn mathematics well only when they construct their own mathematical understanding. To understand what they learn, they must enact for themselves verbs that permeate the mathematics curriculum: "examine", "represent", "transform", "solve", "apply", "prove", "communicate". This happens most readily when students work in groups, engage in discussions, make presentations, and in other ways take charge of their own learning. (National Research Council, 1989, p. 58-59)

In approaches to group work in the mathematics classroom however, peer interaction is frequently associated with cooperative behavior and collaborative skills of the group members. In that, cooperation is often associated with a certain kind of interaction and expectations regarding the respective behavior of the group members, as for example described by Johnson and Johnson (1991), who defined cooperation as: "Working together to accomplish shared goals and maximise one's own and others achievement. Individuals perceiving that they can reach their goals if the other group members also do" (p. 230).

While there seems to be a shared understanding in the scientific community that interaction with peers is an important factor for mathematics learning, the assumption that getting children to work in small groups automatically results in collaborative behaviour is obviously a myth. In her analyses of task-related verbal interactions with respect to mathematics learning in small groups, Webb (1991) found differences in terms of the level of mathematics learning in different group settings:

> The optimum small group setting is one in which students freely admit what they do and do not understand, consistently give each other detailed explanations about how to solve the problems, and give each other opportunities to demonstrate their level of understanding (p. 386).

Furthermore, Yackel, Cobb and Wood (1993) have pointed out the importance of social norms in the classroom, claiming that "the interactions that occur when children work together in small groups are intimately related to the children's mathematical conceptions as well as to the social norms that have been negotiated in the classroom".

Only very few studies on group work have attempted to study in detail the interactions that are taking place (Davidson & Lambdin Kroll, 1991). In the "Handbook of Research on Mathematics Teaching and Learning" published in 1992, Good, Mulryan and McCaslin stated a lack of understanding as to how students acquire problem solving skills with respect to group work. One of the studies, that informed the research reported in this paper, is the work of Kieran and Dreyfus (1998) on collaborative problem solving. With respect to the mathematical discourse of two high-ability 13-year-old boys, they identified five types of interaction in the analysis of the conditions under which the partner directed talk was or was not beneficial.

However, comprehensive scientific evidence in terms of *which* interactive processes underlie the different types of learning that take place during group work is to date yet to be provided. In this context, the classes of social interaction in terms of contingency developed by the social psychologists Jones and Gerard (1967) can provide a helpful starting point for respective research studies in the field of mathematics education.

A Social-psychological View of Interaction in Terms of Contingency

The approach chosen by Jones and Gerard in the late sixties attempted to classify conversation with respect to the way in which the social exchange of ideas—the interaction—occurs and in how far the behavior of one person is affected by that of another person during this process. Basis of their analysis is the dyad—the smallest possible interaction unit:

> Once a conversation begins, social contingency is immediately present. To some extent, thereafter, the next response of person A will be contingent on the last response of person B. B's next response will be in turn be determined by the preceding A response, and so it goes. The plans of each actor continue, however, to exert some influence on the unfolding responses, so that person A in effect responds to his own last response as well as to the last response of B. The response is thus jointly determined by self-produced and socially produced stimulation ... this way of looking at social versus internal determinants is helpful in classifying types of interaction (p. 506).

Continuing their concern with the degree to which social interactions involve true social influence, Jones and Gerard identify four classes of interactions that vary in the patterns of weights to be attached to self-produced versus social stimulation: pseudo, asymmetrical reactive, and mutual contingency. In the following schematic representations of these four classes persons A and B are partners in the interaction that flows through a series of responses. According to Jones and Gerard, "the solid arrows reflect the predominant source of influence or response determination, the dotted arrows reflect the less important or minor source" (p. 506).

The first class of interaction—*pseudocontingency*—can be described as a limited case of social interaction. The existence of a contingency between A and B appears, but in fact social stimuli are only minimally involved, since both actors predominantly follow their own pre-established plans.

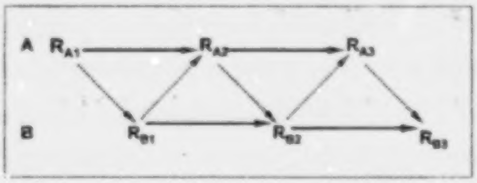

Figure 1. First class of social interaction: Pseudocontingency.

In the second class—*asymmetrical contingency*—the responses of person A are largely determined by self-produced stimuli (plans, strategies), whereas B's responses are determined predominantly by social stimuli produced by person A.

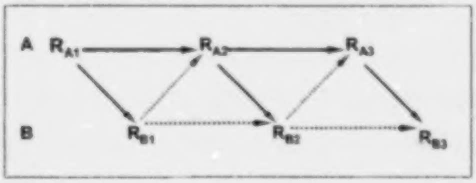

Figure 2. Second class of social interaction: Asymmetrical contingency.

According to Jones & Gerard, "it is at least logically possible that interaction could occur in a sequence in which each actor's response is almost entirely contingent on the preceding response of the other" (p. 510). In this situation neither A nor B follow self-produced stimuli, their responses are rather characterised by a *reactive contingency* and therefore closely related to and dependent on preceding social stimuli.

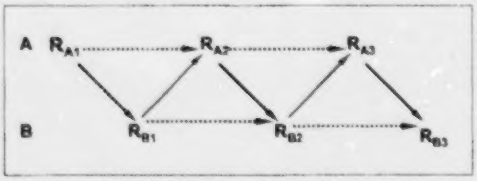

Figure 3. Third class of social interaction: Reactive contingency.

Only the fourth class of interaction is mutually driven. Each response is partially determined by the preceding response of the other and partly by the individual's internal and self-produced stimulation. Interactions on this *mutual contingency* class "thus require that a plan governs the responses of each actor, but the plan becomes continually recast in the light of the other's responses" (p. 511).

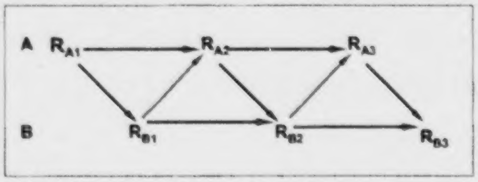

Figure 4. Fourth class of social interaction: Mutual contingency.

The classification of social interaction in terms of contingency as suggested by Jones and Gerard allows a more detailed study of phenomena that are frequently summarised under the heading 'social interaction', because it brings to the forefront the interweaving roles of self-stimulation and social stimulation in dyadic interaction. In the following samples it has been used to show how the application of this dyadic classification can be applied to larger social units—in this case groups of 3 to 5 pupils—to enhance our understanding of the types of interaction that occur during group work in the primary mathematics classroom.

Methodology and Data Analysis

The methodological framework of the research project underlying this paper is based on the interpretative classroom research approach (e.g., Jungwirth, Steinbring, Voigt, & Wollring, 2001) and involves pre-service teachers as teacher researchers. Consequently, the data collection and interpretation phases of the study followed a strict procedure consisting of four stages: video recordings, comprehensive transcriptions with respect to either the full document or selected segments of the recording relevant to a particular research question, the *sequential interpretation* of the data by a team of four student teachers under the guidance of the student teacher in charge of this data sample, and the *specific interpretation* of the results on the basis of relevant literature and research findings by the respective 'student teacher researcher' (for details regarding this procedure see Peter-Koop & Wollring, 2001). Each of the 23 student teachers involved in the study was responsible for the analysis of a specific sub-question in relation to a data sample (i.e., in most cases that data sample consisted of the work of one group of children), while the presentation, connection and discussion of the findings of these sub-studies in a type of meta-analysis was the responsibility of the project supervisor and author of this paper.

For the data collection, one grade 3 and two grade 4 mathematics classes from three different urban schools in a large city in north-western Germany were split into small groups, these groups were simultaneously videotaped while solving the task. Each group was videotaped on four occasions while solving a particular open real-world problem. Two student teachers assisted each group. In turns they either adopted the role of the teacher, assisting the group work if necessary, or they controlled the video recording. Once all groups had finished their work, the children presented, discussed, and evaluated the solution process and the respective results of each group in form of a "strategies conference" which was also videotaped.

The grouping of the children was organised in two different ways. In the first two classrooms (one grade 3 and one grade 4) the mathematics teacher was asked to form mixed-ability groups, because literature suggests that this is an optimal setting for group work. Furthermore, these groups were mostly heterogeneous in terms of gender. In the third classroom (grade 4) however, the teacher suggested that the children form their own groups. She argued that from her experience, the children would be in a position to know best with whom of their classmates they can work well. Thus, in this classroom groups were formed by the children primarily on the basis of friendship. Three out of four groups were gender homogenous. In addition, the members of two of the homogenous groups (one male, one female) were all low achievers.

Context of the Study: Open Real-world Problems

The real-world problems used in the study should intrinsically present challenges and thus motivate peer interaction during the solution process as opposed to problems that can be solved quite easily by an individual student. Therefore, the specific problems should be open real-world tasks that include *reference contexts* for elementary students. The wording of the problems should not contain numbers in order to avoid that the children start calculating with the given numbers without analysing the context of the specific situation. According to these criteria the following problems were chosen/developed and used in the study:

- How much paper does your school use in a month? (*paper problem*)
- Your class is planning a trip to visit the Cologne Cathedral. Is it better to travel by bus or by train? (*cathedral problem*)
- How many children are as heavy as a polar bear? (*polar bear problem*)
- There is a 3 km tailback on the A1 motorway between Muenster and Bremen. How many vehicles are caught in this traffic jam? (*traffic problem*)

Discussion of Selected Results

Due to the complexity of the complete study, in this section selected examples will be introduced in order to illustrate the insights that were gained with respect to occurring types of interaction and their classification. A cursory viewing of the video recordings of the different group works suggested a rather unstructured and unsystematic solution process—especially in the two groups with low achievers. In addition, the student teachers who had videotaped these groups initially were disappointed with the observed course of the group work that they had perceived as "chaotic" and "haphazard". At the same time they expressed surprise regarding the fact that most groups not only "somehow" managed to generate sensible solutions to all four problems but also were able to present and explain their solution to their peers during the "strategies conference".

The systematic sequential and special interpretations of the transcripts of the video recordings provided some insight into this observation. A re-occurring phenomenon was a kind of "interweaving of thought processes". Figure 5 illustrates, with respect to the *traffic problem*, that each of the three girls follows her own line of thought, while comments, statements or questions of other group members are (subconsciously?) incorporated in the individual considerations. As a result of the group discussion, Alwina reduces her original estimation of "more than 100 cars"—obviously because she realised that the longer a single car is, the lower is the number of cars that "fit" in a 3 km tailback (inverse proportionality).

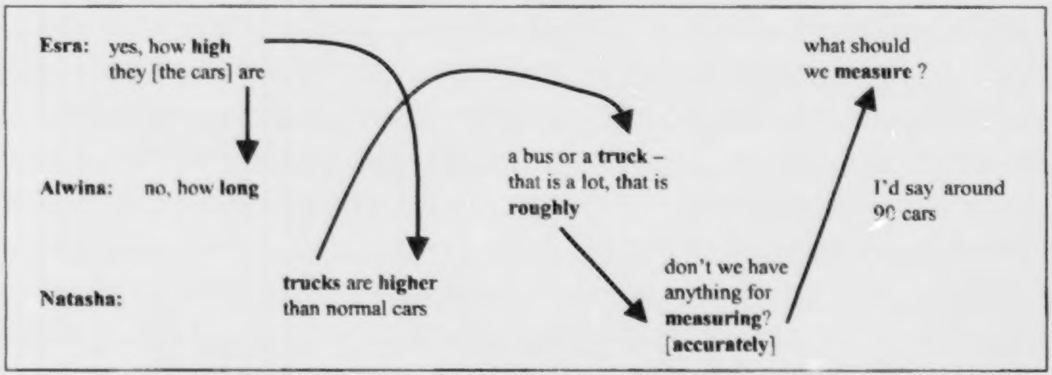
Esra:
yes, how high they [the cars] are
what should we measure ?
Alwina:
no, how long
a bus or a truck – that is a lot, that is roughly
I'd say around 90 cars
Natasha:
trucks are higher than normal cars
don't we have anything for measuring? [accurately]

Figure 5. Example of an "interweaving of thought processes" in the interaction three low-achieving fourth graders working on the *traffic problem*.

A major difficulty encountered during the interpretative data analysis however, was the representation of solution processes and group discussions in a way that allowed theoretical generalisations. Transcript excerpts provided little meaningful information in terms of the quality of the interaction process, and the initial attempts of graphical representation as shown above were not satisfactory in terms of distinguishing different types of interaction. The five types of dyadic interaction identified by Kieran and Dreyfus (1998) in this context helped to classify some, however, not all of the phenomena arising from the sequential interpretative analyses. A vast majority of the observed interaction sequences did not match the typical notion of cooperative work in which the group members on the one hand contribute and carefully explain their own ideas and on the other hand try to understand and appropriately react to the suggestions of others. Instead, the group members frequently expressed spontaneous thoughts, guesses and estimations as to what the solution could be.

The extended adaptation of the four classes of dyadic interaction in terms of contingency by Jones and Gerard to the group work episodes helped to classify the children's interaction processes in different group settings. Asymmetrical contingency, for example, was found predominantly in mixed-ability groups where participation was frequently dominated by high achievers (e.g., Vera in Fig. 6) who took a leading role with respect to the development and execution of the solution process.

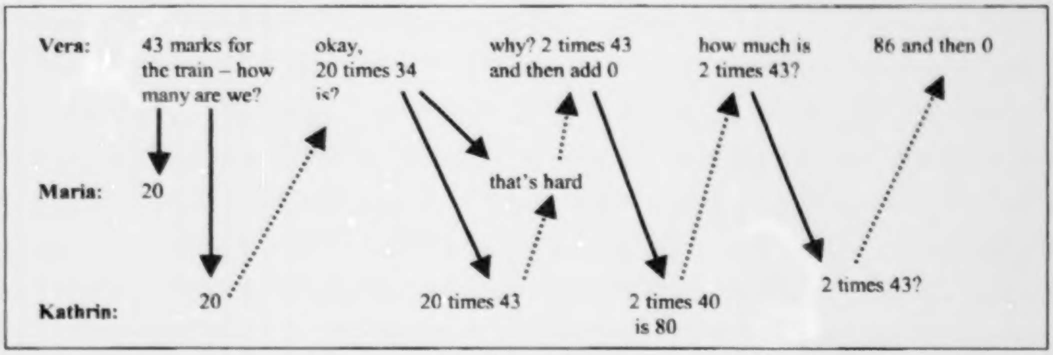
Vera:
Maria:
Kathrin:
43 marks for the train – how many are we?
20
20
okay, 20 times 34 is?
20 times 43
that's hard
why? 2 times 43 and then add 0
2 times 40 is 80
how much is 2 times 43?
2 times 43?
86 and then 0

Figure 6. Example of an asymmetrical contingency in a mixed-ability group working on the *cathedral problem*.

Mutual contingency on the contrary, was the class of interaction that was found least frequently and that occurred mainly among high achievers in mixed-ability groups (the data sample did not include a group in which all members were considered high achievers). The majority of interaction sequences that were found throughout the entire data set, in both mixed-ability groups as well as among low achievers, are characterised by reactive contingency, that is, the children reacted (often very spontaneously) to comments from their peers without developing and contributing their own strategies. A typical example of that class of interaction is shown in Figure 7.

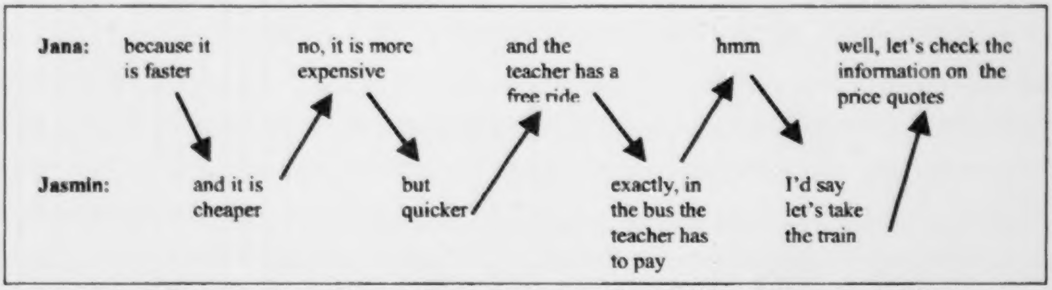
Jana:
Jasmin:
because it is faster
and it is cheaper
no, it is more expensive
but quicker
and the teacher has a free ride
exactly, in the bus the teacher has to pay
hmm
I'd say let's take the train
well, let's check the information on the price quotes

Figure 7. Example of a reactive contingent interaction of two children from a mixed-ability group of four working on the *cathedral problem*.

Conclusions

Since the majority of all group work episodes observed, including the groups of alleged low achievers, proved to be successful (the groups not only found but could also explain a sensible solution), their largely haphazard approaches obviously did not have a negative impact on the quality of the solution. However, group work episodes dominated by reactive interaction were in most cases very time consuming. Furthermore, in these episodes, which often took up to 60 minutes or longer, pseudo-contingent behavior was also a determining factor with respect to group interaction. This might be related to the fact that it is particularly difficult for young children—especially those with mathematical learning difficulties—to enter someone else's "universe of thought" (Trognon, 1993) parallel to their own thinking and mathematical sense making. A further element that was found corresponds to what Kieran and Dreyfus have called "anti-interaction". Supported by verbal signals like "hang on" or "just be quiet" individual members deliberately refused the

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interaction with others for a period of time in order to be able to follow a new thought and think about it without being distracted or disturbed. Such anti-interactive behavior however, should not necessarily be judged negatively. It indicates that also (or especially?) during group work, phases of individual and undisturbed thinking are crucial for mathematics learning. In order to comprehensively understand the conditions of learning in small group settings in mathematics classrooms, further studies with research designs focussing particularly on the individual learning processes are clearly needed.

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Statistical Thinking and Transnumeration

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This study explored transnumeration-type thinking exhibited by eleven and twelve year old students as they worked on a statistical investigation. Some aspects of this thinking are identified and described. The analysis raises issues about the development of such thinking and about how teaching might build on students' thinking strategies.

Statistical thinking is a complex intellectual activity that is "an artefact of civilisation, not part of our natural neural equipment" (Moore, 1998, p. 1257). This particular way of thinking and reasoning must be learnt. Wild and Pfannkuch (1999) in their framework for statistical thinking proposed five types of statistical thinking that they considered were fundamental in empirical enquiry. One of these was transnumeration, a coined word meaning "changing representations to engender understanding". If a modelling perspective is taken of statistics then we have a real world situation from which a statistical model of the situation is developed and transnumeration occurs in three instances: firstly, when quantitative or classification measures that are relevant to the problem are captured from the real world situation; secondly, in the statistical system when multiple representations of the data are employed in an attempt to understand what the data are saying about the real world situation; thirdly, when the statistical summaries are communicated in a form that is understandable to and convincing for the intended audience, and that is related to the original problem situation.

Many questions can be raised about how students learn and understand this type of thinking. Some of these are: What transnumeration-type thinking do students use? Are there transition or developmental stages in this thinking? How is a transnumeration structure of thought built up in teaching? Related research in this area on how students organise, reduce, represent, and interpret data has been conducted by many researchers such as Ben-Zvi and Friedlander (1997), Bright and Friel (1998), Cobb (1999), Moritz (2000), and Lehrer and Schauble (2000). Their research will be drawn upon to elucidate transnumeration-type thinking, a way of thinking that suggests how to change real and statistical data into other types of representations that may allow insights into the data and consequently the real situation.

Background of Study

This exploratory study was conducted by Rubick (2000) for her masters thesis. Her purpose was to ascertain whether all four dimensions of the Wild and Pfannkuch (1999) framework for statistical thinking in empirical enquiry could be used to characterise school students' thinking when they were conducting an investigation. The reason for such an exploration was that the framework had been created from statisticians describing their thinking when involved in projects, and she wanted to see if it were applicable to the

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 567-574). Sydney: MERGA. ©2002 MERGA Inc.

students' statistical thinking at a macro-level. Yoon (2001), with the permission of Rubick, analysed three of the interviews to identify and describe at a micro-level the conceptions of the students using part of the second dimension of the framework, namely "types fundamental to statistical thinking" which included, transnumeration, consideration of variation, reasoning with statistical models, and integrating the statistical and the contextual. The analysis reported here takes three interviews, which all three authors analysed, and delineates some categories of transnumeration-type thinking that can be observed.

Method of Study

Twelve Year 7 and 8 students (11–13 year olds) were randomly selected from the most able students in a high socio-economic level school. The students were put into pairs so that members of the pair were of the same year level and gender, and were known to be able to work together. Using an investigation and protocol created by Watson, Collis, Callingham, and Moritz (1995), the students were given sixteen cards. Each card contained information about one person: the name, age, weight, eye colour, favourite activity and number of fast food meals eaten per week. The students had not worked with multivariate datasets before and were not familiar with scatterplots. The students were first required to read and understand the information on the cards before thinking about what they could investigate. A pencil-and-paper environment was provided with materials such as calculators and graph paper being available for the students. They were interviewed and audio-taped for approximately one hour while they were conducting an investigation of their choice. The students were asked to think aloud as they progressed through the investigation.

Analysis

The focus of this analysis is on transnumeration thinking not on how students engage with the other types of statistical thinking identified by Wild and Pfannkuch (1999). An analysis of the interview data revealed that it was possible to describe the students'

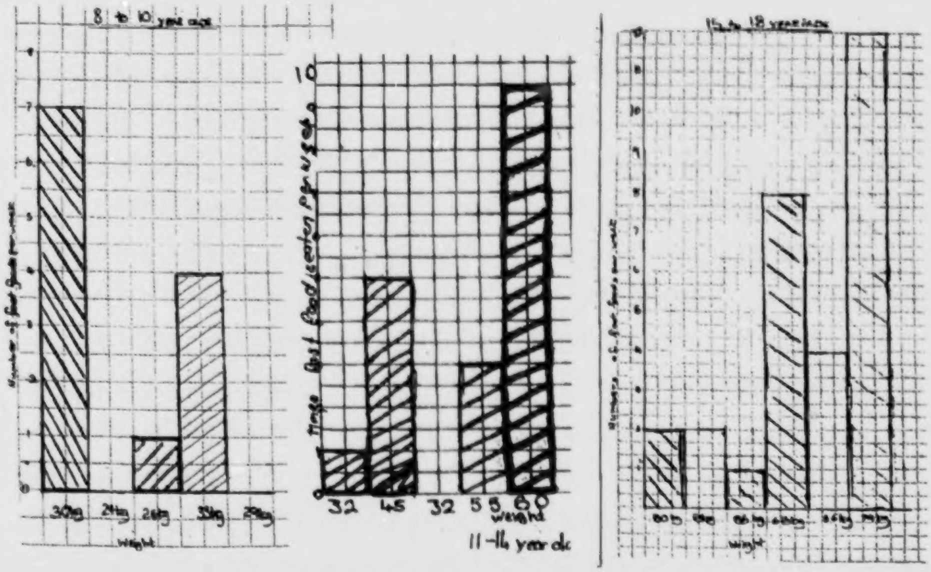

Figure 1. Graphs of Interview One students.

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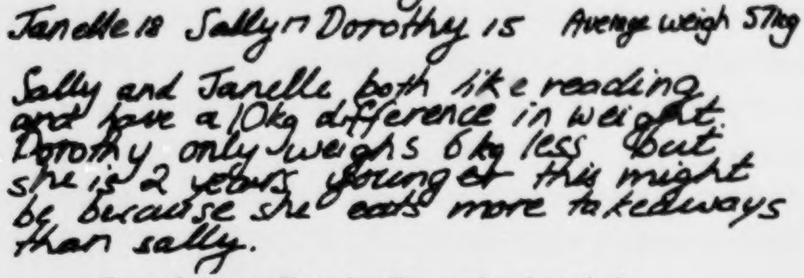
transnumeration thinking within the broad categories: classifying and reclassifying the data; calculating averages and constructing graphs; and communicating findings. To illustrate the range of thinking extracts will be drawn from three interviews. These will be referred to as Interview One students who grouped the data on age (Figure 1), Interview Two students who grouped the data on age and gender and found averages (Figure 2), and Interview Three students who grouped the data on age and activity and found averages and ratios (Figures 3, 4, & 5).


Janelle 18 Sally 17 Dorothy 15 Average weigh 57kg

Sally and Janelle both like reading and have a 10kg difference in weight. Dorothy only weighs 6kg less but she is 2 years younger this might be because she eats more takeaways than sally.

Figure 2. A part of Interview Two students' conclusion.

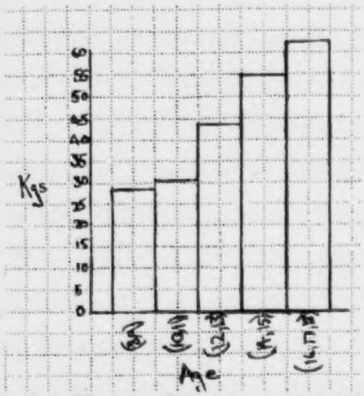


Figure 3. Interview Three students' age and weight graph.

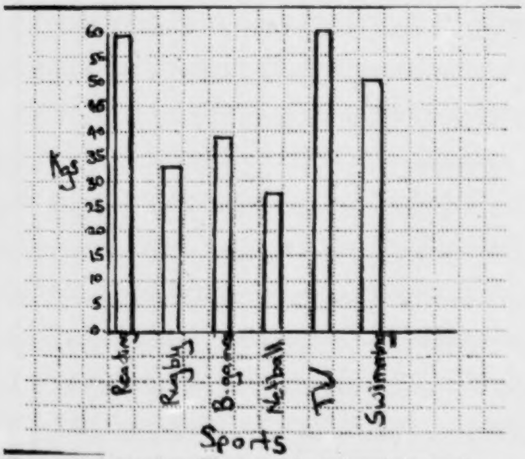


Figure 4. Interview Three students' activity and weight graph.

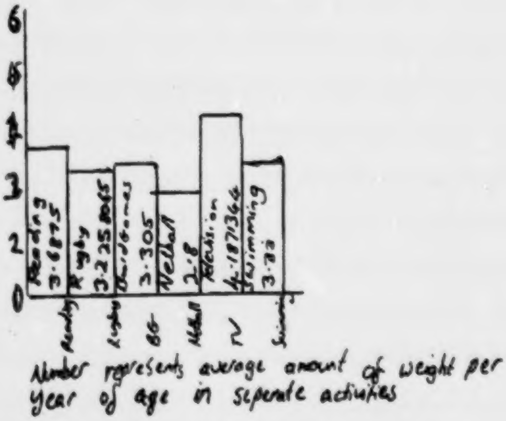


Figure 5. Interview Three students' activity and weight ratio graph.

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After spending some time understanding the dataset these students took weight as the response variable and investigated possible explanatory variables. The concern was whether weight could be explained by the amount of fast food eaten per week (Interview 1), by fast food eaten or activity or age (Interview 2) or by activity or age (Interview 3).

Classifying and Reclassifying the Data

Transnumeration thinking was first prompted through the students sorting the cards on a particular characteristic such as age. The students in Interviews One and Three sorted the cards into groups of active and passive activities but were unable to reclassify the data and define a new variable. The Interview Two students sorted on gender, but did not define it as a variable. Only two students in one interview (not discussed here) defined gender as a variable. The students in these interviews (1, 2 and 3) sorted the data into distinct age groups but only Interview Three students were able to reclassify age group as a qualitative variable. The thinking that is required from sorting the data into groups based on a common characteristic to reclassifying data and defining new variables seems to be an identifiable transition step. The classification, reclassification, or stratification of data, involves not only determining new variables but also understanding how data can be structured into quantitative and qualitative variables. Recognising and representing implicit information in the data do not appear easy for the students. A tool that may help in the classification process is the construction of a table.

The transnumeration of the card data to a table representation seems to indicate there may be three levels of response. When considering all the students interviewed, the tables created by them could be classified as: (1) abbreviated form, very similar to how the original data were presented (two students); (2) some grouping of variables evident, data partially ordered (seven students – includes the students discussed here); (3) close to a conventional format, data ordered (two students). One student did not construct a table but instead established explanations and reasons for the observed phenomena. We believe that the transnumeration thinking that led to the construction of a table was a learnt procedure.

Calculating Averages and Constructing Graphs

Once the data are classified and organised transnumeration thinking is operationalised to consider ways of summarising and displaying the data. The students in Interview One first established three tables for three age groups but then reversed this grouping by drawing lines between individual cases of the same age. This reversal to individual cases may indicate the students were in a transition phase towards dealing with grouped data or may indicate a problem with not knowing how to deal simultaneously with a qualitative variable, age group, and two quantitative variables, fast food and weight. They did not think to calculate averages for each age group. It is debatable whether such recall would have helped in their dilemma of handling three variables.

The students in Interview Two recalled that in statistics you could find or transnumerate data into averages. With this goal in mind they split the data on gender, stratified the data into age groups and then found the average weight of each age group (Fig. 2). This separation of the data into two qualitative variables and one quantitative variable is a considerable step forward in being able to compare the averages. They did not think, however, to compare or graph the average weights. This may be because they recalled that in statistics you could find averages but did not remember the purpose for finding an average, and hence the changes in representation did not lead to more understanding. Other reasons may be that they did not define the variables, or their experience of averages has been entirely with univariate data.

The Interview Three students investigated the data systematically. They dealt with only one qualitative and one quantitative variable at a time. For the qualitative variable they found the average weight for each age group which they then graphed (Figure 3). A similar process was carried out for each activity (Figure 4). When interpreting the activity graph they were surprised at their findings as they thought the average weight for the swimming activity should be lower than the board games; "the only reason why it was high was because we didn't have enough sample … because we only had one swimmer … and he was quite old so he weighed quite a bit". Deliberation about sample size led them to consider that age may be affecting the average weight for each activity so they determined that the way to change the representation was to find the average weight per year of age for each activity and then compare the activities through graphical means (Fig. 5). Reference to sample size and the factoring out of the variable age demonstrates that these students are moving towards fluency in handling data, in making statistical and contextual judgements, and in transnumeration-type thinking.

All the students constructed bar graphs. The students in Interview One constructed three separate graphs (Figure 1) for each age group. Their x-axis has unordered weights since age order is retained and the data for each individual case are presented fully. The retention of information about the individual case occurred also for the students in Interview Two. Their graph (not illustrated), however, only compared two individual-named cases of the same age, on gender, weight and fast food. The students in Interview Three used their calculated averages and ratios for transnumerating into a graphical representation. The individual case has been subsumed into a particular group average or ratio. The handling of quantitative and qualitative variables poses problems for the students when they attempt to trannsnumerate the data through finding averages or constructing graphical displays. These changes in representations give another perspective on the data and can potentially lead to more understanding.

Communicating Findings

In this phase of transnumeration the students changed their statistical summaries and displays into either verbal and/or written communications. In Interview One, in response to interviewer prompts, the students first communicated by reading each graph at a local level. One student lifted the information directly from her graph: "Well the person who weighs 30-kg eats 7 fast foods per week, 24 kg's no fast foods a week, 26 kg's one fast food a week …", whereas the other student read information about age and gender that was not explicitly in her graph: "he was 12 years old and weighed 45 kilograms and he had 5 fast food meals a week …". The first student, with another interviewer prompt about whether the graph might tell her anything else, then reasoned between the data as well as referring to age: "It tells you that people who don't eat fast foods and considering their age at the same time … have less weight than someone who has 7 fast foods a week but is younger." This struggle to change the representation into words involved thinking how the representation might confirm or justify her particular preconceived claim. The transnumeration culminated in their written conclusion which they considered necessary as otherwise "you wouldn't actually know what the graph is about". The conclusion was read out as: "These graphs show the amount of fast food eaten and the weight in various age groups. It proves that the more fast food you eat the more you weigh." The transnumeration was regarded as self-evident by the students at this stage.

The interviewer then transnumerated the separate graphs by putting them side-by-side and asked whether there was anything more they could say. After some time reflecting on why they drew the graphs, one student through now looking at the three graphs as one entity said:

> I just thought of something … The older the children get it seems the more fast food they eat … on average. Because on the 8-to-10-year-olds there's two children who eat no fast food. In the 11-to-14-year-olds there's only one person. In the 15-to18-year-olds there is no people. And the highest amount of [fast] food increases. Like um… in 8-to-10-year-olds the highest amount is seven. In the 11-to-14-year-olds the highest amount is ten and in the 15-to-18-year-olds the highest amount is 12.

The transnumeration of the graphs into this communication at a global level, or reading beyond the data, has facilitated more understanding about the data and context. The argumentation is based on a statistical statement followed by a data-based justification through comparison of maximums and minimums. A contextual evaluation of whether the statement makes sense is not articulated.

The Interview Two students communicated their findings by comparing weights of two individual cases within an age group to test their conjectured "10kg rule" for weight, which stated that they expected people to put on about 10kg each year. If the rule was followed they did not look for reasons whereas if the rule was violated they rationalised the discrepancy by finding a causal factor that explained the difference such as fast food eaten. Thus they changed the "10kg per year" representation into a causal story for the age group. In this way they were engendering understanding of the data. At no stage did they look at the subgroup as one entity. They had a global rule but gave a localised transnumeration and hence their argumentation, statistically and contextually, remained flawed.

An Interview Three student when asked to give a summary of his age and weight graph first read the information "… the 10–11[year-olds] had 30.5kgs as an average …". When asked for further information he gave a global transnumeration: "as they grow older they obviously get heavier and weight and … age have something to do with each other". Later on he mused: "I'm just thinking. There's like bigger leaps for some groups (he calculates the weight differences between each adjacent age group) … like a big difference between 10–11 and 12–13. That's probably where the growth spurt is … around 12 and 13 [years]". A shift from a global noticing of the non-linearity of the graph to a local data-based justification followed by a confirmatory contextual interpretation, is evident in this communication. The effectiveness of the transnumeration into a communication is promulgated from both a global and local perspective of the data, a comprehension of the graphs over the range of levels identified by Curcio (1987), and a form of argumentation based on statistical statements, data-based justifications, and a contextual evaluation.

Discussion

Transnumeration thinking for classifying data into relevant subgroups for comparison purpose was present. This type of thinking requires both statistical and contextual knowledge. The connecting step to then transnumerate the subgroups into defined and new entities, and the changing of quantitative variables into qualitative variables, was not well developed in the students. They found it a struggle to re-represent data into another grouping perhaps demonstrating the tension between focussing on individual cases and focussing on variables that may describe group behaviour. Bright and Friel (1998, p. 67) conjecture that "tables may play an important role as an intervening representation that can smooth the transition between representing raw and reduced data". We think, however, that the representations that re-aggregate or reclassify the raw data, which may pertain to the future analysis or to the question posed, play a role in table representations. Hancock, Kaput and Goldsmith (1992) also found that students lacked awareness of the need to represent implicit information in the data, such as gender from the given names, which they attributed to a lack of knowledge about data structures. We would conjecture that the reclassification of given data is not a feature of the taught curriculum.

The students displayed a range of responses when they constructed tables from raw data. Recognising, developing and implementing criteria for an effective classification procedure for their tables was not easy for these students, an aspect also documented in the research of Lehrer and Schauble (2000). Little research appears to have been conducted on students' construction and interpretation of statistical data tables. This study has raised questions about how students perceive tables. We believe, however, that a synergy between data, contextual knowledge, and statistical knowledge informs this transnumeration-type thinking about how to structure data.

In the structuring of data there is an intertwinement of local and global transnumeration-type thinking throughout the process of a statistical investigation. The seamless interchange between local and global thinking can be illustrated by the following hypothetical scenario. When students are presented with raw data one individual-case entry is considered at a local level. From this starting point the interchange of perspectives occurs: data are sorted on age and age becomes one global entity; data are split into local age subgroups; data are globally redefined as age-group and perceived as one entity again; each age group average is calculated locally; the averages are perceived as one global entity; each age group average is graphed and interpreted locally; the graph is perceived as one entity and interpreted globally. From such a scenario it may be possible to become aware of the current thinking of students and, if necessary, scaffold it into the next stage.

When Moritz (2000) gave bivariate and multivariate data situations to similar-aged students to graph he identified single, double, and series comparision graphs of individual cases as the possible outcomes. Students in the current study produced double comparison (Interview 2) and series comparison (Interview 1) of individual cases, and series comparison of averages and rates (Interview 3). This raises the issue of whether there is a possible developmental path for comparison of averages since the students in Interview One produced separate graphs for each age group, and the students in Interview Two calculated averages but did not graph them whereas the Interview Three students did.

Another issue pertaining to transnumerating data into graphical representations is the handling of quantitative and qualitative variables. This includes learning the types of representation for such variables, the number of variables it is possible to deal with within one representation, and factoring out or controlling variables. The students in the current study created their own graphs and ways of dealing with such variables, and were able to communicate their findings at a higher level than their representations might indicate. This suggests that students at this school level should be dealing with multivariate data, which is supported by the research of Moritz (2000). Students should be encouraged to create their own representations before being introduced to conventional ones since this will not only enable them to build up an understanding of how to manipulate and change representations but also foster creative thinking about how to produce innovative graphical representations for insights into data. This latter type of thinking was identified by Ben-Zvi and Friedlander (1997) as a possible high level of thinking that is desirable in statistics. The curriculum and teaching should encourage students to think about how to handle multivariate data..

The communication of findings was consistent with the three levels of comprehension of graphs identified by Curcio (1987), that is, reading the data, reading between the data, and reading beyond the data. For example, in Interview One a student was able, with interviewer prompts to transnumerate the graphs into a communication that shifted through each of the three levels. At the third level her language became statistical with her reasoning justified through comparing minimums and maximums within the dataset. A similar type of argumentation with a statistical statement, data-based justification to provide evidence for the statement, followed by a contextual evaluation has been described by Cobb (1999). The students in Interview Three referred to sample size but generally the students were not at the stage of generalising statistical statements and justifying statements through considering sample size. They were also not at the stage of considering such issues as whether the perceived pattern was real or random, or evaluating statistical or contextual assumptions. The communication of findings, which is a transnumeration of the statistical summaries into an evidence-based conclusion that will convince the intended audience, should be the focus of more research. In particular, research should attempt to describe the developmental levels for the type of answers that might be expected from students.

Conventional teaching at the middle-school level eschews the handling of multivariate datasets yet the ability of these students to perceive messages in such data with non-conventional graphs was quite remarkable. More research is needed on the development of transnumeration-type thinking and how transnumeration-type thinking strategies might be promoted and built up in the teaching process and curriculum. Extracting implicit messages from data requires a constant dialogue between the data, the context, and the student. Transnumeration-type thinking acts as a catalyst for this conversation.

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Monitoring Effective Use of Computer Algebra Systems

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Computer Algebra Systems (CAS) potentially offer students a powerful tool. However the mere availability of CAS does not enhance teaching and learning; students (and teachers) need to use it effectively. This paper presents a framework for components of Effective Use of CAS and illustrates its use as a guide for monitoring the progress of students. The results presented highlight the importance of considering the interaction between technical and personal aspects of Effective Use of CAS.

Background

Since their development in the 1970's and their introduction into some tertiary teaching and learning in the 1980's, Computer Algebra Systems (CAS) have been recognised as highly valuable for doing mathematics and potentially valuable for teaching and learning mathematics. In 1989, after reflecting on her experience of observing undergraduate students using CAS for doing and learning mathematics, Heid listed the characteristics of symbolic manipulation systems that could have potential for changing the content and processes of mathematics taught in secondary schools. She summarised these characteristics as follows:

1. Symbolic-manipulation results are exact and free of manipulation errors.
2. Symbolic-manipulation results are quickly generated.
3. A wide range of symbolic capacities is available within a single environment.
4. Symbolic-manipulation systems can handle more complicated problems than most students can do by hand. (1989, p. 411)

Studies in the intervening years, involving both tertiary and secondary mathematics classes (for example Atkins, Creegan and Soan, 1995; Bennett, 1995; Mayes, 1993; McCrae, Asp and Kendal, 1999; Pierce, 1999; Lagrange, 1999) have supported Heid's early findings, but also point out that the mere presence of CAS in a classroom does not mean that its potential benefits will be realised. Students must learn to use both hardware and software effectively. Artigue (2001) calls this process, by which such available technology becomes a powerful tool, 'instrumental genesis'. The learning this process requires presents a new, additional challenge for students. Atkins, Creegan and Soan (1995) expressed concern that students learning new mathematics with new technology may be distracted by the overhead of learning to use the technology while Lagrange (1999) observed that these 'technical difficulties in the use of CAS replaced the usual difficulties that students encountered with paper/pencil calculations' (p. 144).

To benefit from the availability of CAS, students must not only be able but also willing and discriminating in their use of this new technology. Arnold (1995) observed that the participants in his study showed a range of levels of engagement with the technology. He found that students' use of algebra software was sometimes impeded by their beliefs

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about mathematics and their perceptions of what was valuable. Arnold reported that students who viewed mathematics as 'answer-based' dismissed the possibilities for exploration of mathematical ideas that CAS provided. They valued unaided individual effort and devalued the use of technology. Lagrange (1996) also commented that in his experience not all students wanted to use CAS. They did not want to be relieved of pen and paper work and that many, in fact, enjoyed doing routine calculations. The present authors' previous studies (Pierce and Stacey 2001a, 2001b) provide examples of the individual nature of students' responses to the availability of CAS. This is clearly shown in the following students' comments.

> **Student A:** Sometimes I use pen and paper and (later) find you can do it on the computer. Then I prefer the computer but I don't start on the computer first or I get confused. (2001b p.16)
>
> **Student B:** I think it (CAS) actually helps me learn new things because when there are new things that I'm learning, while I'm finding them difficult, I can use DERIVE and go through the steps. With more practice, and seeing DERIVE go through it, I pick it up myself and then I can feel confident doing it myself without a package. (2001a p.9)

The value of CAS, like any new tool or aid, depends on how effectively it is used. Employing CAS to do mathematics requires the student to become familiar with both the hardware and the software associated with this technology. This presents the student with some learning overhead on top of learning mathematics. Students' success in obtaining benefits from the use of CAS will depend on how effectively they learn to use this technology. The efficacy of their use will depend on both technical and personal aspects: whether the student can operate the program with a minimum of difficulty; their attitude towards the use of CAS; and the manner and purpose of their partnership with CAS. In the light of the earlier literature and our previous experience of students' learning in a CAS environment, the first named author decided to develop a framework which might act as a guide for monitoring students' progress in Effective Use of CAS.

Framework for Effective Use of CAS

The aim of the framework, set out in table 1, is to highlight aspects of CAS use that determine how effectively a student is able to employ CAS both to do and to learn mathematics. The framework may provide a guide for systematically examining whether students are using CAS effectively. It may also provide a basis for planning and reflecting on teaching content and practice. It is designed for use by researchers and teachers, to structure observations about students' progress and to identify features of teaching which contribute to their progress. The purpose of the framework is not to provide a list of categories such that an observation instrument can be created with each item relating to one, and only one category. This may be desirable but does not seem possible.

The framework divides Effective Use of CAS into two aspects; the students' ability to handle the *technical* aspect of their CAS and the *personal* aspect, encompassing their attitudes and manner of use of CAS. Each of these is further divided into elements, which are discussed below and then illustrated by a set of common instances. Whereas the aspects and elements are intended to be generally applicable across levels of mathematics studied and types of students and courses, the common instances are specific to the setting where the framework is being used. The common instances of table 1 were developed for teaching an introductory functions and calculus course.

Table 1
Effective Use of CAS Framework

Aspects	Elements		Common Instances	
1. Technical	1.1	Fluent use of program syntax	1.1.1	Enter syntax correctly
			1.1.2	Use a sequence of commands and menus proficiently
	1.2	Ability to systematically change representation.	1.2.1	CAS plot a graph from a rule and vice versa
			1.2.2	CAS plot a graph from a table and vice versa
			1.2.3	Create table from a rule or vice versa
	1.3	Ability to interpret CAS output	1.3.1	Locate required results
			1.3.2	Interpret symbolic CAS output as conventional mathematics
			1.3.3	Sketch graphs from CAS plots
2. Personal	2.1	Positive attitude	2.1.1	Value CAS availability for doing mathematics
			2.1.2	Value CAS availability for learning mathematics
	2.2	Judicious Use of CAS	2.2.1	Use CAS in a strategic manner
			2.2.2	Discriminate in functional use of CAS
			2.2.3	Undertake pedagogical use of CAS

Elements of the Technical Aspect

The *technical* aspect relates to students' ability to use their own CAS to achieve mathematical goals. *Fluent use of program syntax* can be summarised as the changing of conventional mathematics to CAS syntax and knowing 'which buttons to push' or 'which commands to execute' when using the program to perform mathematical calculations. Common Instances of fluent use are seen when students enter syntax correctly or use a sequence of commands and menus proficiently. *Ability to systematically change representation* can be seen when students use CAS to, for example, plot a graph from a given rule, plot a graph from a table. Changing representations is highlighted because, in common CAS, it involves moving from one module of the software to another, whereas the element fluent use of program syntax applies within a module. *Interpreting CAS output* can cause difficulties that will lessen the Effective Use of CAS. In this paper, 'interpret' refers to the specific understanding needed when the format of results is different from the conventional pen and paper presentation, or is constrained by the limitations of the screen, rather than being able to make mathematical or real-world deductions from an output

Elements of the Personal Aspect

A *positive attitude* towards the use of CAS is seen when students value the availability of CAS as a tool for both doing and learning mathematics. Students with a positive attitude could be expected to persist with CAS use and overcome initial technical difficulties due to lack of familiarity with the peculiarities of the program. Students with a negative attitude towards the use of CAS for doing mathematics are more likely to avoid its use. The manner in which students approach the use of CAS can be described as being anything from a passive observer to fully involved participant, who displays *Judicious Use of CAS*. This is seen when students plan to use CAS in a strategic manner, rather than randomly "trying things out". They will selectively choose to use CAS for more difficult or time consuming problems, doing other problems by hand so that their *functional* use (aiming to get answers to problems) is discriminating. Furthermore, they will undertake *pedagogical* use, by using CAS to explore relationships or patterns in order to increase their own understanding of mathematics.

The Study: An Illustration of Use of the Framework

A study was undertaken to examine and monitor students' use of CAS in both doing and learning mathematics. The group studied consisted of 21 undergraduate students taking a 15-week (13 weeks teaching) introductory calculus course, where CAS (DERIVE 2.55) was available for teaching, learning, and assessment tasks. It is fully reported by Pierce (2001). The aim was to use the framework as a basis for considering whether students' Effective Use of CAS improved during the course and, if there was differential change amongst the students, which students improved in what aspects and why. Surveys, interviews and observation were used to collect data for class results and detailed case studies.

Technical Difficulties and *Judicious Use of CAS* surveys were administered at the conclusion of laboratory classes in weeks 1, 7 and 13. Students were asked to reflect on their use of CAS during that laboratory session and respond to a series of statements. The statements relating to possible difficulties (see table 2) had the option of 'not applicable' as well as a 5-point frequency scale from 'never' to 'every time'. The Judicious Use statements (see table 3) were presented as a single multiple-response item. In week 1 extra questions were asked concerning students' prior use of technology for mathematics and in week 13 they were asked to respond to statements concerning when they choose to use CAS and if they find CAS helpful. These responses, with those from interviews conducted in weeks 13-15, provided basic information about students' attitude towards using CAS.

Table 2
Statements from Technical Difficulties Survey with links to Effective Use of CAS framework

Possible area of difficulty	**Framework Element**
1. Authoring , using /, ^ , *, +, - symbols	1.1
2. Using brackets to force the structure expressions	1.1
3. Using syntax for commands eg FIT, VECTOR, F(x)	1.1 1.2
4. Using sequences of commands eg to substitute	1.1
5. Interpreting the results of the soLve command	1.3
6. Obtaining exact or approximate solutions	1.3
7. Working from the screen to ordinary maths symbols	1.3
8. Moving to a graph window	1.2
9. Obtaining a window which shows the graph required	1.2
10. Zooming to show key features of a graph	1.2
11. Copying from the screen for sketch graphs	1.3
12. Working out the scale of a graph	1.3

Table 3

Judicious Use of CAS survey items with links to Effective Use of CAS framework

Judicious Use of CAS survey statements	**Framework Instances**
Please tick as many of the statements below as apply.	
Today I have used DERIVE to:	
Find answers if the computer was suggested.	2.2.2
Explore problems if the computer was suggested	2.2.3
Explore variations on the set problems.	2.2.3
Explore, other than when directed but on the same topic	2.2.3
Explore other mathematics not on today's topic	2.2.3
Find answers I could 'easily' have done with pen and paper.	2.2.2
Find answers to 'hard' questions	2.2.2
Find answers to time consuming questions.	2.2.2

Observations of CAS use were recorded during class sessions and during examinations. The teacher/researcher made observations notes during laboratory classes, noting the way students were using the CAS and the nature of the questions they asked. The aims of this observation were to determine: if students had technical difficulties in using CAS; the 'manner' in which they approached its use; if they demonstrated discriminating functional use; and the extent to which they used CAS for pedagogical purposes. Discussions between the teacher/researcher and students during class sessions also provided information about their attitude to the use of this technology. The observation data were used to confirm the classification of students' functional and pedagogical use and to establish their profile of 'manner of use'. The extent to which students used CAS in a strategic manner was determined from their comments and the researcher's observations. Consideration of the students' survey responses and these observations provided cross validation.

Results and Discussion

All but one student had studied mathematics in year 12 but 7 only took a lower level course (V.C.E. Further Mathematics). All students had previously used scientific calculators and forty-one percent had also used graphical calculators for school mathematics. While thirty -eight percent had made some use of computers for school mathematics, only one student had previously used a Computer Algebra System.

Technical Aspect

It had been expected that students' ability to adapt to the use of this 'new' technology may relate to their general computing skills but no student experienced very high levels of technical difficulty. Only a student who claimed to have 'excellent' general computing skills showed any obvious advantage in using this 'new' software. Students' level of self-reported (table 2) and observed technical difficulty fluctuated during the course. Sufficiently similar results were found for each of the three elements of technical difficulty that, in this paper, they will be considered together. Total technical difficulty was scored out of 30 (A composite score equally weighting 1.1, 1.2 and 1.3 where zero indicated no difficulties i.e. low scores were better). At the end of week 1 half the group scored between 0 and 10 and the rest 10 to 15 and by mid course all scored below 10 and most below 5. However late in the course while half the group scored below 5 the other half scored between 15 and 20. Two factors may explain this result. First there were new difficulties to be overcome each time new mathematics was introduced that required new commands and syntax. Functions and then calculus required similar technical skills but trigonometry (taught late in the course) presented new syntax and interpretation hurdles. Secondly some students had few difficulties earlier in the course because they had avoided using CAS. In this short course those 'late starters' were experiencing difficulty in the last weeks. Observation showed that, in general, those who practised using CAS overcame technical difficulties and thus the personal aspect of Effective Use of CAS impacted on the technical aspect.

Personal Aspect

Based on their comments in class throughout the course and their responses to surveys and at interviews, the group could be divided into those with a positive and those with a negative attitude towards the use of DERIVE 2.55 for learning mathematics. Students who began the course with an open mind and an even mildly positive attitude showed improvement during the course in their Effective Use of CAS. As they gained confidence in using the program, they widened the range of use of the program and, as a consequence, their attitude to the place of CAS in their learning became more positive. On the Post-course survey even the three students who initially had a strong negative attitude towards the use of CAS each suggested that they had gained some appreciation for the place of CAS in doing some problems quickly. However, their attitude could only be described as 'less negative' than early in the course. These students with a negative attitude did not necessarily experience greater technical difficulty but they did make more limited use of CAS. It should be noted that a positive attitude did not preclude technical difficulties. Some students with positive attitude still reported experiencing a wide range of technical difficulties during the course. To attain a high level of Effective Use of CAS each student must learn to operate the program correctly. Feeling good about its use is not sufficient.

No student wanted to replace all by-hand mathematics with the use of CAS. Each student said that they preferred to do 'easy' problems by-hand. However, perceptions of what constituted an'easy' problem differed from student to student. There was also observable change during the course. Students sometimes used CAS to search for a general pattern by doing a series of examples in which they systematically changed the value of one parameter. Once a 'rule' had been established they would move to doing this processing mentally or by-hand. Most students became discriminating in their functional use of CAS, choosing to use CAS when they thought the problem would be time consuming or difficult. For example as they gained speed and confidence with differentiation they used CAS to differentiate rational functions but not polynomials. It was interesting to see that the majority of students liked to try to work problems by hand and would often choose to work in parallel with CAS, using its facility to check their by-hand work. In these ways students increased their Judicious Use of CAS.

Observation and students' survey responses indicated change during the course in both students' intent (*functional* or *pedagogical*) and manner of approach (*passive* – just observing others, *random* – trying anything, *responsive* –blindly following directions or *strategic* – following their own reasoned plan). As might be expected, students initially used CAS only for functional purposes and this use was non-discriminating. In fact they used CAS for simple problems which they might easily have done mentally or by -hand. At this stage most students were learning to use the program rather than learning new mathematics. In order to check their CAS results, students often chose to use CAS for questions that they felt they could confidently do by-hand. They wanted to be sure that they were using the CAS program correctly. All students progressed to become more discriminating in their functional use (as described above) and to undertaking some pedagogical use of CAS, at least when this was explicitly suggested. Some students took the initiative to use the facility to follow their own variations on problems in order to further their own understanding, taking a strategic approach and extending their pedagogical use of CAS.

Early in the course 40% of the group were passive or random in their approach to CAS and of the 60 % who sometimes used it strategically only 25% sometimes initiated such planned used without teacher direction. By emulating examples (for course details see Pierce, (2001)) and with teacher encouragement, by late in the course 90% of students used CAS in a strategic manner and 67% of these students would sometimes initiate this planning without teacher guidance. Only 10% persisted with a passive or random approach to the use of CAS. By late in the course most students were operating judiciously.

The detailed case studies emphasised that student response to the availability of CAS, and hence their Effective Use of CAS, was quite individual. The following brief examples illustrate this complexity. Sam and Jocelyn provide us with examples of students with a negative attitude towards CAS. Both avoided the use of CAS. Sam had considerable technical difficulty throughout the course and did not progress to using CAS in a strategic manner. Jocelyn was honest and open about her feelings when she said that computers scared her. In fact she experienced a low level of technical difficulty when she did use CAS but this did not change her attitude. Jennifer valued the availability of CAS and practised her CAS use even though she experienced technical difficulties each time new mathematics was introduced. Her difficulties with CAS syntax were linked to her difficulties in using conventional mathematical notation. Louise had a positive attitude towards CAS use but was not serious in her engagement with the mathematics and so continued to use CAS in a mindless random manner throughout the course. Stephen could do most of the work easily by hand and so used CAS less. He talked about being frustrated by technical difficulties but observation suggested that he actually experienced few problems. Stephen was positive about using CAS to extend his learning and to do harder problems. These examples suggest that the various elements of the framework should not be considered in isolation and that both Aspects must be addressed if CAS is to become a useful tool.

Conclusion

The purpose of this paper has been to describe what constitutes Effective Use of CAS, to present these constituents in an organised framework and to demonstrate the value of the framework by using it to describe the evolution of the Effective Use of CAS of a group of students in a functions and calculus course. The division of Effective Use of CAS into two basic aspects, the *technical* and the *personal*, seems robust. The intentions of these two aspects seem readily accessible and data can be gathered on each of these aspects separately. Even in one class, students varied considerably on each of the dimensions. The data showed some independence of the technical and personal and that students with positive attitude could be technically strong or weak. On the other hand, the study has shown that the personal and technical aspects influence each other over time: students with positive attitudes, for example, practise using CAS and so their technical ability improves. The data relating to the technical aspect were more 'user friendly' interfaces and hand-held CAS calculators, but, based on more recent teaching experiences, we still expect that the range of technical difficulty within a class is potentially wide. Overcoming technical difficulties is still a major consideration when teaching with CAS.

The division of the personal aspect into two elements has been successful. Once again, the two components are independent in the sense that the study found students with positive attitudes who were strong in their Judicious Use of CAS, using it strategically, and with discrimination while, on the other hand, a student with a positive attitude often used it mindlessly and randomly thus exhibiting low Judicious Use of CAS. The framework was also useful to point out to the teacher/researcher how her teaching could be altered to further emphasise the development of Effective Use of CAS.

The results of this study only serve to emphasise that a CAS, new in the box, merely has potential. The process of a user realising its capacity and going further to make it a sophisticated instrument in their hands (the process Artigue (2001) calls "instrumental genesis") still needs attention. In this study the framework was sufficiently successful in providing a structure for recording and analysing important components of the process to recommend its use in larger studies.

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Student Difficulties in Abstracting Angle Concepts From Physical Activities with Concrete Materials

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A Year 3 angles unit using the Teaching for Abstraction paradigm was developed on the basis of earlier empirical research and trialed by 12 NSW teachers. Analysis of data submitted by the teachers showed that the unit was generally successful but a number of minor areas for improvement were identified. An analysis of independent classroom observations of students working on the angles lessons is reported in the present paper. It was found that the difficulties students experienced could be classified as matching, measuring, drawing, and describing errors. The results clearly support the hypothesised steps in learning by abstraction.

The angle concept is exceptional because it arises in so many different contexts. For example, angles are not only used to describe the shape of the corner of a geometrical figure but they are also used to specify a direction, an amount of turning or opening, and an inclination or slope. Mitchelmore and White (2000a) showed how the multifaceted nature of the angle concept makes it a difficult concept for children to learn, and proposed a theory of learning angle concepts by successive abstraction and generalisation. They proposed that young children initially recognise superficial similarities between physical situations and abstract everyday concepts such as table corner, road junction, door, and roof. Gradually, they recognise deeper similarities between these objects and abstract more general concepts such as corner, turn, and slope. They may be said to have formed a general abstract angle concept only when they see that all these concepts in turn have an even deeper similarity.

Mitchelmore & White (2000a) showed that children can identify so-called *2-line angles* (e.g., corners of a room, road intersections, pairs of scissors), where both arms of the angle are visible, as early as Year 2. On the other hand, although young children seem to understand very well a number of *1-line angle* situations (doors, windscreen wipers, ramps, etc.) and *0-line angle* situations (e.g., doorknobs, pirouettes, wheels), even by Year 8 many students cannot interpret these situations using angles. To do so, students have to recognise that these objects are similar in that they can all be regarded as consisting of two linear parts that meet at a point and that the relative inclination of these two parts has a crucial significance. For 1- and 0-line angle situations, one or both linear parts have to be imagined or remembered.

The same researchers (Mitchelmore and White, 2000b) proposed that the teaching of angle should consist of guiding students to make the abstractions and generalisations just described. They advanced three principles of their method, which they called *Teaching for Abstraction*:

- *Familiarity.* Students should first become familiar with a variety of angle contexts.
- *Similarity.* Teaching should then focus on helping students recognise the similarities between these contexts.

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 583-591). Sydney: MERGA. ©2002 MERGA Inc.

- *Reification.* Activities should be undertaken whereby the recognised similarity becomes abstracted to an angle concept which can be operated on in its own right.

The present paper describes an attempt to put this theory into practice, focussing on the difficulties which students must overcome.

A Year 3 Angles Unit

Following exploratory research using one-to-one and small-group teaching experiments (White & Mitchelmore, 2001a, 2001b), an experimental Year 3 unit covering the angles material in the 2D Space strand of the K-6 mathematics syllabus was developed for the NSW Department of Education and Training and trialed during 2001.

Content

The unit (Mitchelmore & White, 2001a) consisted of 10 lessons, as summarised in Table 1. The first 5 lessons dealt with 2-line angles and the second 5 lessons dealt with 1-line angles. It was judged that 0-line angles would be too difficult for Year 3.

Table 1
The Content of the Angles Unit

Lesson	Lesson
1. **Pattern block corners** Matching and comparing pattern block corners, introducing the word "angle"	6. **Clocks** How a clock uses angles to show the time
2. **More corners** Corners in the classroom, right angles	7. **Doors** Interpreting the opening of a door in terms of angles
3. **Comparing corners** Making an angle tester to informally measure corners	8. **Slopes** Interpreting slope in terms of angles
4. **Scissors** Using angles to describe the amount of opening of a pair of scissors	9. **Angles, angles everywhere** Matching angles across all the situations investigated in this unit
5. **Scissors-like objects** Measuring and matching angles in other objects with two parts connected by a pivot	10. **Creating angles** An open-ended task involving a variety of different angles

The unit followed the principles of Teaching for Abstraction in the following ways.

- *Familiarity.* Students explored a variety of angle contexts (corners, scissors, clocks, doors, slopes) separately to learn about their crucial, angle-related features.
- *Similarity.* Teaching employed frequent direct and indirect matching of angles:
  - In *direct matching*, the angle in one context was physically superimposed upon the angle in the other context. For example, the corner of a pattern block was fitted into the corner of a window frame.

  - In *indirect matching*, an intermediate angle-like object was used to indicate how the angles in the two contexts were similar to each other. For example, a bent straw was used to show that many angles in the classroom were "the same".

  Matching was particularly crucial in the case of 1-line angles. The approach used was to help children see how to interpret each position in terms of the angle formed between the object and some "neutral" position (12 o'clock on the clock, the closed position of the door, and the horizontal direction for a slope).

- *Reification*. Several activities were aimed towards abstracting the angle concept from the concrete situations which were seen to be similar:
  - Angles were *measured* by finding how many "little angles" (30°) fitted into them. The "angle tester" shown in Figure 1 was developed for this purpose.
  - Angles were *drawn* at approximately the correct size and (in the case of clocks and slopes) the correct orientation.
  - Children *described* how different contexts were similar, where the lines and vertex of an angle were, and what an angle was.

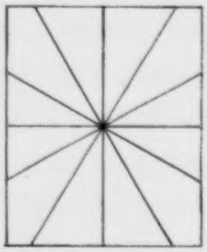

Figure 1: A primitive angle tester made by folding a sheet of paper. Each "little angle" is 30°.

The Trial

Participants. A total of 12 teachers from five DET schools in Sydney trialed the angles unit. The teachers first attended a one-day workshop at which the researchers outlined recent research on student understanding of angle concepts and the teachers worked through many of the student activities and role-played the administration of an assessment interview. They then administered the assessment interview to a target group of eight students, taught the unit at the rate of about one lesson per week, and re-administered the assessment interview to the target students. Finally, the teachers met with the researchers for a one-day de-briefing workshop.

Data collection. Data was collected from two sources.

1. The teachers provided comments on individual lessons and the unit as a whole during the de-briefing workshop. The data from their assessment interviews, together with a folder of work samples for each of the students interviewed, were also collected and analysed.
2. The first author visited each teacher twice. During each visit, she observed one lesson, writing field notes on the flow of the lesson and the behaviour of the target students; interviewed the target students in groups about what they had learned; and

discussed the lesson with the teacher. The first author also viewed videotapes of several student assessment interviews.

Results. An analysis of the data collected from teachers was reported in Mitchelmore and White (2001b). To summarise, teachers' assessments showed that their students' understanding of both 2-line and 1-line angles had improved substantially over the course of the teaching. By the end of the unit, almost all students seemed to have learned to identify and compare 2-line angles, but only about a half of them were able to do so in 1-line angle situations. Teachers felt that the variety of material, the sequential nature of the lessons, and the hands-on nature of the activities were the best features of the unit but indicated several minor areas for improvement.

The teachers' data gave us very little indication of what actually happened in the classroom. It is the purpose of the rest of this paper to provide this extra dimension by analysing the data collected during the class visits. The focus is on the difficulties students experienced in abstracting angle concepts from the various activities they undertook.

Student Difficulties

The lesson field notes, along with transcriptions of the student and teacher interviews and the students' work samples, were first analysed to identify the episodes where students experienced or reported difficulties. (Unfortunately, no examples of Lessons 8 or 10 were observed.) These episodes were then sorted to identify common themes. It was found that the difficulties could be classified into four categories: matching, measuring, drawing, and describing. It may be noted that these four categories of activity are precisely those which were used to implement the similarity and reification principles of Teaching for Abstraction.

Matching

Students experienced few difficulties in matching 2-line angles, either directly or indirectly, when both lines were clearly defined. However, even at the end of the unit, some students could not do so accurately. For example, one student placed the triangular pattern block in a corner as shown in Figure 2 and considered it a match. Another student tried every block until she finally chose the square block as matching the right-angled corner of the desk.

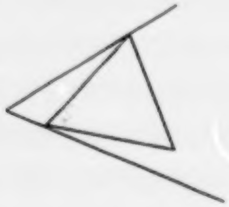

Figure 2. Incorrect matching of two corners.

The scissors caused most students some difficulty. Although the pivot was a focus of the teaching (and the assessment results showed that almost all students identified it as the vertex of the angle formed by a pair of scissors), most students matched the edges of the pattern block with the edges of the blades and therefore did not match the vertex of the pattern block with the pivot.

Many students showed the well-documented error of confusing angle with length or area at some stage during the unit. The angle tester was particularly effective in highlighting the distinctiveness of the angle concept. At the start of Lesson 3, most students thought that the angles at the top and bottom of the rectangle in Figure 1 were greater than those at the sides (either because the areas were greater or the lines longer). By the end of the lesson, as a result of a number of matching activities (for example, fitting a 30° corner of a pattern block into each angle, and folding the angle tester into twelve), most students had realised that the angles were indeed equal. Nevertheless, some students still had difficulty accepting this fact.

As expected from the analysis of student assessments, students had many more difficulties matching 2-line and 1-line angles. For example, many students could open a door to 60° but could not place a triangular block in the door in such a way as to indicate the angle. As a result, many students did not succeed in distinguishing angle from length in 1-line situations. For example, in comparing the openings of two doors, one student said "The big door has the bigger angle" even though the small door was open to a larger angle.

Measuring Angles

Students had little difficulty using the angle tester or pattern blocks to measure the size of 2-line angles (except for the scissors problem mentioned above). Students who had not conceptualised 1-line situations as involving angles experienced the expected difficulties. For example, students who had not made the connection between hours and "little angles" (30°) on the clock found it difficult to check the angles. On several occasions, students suggested checking whether angles on a clock were the same by using a ruler. They seemed to think they were measuring something, so a ruler must be necessary.

Another type of difficulty was revealed in Lesson 6. There were some who measured an angle by counting the lines between the hours. For example, the time between 5 and 10 o'clock was found by counting "6, 7, 8, 9", giving the answer 4 hours. (A similar difficulty is reported in respect to length measurement by Bragg & Outhred, 2000). Some students always counted the hours and/or angles from 12, stating that there were 10 hours between 5 and 10.

Drawing Angles

The many drawing exercises in the unit threw light on a number of difficulties which students experienced. Examples occurred in Lessons 1, 2, and 6.

In Lesson 1, students were asked to make "star patterns" out of pattern blocks by fitting the same corner around a central point, and then to draw the pattern at the centre of each star. Most students successfully made one or more stars, but many could not draw the resulting pattern. Some drawings are shown in Figure 2. Figure 2a shows the angles well. In Figure 2b, the student drew the whole blocks rather than the angles at the centre. Some students could not draw the shape at all (Figure 2c).

In Lesson 2, students found angles in the classroom, matched them with pattern blocks or a bent straw, and then drew the angles. Two groups of students drew an eraser as their example of an angle. They did not mean the angle between the flat sections of the eraser—the angle they drew was the curved part that changes shape as the eraser is used. In the observed classrooms, all the other angles found did consist of two linear parts. However, many students drew the complete objects and often did not indicate the angle at all (Figure 3). Others drew the actual straw in 2 or 3 dimensions, even including the stripes.

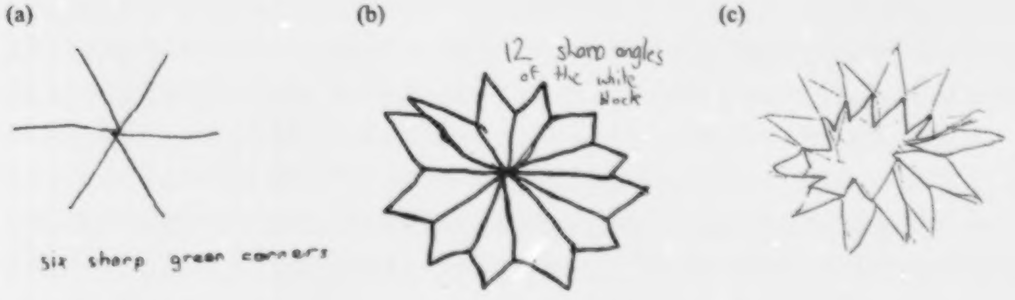
(a)
(b)
(c)
12 sharp angles of the white block
six sharp green corners

Figure 2. Examples of student work from Lesson 1.

Another type of difficulty was indicated by two students who found an obtuse angle on the blackboard tray, were able to make the angle with a straw, but then drew an acute angle. When it was suggested that perhaps they had not drawn the angle accurately, they correctly matched the straw again but still drew an acute angle.

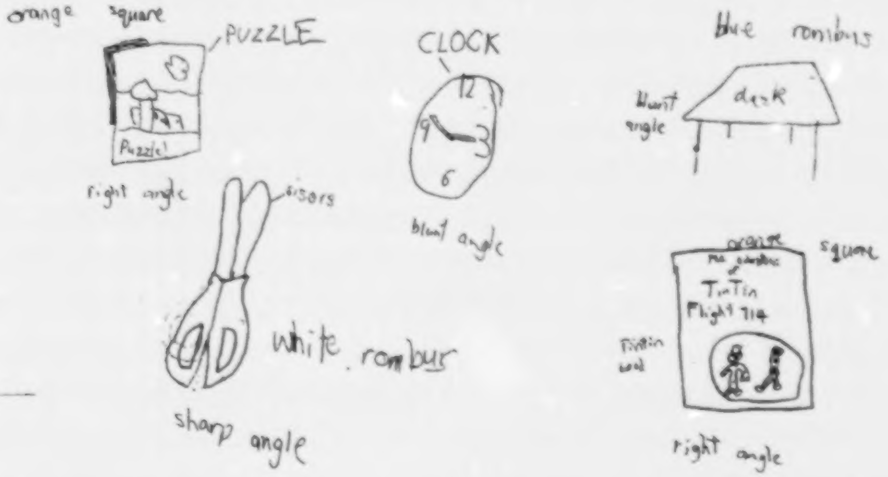
orange square
PUZZLE
Puzzle!
right angle
CLOCK
12
9
3
6
blunt angle
blue rombus
desk
blunt angle
sisors
white rombur
sharp angle
orange square
The adventures of TinTin Flight 714
Tintin and
right angle

Figure 3. A student's sketch of angles in the classroom.

In Lesson 6, the students made a one-handed clock (using the angle tester for the face and a pencil for the hand) and investigated the angles turned by the hand between various times. On a worksheet, they were given a start and finish time and had to indicate the number of hours and "little angles" from the angle tester, and then draw the angle. Some students drew the same (obtuse) angle for each example, illustrating a similar difficulty to the students who could not copy the blackboard angle in Lesson 2. Other students drew the angles with the correct size, but drew them all from 12 o'clock instead of copying the

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orientation of the hands. When asked how she had checked that there was a right angle between 1 and 4 o'clock, one student replied:

> I looked at the clock and I knew that 3 was, that 12 to 3 was—makes a right angle. Then I imagine it was 3 o'clock, like that with my fingers, like a L, and that's how I got to do the right angle.

Describing Angles

In sections of Lessons 5 and 9, discussions were planned to focus on the three crucial features of an angle (2 lines, vertex, and opening). Unfortunately, due to pressure of time or because teachers did not appreciate their significance, most teachers omitted these two sections. However, the topic was covered in the post-lesson interviews when students were asked, "What have your learnt?" Here are some typical responses:

> S_1: [An angle is] not the outside of lines where the lines spread out, it's where the point is.
> S_2: I learned that even though some shapes are different sizes and different shapes that the angles can be the same.
> S_3: I learned you can change the thingy, like on the scissors you can change the angle. But on some things like a different one, a cardboard or something like that, you can't.
> S_4: If a house door is open a particular angle and the house falls over and the door stays the same, it's the same angle.

Such responses suggest that students were beginning to abstract a general angle concept, but were having difficulties expressing their understanding.

Discussion

The angles unit was clearly effective in teaching students about angles. The analysis of student difficulties shows that student learning could be effectively mapped using the four categories of matching, measuring, drawing, and describing. Since these categories match precisely the pedagogical approaches used in the unit which were intended to promote abstraction, the results lend support to the theory of Teaching for Abstraction and show that it can be used as a practical basis for designing instructional sequences.

Student errors in matching different angle contexts illustrate the crucial significance of matching as a first step in learning to interpret a context using angles. Students who could not match the angle in a new context (e.g., doors) with the angle in an existing context (e.g., pattern blocks) would not have identified in the new context the two lines, the vertex, and the opening which are the essential features of an angle. So they could not be expected to measure, draw, or describe the angles in that context.

Children's drawings show how difficult it is for them to isolate the linear parts of a concrete object which form the arms of the angle. It is clearly a lengthy process to come to regard edges, joins, limbs, and pointers— which may be wide and anything but linear—as lines. Many objects which form the arms of an angle do not *look* like lines at all. They can only come to be regarded as lines when the child recognises the angular similarity between the situation where they occur and a more familiar situation which is known to involve angles and where the lines may be more obvious.

It seems a little easier for students to learn the significance of the vertex and the opening of an angle. The importance of the vertex is most clearly illustrated by children's difficulties in appreciating the significance of the pivot in a pair of scissors. The vertex is more clearly defined in other contexts (even doors) where there is no distraction like the intersection of the two blades on a pair of scissors. In most contexts, the opening aspect is also fairly obvious. Students who do not copy the size of an angle correctly have clearly not yet recognised the significance of the opening. Students who do not copy orientation correctly (e.g., when drawing a clock) may not recognise the significance of the orientation or they may be deliberately transforming the angle to a standard position.

Once students recognise that a context involves angles, they seem to have little difficulty with measuring angle size. Relating angle measurement to the basic concepts of measurement (as is done in *Count Me Into Measurement* [Outhred, Mitchelmore, McPhail, & Gould, in press]) might help those students who count markers instead of spaces. It would not seem a difficult task for students to refine the angle tester (Figure 1) to obtain the standard protractor for more accurate angle measurement.

This analysis of students' difficulties has indicated the important points to emphasise in teaching about angles. In each context, the students must clearly recognise the vertex, lines, and opening aspects of the angle. It is not necessary for them to be able to do so in words—clear verbal definitions seem to follow a long time after understanding. But students should be able to point out the three crucial features and use them in matching. The classroom observations have also confirmed the value of drawing activities in the learning of spatial concepts, both as a way of drawing students' attention to vital features and as a means for the observer to assess students' understanding.

While most students in the present study seem to have abstracted the concept of 2-line angles, and many students abstracted 1-line angles, it is evident that abstracting these concepts is difficult for Year 3 students. The teachers told us that the students needed more time on the angles in each context, and our analysis confirms this need. The unit described in this paper has therefore been revised into one unit of 8 lessons for Year 3 (mainly on 2-line angles) and another unit of 8 lessons for Year 4 (mainly on 1-line angles). The revised angles units will be trialed in 2002.

Acknowledgments

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Children's Difficulties With Base-Ten Numbers: "Face-Value" and "Independent-Place" Constructs

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Sixteen Year 3 students, in groups of four, worked collaboratively on numeration tasks, using either conventional blocks, or computer software. Participants were interviewed individually before and after group sessions to ascertain place-value understanding. From a teaching experiment design, using a grounded theory approach, emerged previously reported categories of response, including counting approaches, grouping approaches, and "face-value" thinking. A previously unreported construct emerged, the "independent-place construct". Teachers of young children should familiarise themselves with these common approaches to place-value tasks.

The study reported on here formed the basis of the author's doctoral thesis (Price, 2001); aspects of it have previously been reported by the author (Price, 1996, 1997, 1998a, 1998b). In this paper the author introduces a previously unreported construction made by children who are not secure in their understanding of base-ten numeration, the "independent-place construct", and compares it with the "face-value construct", observed by a number of researchers.

Place value is a term used frequently in mathematical writing, referring to a central feature of the base-ten numeration system: individual digits in base-ten numbers assume a value based on their position in the entire number, relative to the rightmost place, or the place to the right of the decimal marker, where present. Unlike other numeration systems, such as the Roman and Egyptian systems, a place-value system uses a small number of symbols to represent a variety of values, depending upon the position of each symbol. This gives a place-value system a huge advantage over non-place-value systems, in being able to represent an unlimited variety of quantities. In other words, any finite amount can be represented using merely the symbols 0 to 9, with a few other symbols such as "-" and a decimal marker symbol.

Place-value understanding is arguably the most important topic in the primary mathematics curriculum. A robust understanding of place-value concepts enables a person to manage the demands of problems from a wide range of mathematical topics, especially those that involve mental computation. The importance of the development of numeracy skills in children's education is well recognised; lists of numeracy sub-skills (e.g., Queensland School Curriculum Council, 1999) typically include proficiency in use of the base-ten numeration system. However, it can seem somewhat incongruous that a system which incorporates at its heart a small number of simple rules should be so difficult for teachers to teach, and for students to learn (Jones & Thornton, 1993; Ross, 1990). Thus research into children's learning of place-value concepts has an on-going, important role in mathematics education research.

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Review of the Literature

Researchers of children's mathematical understandings commonly deduce children's mental models, or "inaccessible mathematical realities" (Cobb & Steffe, 1983, p. 93) based on their performances on mathematical tasks (Davis, 1992; Resnick, 1983). The mental models of interest here are children's models for multidigit numbers; the term adopted in this study is *conceptual structures* (see also Fuson, 1990, 1992).

A number of researchers into children's development of place-value understanding have proposed schemes to describe how such understanding develops (e.g., Cobb & Wheatley, 1988; Fuson, 1990; Fuson & Briars, 1990; Fuson et al., 1997; Miura, Okamoto, Kim, Steere, & Fayol, 1993; Resnick, 1983; Ross, 1989, 1990). Such schemes have a number of features in common. Principal among these is the idea that children pass through a number of fairly well-defined stages, as their thinking develops from early knowledge of counting number sequences to more mature understanding of multidigit numbers and how they can be used in problem solving. One such scheme, proposed by Ross (1989, 1990) is notable for the number of other authors who have either referred to it or duplicated some of Ross's methods. The author also used some of these methods, and reached a number of conclusions regarding Ross's claims, outlined in this paper.

Digit Correspondence Tasks and Face-Value Thinking.

Ross (1989) proposed a five-stage model for the development of place-value understanding, based on her study involving 60 students, 12 each from Grades 2 to 5. She assessed children's place-value understanding based on their performance on a series of tasks involving written symbols and concrete materials. Results from one task, a "digit correspondence" task, were particularly revealing: when asked to explain the referents for each digit in a two-digit number, many students responded with a "face-value" interpretation of the digits—stating that each digit in a multidigit number represented only its face value. Ross concluded from these and other results that a surprising number of students were at an early stage of development of place-value understanding, and exhibited faulty understanding of place-value rules underlying the base-ten numeration system. Digit correspondence tasks have since been used by other researchers, including Carpenter, Franke, Jacobs, Fennema, and Empson (1997); Fuson and Briars, (1990); and Miura et al., (1993), as well as in this study. Digit correspondence tasks are described in the Methods section, and results from Ross's study and this study are compared later in the paper.

Face-value thinking is possibly the most common misunderstanding of multidigit numbers held by young children, and is described in many studies reported in the literature (e.g., Bednarz & Janvier, 1982; Cobb & Wheatley, 1988; Fuson & Briars, 1990; Fuson et al., 1997; Hughes, 1995; Miura et al., 1993; Sinclair & Scheuer, 1993). This faulty thinking about multidigit numbers has been given the status of a "conceptual structure", or "construct" by some authors, who posit that it indicates a particular understanding of numbers in a place-value system held by many children.

Methods

Due to the nature of the interactions under investigation, and the necessarily imprecise view of children's thinking captured, this was an exploratory study, designed to explore children's mental models for base-ten numbers as they interacted with materials, a teacher and each other. A grounded theory approach (Strauss & Corbin, 1990) was adopted, involving four main phases: (a) review of literature, (b) open coding of data, (c) axial coding of data, and (d) final integration of categories to form theory.

This study was conducted using a teaching experiment design, in which the researcher acted as the teacher of small groups of participants (Confrey & Lachance, 2000). A total of sixteen Year 3 students worked cooperatively in groups of four for a series of 10 sessions. Each group comprised two girls and two boys, all of whom were assessed as either of high or low mathematical achievement, based on the previous year's *Year Two Diagnostic Net* (Queensland Department of Education, 1996). The groups worked on a series of tasks designed to reinforce knowledge and understanding of two-digit numeration, and to introduce the hundreds place. Two groups used conventional place-value blocks (known to many Australian teachers as Multibase Arithmetic Blocks, or MABs) to represent numbers, as directed in various tasks. The other two groups used computer software designed by the author to represent numbers via pictures of place-value blocks which can be manipulated via various on-screen tools (see Price, 1997, for further details).

Each participant was interviewed before and after the teaching sessions, to assess any changes that occurred in participants' knowledge and understanding of place-value concepts during the study. The two interviews were parallel in form, each comprising 9 questions in five categories: number representations, skip counting, number relationships, digit correspondence, and novel tens grouping.

Digit Correspondence Tasks

As mentioned previously, digit correspondence tasks have been widely used in place-value research since they were used by Ross (1989). Digit correspondence tasks are a useful tool for investigating children's understanding of multidigit numbers, as they clearly indicate "face-value" thinking, as described later.

In a digit correspondence task, the subject is shown a collection of objects, and asked to count them. In each of the interviews used in this study, each participant was asked to count a collection of pop sticks—24 in the first interview. The participant was then asked to write the symbol for that number, and was then asked about each of the two digits "Does this part of your '24' tell you anything about how many sticks you have?" Asked this question, children with poor understanding of the base-ten numeration system will typically state that the '4' represents four sticks, and the '2' represents two sticks. Ross (1989) gave no indication that she pursued participants' understanding further in this situation; in the study described here, if a participant gave a face-value response, the interviewer asked the participant about the remaining sticks, to explain why they were not included in the sets of sticks apparently represented by the two separate digits.

In a modified digit correspondence task, also reported by other researchers, a child is provided with materials which could give misleading perceptual cues about the values represented. For example, a collection of 13 counters can be shown, and the participant asked to share them evenly among 3 cups, leaving one counter left out. In this situation, a child holding a face-value construct for two-digit numbers might be prompted to respond that the '3' in '13' stands for the three cups, and the '1' stands for the single counter left over.

During certain interview questions, including digit correspondence tasks, the researcher introduced counter-suggestions, as a means of testing the robustness of participants' beliefs. Thus, if a participant responded to a digit correspondence task to say that the tens digit represented collections of ten units, the interviewer asked if it were possible that the digit represented only its face value. While this method does not lead to a "naturalistic" idea of what a child believes, it is felt that it can reveal a level of tentativeness in the ideas of some children.

Results

Space limitations here restrict the reporting of results from this study. In this section a focus is made on data that lead to identification of apparent face-value thinking, especially digit correspondence tasks described earlier, together with other general errors made when answering questions regarding materials, written symbols, and number names.

The Face-Value Construct

Four types of question in the interviews evoked face-value responses from participants. Briefly, these questions asked participants to: (a) compare a collection of blocks with a written symbol, such as 1 ten, 3 hundreds, and 6 ones with '136'; (b) compare written symbols with similar digits, such as '259' and '295'; (c) identify the referents for the digits in a two-digit number, after counting a collection of pop sticks; and (d) identify the referents for the digits in a two-digit number such as '13', in light of objects grouped so that there were (in this example) three collections and one unit left over.

Table 1 indicates the prevalence of responses indicating face-value thinking among the study's participants, in response to these types of interview question. Numbers in brackets indicate the number of times that participants accepted incorrect face-value suggestions from the interviewer.

Table 1
Face-Value Thinking by Participants in Response to Certain Questions

	Mathematical Achievement Level		
Question	High	Low	Total
3-digit block representations	0 (0)	5 (3)	5 (3)
Comparing pairs of numbers	0 (0)	8 (7)	8 (7)
Digit correspondence task	0 (1)	12 (0)	12 (1)
Modified digit correspondence task	3 (3)	5 (4)	8 (7)
Totals	3 (4)	30 (14)	33 (18)

Note. Values not in parentheses represent incidents of face-value interpretations initiated by participants. Values in parentheses represent face-value interpretations suggested by the researcher and accepted by participants.

It is clear from the data in Table 1 that participants of high mathematical achievement levels did not often volunteer incorrect face-value interpretations of digits, and were reluctant to accept the researcher's counter-suggestions. On the other hand, participants of low mathematical achievement levels exhibited faulty face-value thinking comparatively often. One surprising result is that the modified digit correspondence tasks, which might be expected to induce a greater number of face-value responses, did not in fact do so. Apart from responses to the interviewer's counter-suggestions, participants were more likely to give face-value responses to the "standard" digit correspondence task, than to the one in which misleading perceptual cues were present.

Many participants gave face-value responses to digit correspondence tasks. Figure 1 illustrates a response pattern that was repeated by a number of participants when asked about the referents for the digits in the symbol '24': participants stated that the '2' represented two sticks, the '4' represented four sticks, and the remaining 18 sticks were "left-overs". It is an indication of the prevalence of face-value thinking among children of this age that, over the two interviews, 10 of the 16 participants gave responses of this type, in which they attempted to explain why there were leftover sticks when the referents for the two digits were removed from the collection. However, it is important to note that not one participant consistently used face-value explanations in answering interview questions, but participants instead exhibited a variety of ideas about numbers, with which they attempted to explain numerical situations posed to them.

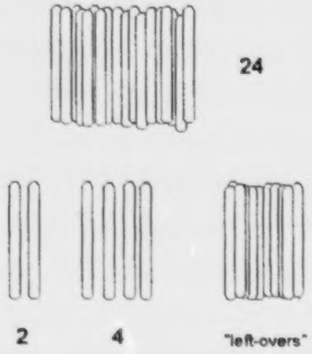

Figure 1. Typical "face-value" response to digit correspondence task.

Other Errors Made on Place-Value Tasks

The tendency of many children to explain that each digit in a multidigit number represents only its face value is just one, rather obvious, error of many made by children when dealing with multidigit numbers. In this study a large number of errors were observed. Listed below are six types of error of particular interest in light of the later discussion:

1. *Block handling errors*, such as using "incorrect" blocks to stand for each place in a multidigit number.
2. *Naming errors* of various types, such as using place names as labels only with no value attached to them.
3. *Trading errors*, such as trading a block of one size for one of another size, such as 1 ten for 1 one.
4. *Errors regarding position of blocks*, in which participants regarded the position of blocks as significant in determining the value represented by the blocks.
5. *Concatenation errors*, in which symbols were written by joining the digits for each set of blocks, such as '617' for 6 tens and 17 ones; or concatenating names, such as naming 4 tens and 12 ones as "forty-twelve".
6. *Symbol errors*, such as writing the number two hundred and four as '24'.

Discussion

The Independent-Place Construct

It would be easy to regard children's mathematical errors such those described in the previous paragraphs as signifying that their mathematical thinking is not highly developed, or to believe that their responses are unthinking, almost random responses to tasks which they cannot handle. However, a pattern emerged from this study's data that has led the author to propose the existence of another construct for multidigit numbers in the minds of many children of this age. A common element in the errors made in this study is that participants did not indicate that they perceived any link connecting "hundreds," "tens," and "ones" places, but regarded them as independent categories of quantity with separate names, separate digits, and separate block representations. The label "independent-place construct" has been coined to refer to this pattern of responses.

Evidence for the independent-place construct in this study. Each of the errors listed previously can be explained in light of an independent-place construct. For example, at various times participants were observed to choose the "wrong" block when representing a number. So, for example, Kelly was observed to use 2 ones and 8 tens to represent the number '28'. When asked if she could represent the number in a different way, she reversed the position of the same blocks, moving the 2 ones to the right and the 8 tens to the left. In another example, several participants traded a ten-block for a single one-block, indicating that they regarded the action as some sort of "block-for-block" trade, rather than a trade of equal values. Concatenation errors involved participants in joining together the names of two places represented by blocks of different sizes, as in "eighty-eleven" for 8 tens and 11 ones. When participants omitted the internal zero from a number such as '204', one interpretation is that they simply wrote one digit for each digit name encountered as they spoke the name: When asked to write '281', a child can write a digit for each number, thus: *two* [2] *hundred and eighty* [8] *-one* [1]; in the case of '204', the zero is not mentioned: *two* [2] *hundred and four* [4]. Such responses can be explained in terms of simple associations made between digits, names and blocks, with no regard for the meanings behind these representations.

The digit correspondence task responses described earlier also give support to idea that some participants were using an independent-place construct. It is possible that children giving such responses were attempting to explain the unexplainable within their knowledge of numbers: that each digit represents a finite quantity based on its face value, and that together the digits represent the entire quantity. There was evidence in the responses of some participants that they were uncomfortable with two conflicting ideas, namely that (a) each digit had a face value apparently indicating how much it represents, and (b) the sum of resulting face-value quantities does not equal the entire collection represented by the multidigit number. As two participants put it when attempting to explain the 18 sticks left out of a collection of 24 after taking out groups of 2 and 4, "Well they're nothing then if that's how that is", and "Well, they would but they're not included in … these things". One gains the impression that the children understood more than a face-value interpretation gives them credit for, and that they were attempting to deal with an internal conflict caused by their limited knowledge of how two simple symbols represent complicated place-value ideas. Furthermore, the fact that no participant consistently used face-value responses throughout the interviews implies that the view proposed by Ross (1989) and others, that children operate at particular levels with regards to multidigit numbers, needs to be reassessed.

Evidence for the independent-place construct in the place-value literature. There is considerable evidence in the place-value literature to support the idea of an independent-place construct. For example, Cobb and Wheatley (1988) described some of their participants as regarding tens as "abstract singletons", indivisible, separate units that could be counted apart from one units, but not as collections of 10 ones.

Several researchers (e.g., Cobb & Wheatley, 1988; Fuson & Briars, 1990; Fuson et al., 1997) have noted children adding or subtracting by considering each column separately, leading to results such as "16 + 9 = 115" and "28 + 36 = 514". Fuson et al. (1997) commented: "The physical appearance of the written multidigit marks as single digits … may combine to seduce children to use a concatenated single-digit conceptual structure even if they have a more meaningful conception available" (p. 142). Fuson, Fraivillig, & Burghardt (1992) posed four-digit addition questions to second-graders, who proceeded to add like blocks, even though they had not previously attempted such questions. This author suggests that completing such a task is quite achievable for a student, if an independent-place construct is adopted: If each place is considered to be separate from every other, then addition of like blocks, or symbols in like columns, enables quite efficient computation of otherwise challenging questions.

Comparing face-value thinking and the independent-place construct. Though there are similarities between the face-value construct and the independent-place construct, differences between them are significant when considering how young children regard multidigit numeration tasks. It is the author's contention that the independent-place construct is more useful than the face-value construct in describing children's thinking regarding multidigit numbers, and reveals a different character to their thinking regarding base-ten numbers.

Face-value thinking, where it truly exists, is a serious obstacle to understanding the base-ten numeration system. Children who actually believe that each digit only represents its face value fail to address the core feature of a place-value numeration system, that each digit represents a certain quantity according to its position in the number. However, data in this study suggests that in at least some cases, children are instead attempting to treat each digit as separate from the others, leaving open the possibility that they have some idea that different quantities are represented, though they may be unable to manage transitions between places.

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The Role of Professional Development in Using Calculators in a Sample of Queensland Primary Schools

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A survey ascertained teachers' knowledge about and use of hand-held calculators in primary mathematics classes. This paper focuses on 341 teachers' experiences of Professional Development(PD) on the use of calculators in teaching mathematics. Only nine had experienced such a workshop in the past two years while 10% had during their career. Approximately 40% regarded themselves as poorly prepared for calculator use in their teaching. A face-to-face hands-on PD session conducted by an 'expert' was the teachers' choice to develop their calculator pedagogy.

This study was motivated by a concern of the authors with the attitudes and ideas held by their pre-service teacher education students who were preparing to become early childhood and primary teachers. Many of the students were quite negative towards the incorporation of calculators into the mathematics curriculum and voiced the standard reasons about cheating and making students lazy thinkers (Biddulph, 1996). Furthermore these students had very little knowledge of how standard four-function calculators could be used for other than simple calculations. The authors wondered whether these negative attitudes and concerns were widely held amongst practicing teachers into whose classrooms many of these students go for professional experience rounds, and whether these supervising teachers had limited knowledge and experience with using calculators in their mathematics lessons. Hence it was decided to survey teachers in a sample of state primary schools in Queensland to seek answers to these questions.

Background to the Study

Sparrow and Swan (1997, p. 1) report that "very few surveys of calculator use in primary schools have been attempted" and nothing much has changed since their 1997 report on calculator use in Western Australian primary schools. In their survey it was reported that "almost three-quarters" of those surveyed agreed with the recommendations of *A National Statement on the use of Calculators for Mathematics in Australian Schools* (Australian Association of Mathematics Teachers, 1989, p. 7). Reys, Reys, and Wyatt (1993) found similarly that between 1979 and 1990 the use of calculators in one state in the USA remained largely unchanged. A study of high school principals and mathematics teachers in New York State (Ostapczuk, 1995) found that less than 50% of mathematics teachers routinely used calculators in classroom instruction and that more than 70% had no in-service available to them on calculator use. The great majority of the teachers were aware of the position statement of the National Council of Teachers of Mathematics on the use of calculators in schools.

The main documents that influence teachers in their lesson preparation in Queensland are the *Year Level Sourcebooks* (Department of Education, Queensland, 1989) and the *Years 1-7 Syllabus Support Document* (Anderson, 1995). These documents both support the views expressed in the *National Statement* that "all students use calculators at all year

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levels". The *Year Level Sourcebooks* go on to explain the role of calculators in problem solving and the development of positive attitudes to mathematics. It was regarded as opportune to conduct the survey to determine the current situation in Queensland state schools.

Methodology

Subjects

A stratified random sample of 159 state primary schools was taken from the 11 Education Districts in Queensland north of the Tropic of Capricorn. Within each district the schools were grouped into bands to give a range of school sizes and then the sample chosen randomly from each band. Details of the distribution across bands are shown in Table 1.

Table 1
Distribution of Schools Across Districts and Band Size

	Band[a]						
Education District	5	6	7	8	9	10	11
Emerald	5	2	3	3			
Longreach	7	1	2	3			
Mackay Hinterland	6	2	2	3	1		
Mackay North	3	3	1	3	2		
Mount Isa	2	2	1	2	2		
Townsville Burdekin	4	2	3	3	3	1	
Townsville North	6	3	3	1	2		
Tablelands	7	4	3	2	1		
Cairns & Cape York	7	5	3	3	6	1	1
Thursday Island	2	4	1	1			
Rockhampton	8	3	3	2	2	1	
Total	57	31	25	26	19	3	1

[a] as a general rule, increasing band size indicates larger school size but there can be other factors involved.

It is worth noting here that most of these districts contain small to medium sized schools many of which are in rural areas and some in quite isolated areas. Furthermore, only four of the districts are centred on the major Queensland cities of Townsville, Cairns and Rockhampton. There are very few band 10 and 11 schools across these Districts and only one band 11 school and three band 10 schools were chosen in the sample.

Instruments

A survey questionnaire consisting of 23 questions was developed and field tested with a group of local teachers. This survey was based on instruments used in previous studies in New South Wales (Howard, 1992) and Western Australia (Spanner & Swan, 1997) and adapted to suit the Queensland context. School administrators were asked to distribute the survey to all teachers within their schools and each teacher was given a return envelope to send the questionnaire individually to the researchers. Information was sought on the following areas: (a) grade currently taught; (b) years of teaching experience; (c) calculator professional development opportunities; (d) school policy on calculator usage; (e) when calculators should be introduced into the mathematics curriculum; (f) planning for calculator use in mathematics classes; and (g) areas in the mathematics program where calculators are included.

In this paper the data related to the teacher characteristics and aspects and issues associated with professional development are reported. (see Appendix for the questions related to this paper)

Data Analysis

Data from the surveys were entered into a database and analysed with the assistance of the Statistical Package for the Social Sciences (SPSS) Release 9.0.1 (1999).

Results

Approximately 1 500 surveys were distributed to the 159 schools in order to allow every teacher in the smaller schools and around 75% of the teachers in most of the larger schools to respond if they so desired. Of the 1 500 distributed a total of 341 surveys were returned by teachers who chose to respond. This represents a return rate of 22.7% which is not high and that may have been caused by the timing of the survey that occurred in the latter half of term four of the school year.

Table 2
Return Rate for Different School Districts

Education District	Number of primary schools Bands 5 - 11	Number of teachers responding to the survey
Emerald	13	13
Longreach	11	16
Mackay Hinterland	14	39
Mackay North	13	30
Mount Isa	9	16
Townsville Burdekin	14	41
Townsville North	15	40
Tablelands	17	25
Cairns & Cape York	26	64
Thursday Island	8	9
Rockhampton	19	48
Total	159	341

Table 2 indicates the number of schools in each district and the number of teachers in each district who responded to the survey. It reveals a good spread of teachers across the 11 districts that vary considerably in area and population. The 341 teachers who returned surveys were from 93 different schools and this represents a return rate of 58.5% of schools.

Grade Currently Taught

Table 3 is the distribution of the grades currently taught by the teachers answering the questionnaire. The category of composite was included as teachers were asked to indicate specifically if they taught a composite grade.

Table 3
Distribution of Teachers by Grade Level

Grade	Frequency	Frequency %
Preschool	6	1.8
1	35	10.3
2	22	6.5
3	35	10.3
4	25	7.3
5	26	7.6
6	23	6.7
7	30	8.8
Composite	130	38.1
Non-teaching staff	2	.6
Special Education	2	.6
Learning Support	4	1.2
Music	1	.3
Total	341	100.1

The distribution across the seven grades of the primary school was fairly consistent. What is somewhat surprising is the large number of teachers with composite grades – over one third (38.1%) of the teachers surveyed. It is also interesting to see six preschool teachers completing the survey as well as non-teaching and learning support and special education staff.

Years of Teaching Experience

Respondents were asked to indicate the length of their teaching experience. The distribution showing the numbers in the age brackets provided on the questionnaire is shown in Table 4.

Table 4
Distribution of Teachers by Length of Teaching Experience

Number of Years	Frequency	Frequency %
0 – 5	67	19.6
6 – 10	64	18.8
11 – 15	70	20.5
> 15	138	40.5
Total	339	99.4

The high percentage of people who have been teaching for more than 15 years represents many of the teachers who would have been in the education system when the current Queensland Mathematics syllabus was introduced in 1987. This syllabus was the first to give specific suggestions on the inclusion of calculators for topics such as long division and multiplication by two digit numbers.

In-service Sessions on Calculator Use

Of particular interest to the authors was the recency and number of any in-service sessions dealing with calculator use that the teachers had undertaken. They were asked whether they had participated in any such sessions in the last two years and only nine people (2.6%) indicated that they had. The number of sessions attended ranged from one to four with the median being two. Of the remainder of teachers who indicated that they had not been to an in-service workshop on calculators, the principal reasons given fell into the following main categories shown in Table 5.

Table 5

Main Reasons for Non-attendance at Calculator In-service Workshops

Reason	Frequency
No sessions were offered or no opportunities arose to attend such a session	229
Calculator in-service was not seen as a priority by the teacher or the school	17
No particular reason for a 'NO' answer	19
No response at all for a 'NO' answer	38
Isolation and distance were a major factor	9
Teachers did not have a need to attend a calculator in-service workshop	6
Teaching less than two years	5

Clearly the lack of availability of workshops on calculators in teaching was the principal reason that about two thirds of these teachers did not or could not participate in a calculator in-service. While only nine mentioned 'distance' as a reason, it is likely to be a major contributing factor to the lack of offering of in-service workshops by some districts that are large in area and lightly populated.

Opportunities for Becoming Proficient at Using Calculators as a Teaching Aid

Teachers were given a range of options to respond to when asked about opportunities that had been available to become proficient at using calculators as a teaching aid. As indicated in the distribution of responses shown in Table 6, approximately 69% of the teachers had some opportunities to develop proficiency in using a calculator as a teaching aid with about half of these referring to their initial teacher education as the time when this occurred. Clearly for people who have been teaching more than 10 years (~62%) this is a long time ago. Only 10% of the respondents indicated that they had been involved in professional development workshop that were specifically devoted to calculator use in teaching mathematics.

Table 6

Frequency of Opportunities at Becoming Proficient at Using Calculators in Teaching

Event/opportunity	Frequency	Frequency %
Initial Teacher Training	120	35.2
Professional Development Workshop	35	10.3
Informal Discussion with Colleagues	65	19.1
None	104	30.5
Other	14	4.1
Total	338	99.1

As a follow up to this question the teachers were asked to indicate how well prepared professionally they felt they were to use calculators in their teaching. Their judgements are shown in Table 7.

Table 7

Judgment of Professional Preparation to Use Calculators in Teaching

Level	Frequency	Frequency %
Very well prepared	10	2.9
Quite well prepared	128	37.5
Well prepared	51	15.0
Poorly prepared	115	33.7
Very poorly prepared	24	7.0
No response	10	2.9
Total	338	99.0

Fifty-five percent of these teachers judged that they were well to very well prepared to use calculators in their teaching. While this is an encouraging figure it still a major concern that about 40% of these teachers view themselves as poorly prepared to use calculators in their teaching.

Table 8

Preferred Method of Preparing Teachers to Use Calculators in the Classroom

Method	A book on how to develop personal skills with the calculator	A book on how to teach with calculators	PD on the use of the calculator conducted by an expert	A book on calculator activities for students	A video on the use of calculators in the classroom
Usefulness	29	106	250	159	95
Ranking [a]	5	3	1	2	4

[a] 1 = most useful, 5 = least useful.

When asked to rank five possible methods for assisting teachers to prepare for the use of calculators in their teaching, there was a diversity of rankings. Table 8 shows that the teachers strongly favoured Professional Development as the way to help them prepare to teach with calculators. The least useful way from the teachers' perspectives was the availability of a book that would develop their personal skills with a calculator.

A further examination of Table 8 reveals the order of decreasing usefulness as: PD on the use of a calculator, a book on calculator activities for students, a book on how to teach with calculators, a video on the use of calculators in the classroom, and a book on how to develop personal skills with the calculator.

Discussion

Findings in this survey have some clear implications for the school system in Queensland both on a local level and a district level. When asked for reasons for not participating in Professional Development workshops on calculators a number of the teachers indicated that distance was a problem. Furthermore, they indicated that their isolation made it difficult if not impossible for Districts to set up a workshop that was accessible to them. The lack of availability of substitute teachers required to release these regular teachers for the day is an added complication if such a workshop is set up during school time. There is a need to investigate ways of overcoming the problems of distance when it comes to delivering professional development.

The teachers made it quite clear that they preferred face to face interaction with an 'expert' for Professional Development. Given that this may not be feasible as suggested above, and given that approximately 20% indicated that their proficiency was helped by informal discussions with their colleagues, it would appear that modern technology could play a valuable role in overcoming the problems of distance in delivery of professional development. The use of video-conferencing that approximates face-to-face interaction has potential is presently used extensively by rural doctors for in-service activities and has potential. Trialling and evaluation of the internet, DVD and CD ROM in the delivery of such workshops is of high priority and not just for mathematics education.

University mathematics educators have a challenge with the large number of composite classes identified in this study. It is imperative that time is taken in developing pre-service teachers' skills in planning for and implementing mathematics programs in multi-age classrooms.

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Appendix

Ques. 1. What Grade(s) are you currently teaching?
If you teach a composite class please indicate what grades.

Ques. 2. How long have you been teaching? (please circle)

- 0 - 5 years
- 6 - 10 years
- 11 - 15 years
- more than 15 years

Ques. 3. Have you participated in an in-service session on calculator usage in mathematics in the last two years? (Circle your response)
YES (go to Ques. 4) NO (go to Ques. 5)

Ques. 4. If YES, how many sessions?

Ques. 5. If NO, any reason?

Ques. 6. What opportunities have you had for becoming proficient at using calculators as a teaching aid? (Tick all those that apply)

- Initial teacher training
- In-service/ professional development workshop
- Informal discussion with colleagues
- None
- Other. Please specify.

Ques. 7. How well are you prepared professionally to use calculators in your teaching? (Circle one)

Very poorly prepared	Poorly prepared	Quite well prepared	Well prepared	Very well prepared

Ques. 8. Listed below are five possible methods to prepare teachers for the use of calculators in their teaching. (Please rank these from 1 to 5 where 1 is the most useful and 5 is the least useful. Do not use equal ranks)

- A book on how to develop personal skills with the calculator
- A book on how to teach with calculators
- In-service/professional development on the use of the calculator provided by an expert
- A book on calculator activities for students
- A video on the use of calculators in the classroom

Investigating Textbook Presentations of Ratio and Proportion

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Many topics within the middle school mathematics curriculum connect to the concept of proportion. Interpretation of proportion situations and understanding of methods for solving proportion problems provides a structure than can be applied to other related topics. As a major resource for secondary mathematics, the extent to which popular textbooks link proportion-related topics was the focus of this study. Our analysis revealed little connectivity of ideas, confusing definitions and frequently illogical calculations. Questions are raised as to the messages texts send to students.

Introduction

In the middle years mathematics curriculum, many topics of study require proportional reasoning skills. For example, proportional reasoning is required in the study of the geometry of plane shapes, in trigonometry, in applications of percentage, as well as for the usual rate, ratio and proportion applications. According to Lesh, Post and Behr (1988) proportional reasoning is a prerequisite for the further study of mathematics: "Proportional reasoning is the capstone of children's elementary school arithmetic and the cornerstone of all that is to follow" (p. 93-94). The development of proportional reasoning then, can be seen as an important goal of primary school mathematics. However, research indicates the elusiveness of such a goal, as many students struggle with proportion-related topics (Behr, Harel, Post & Lesh, 1992; Ben-Chaim, Fey, Fitzgerald, Benedetto & Miller, 1998; Lo & Watanabe, 1997).

In this paper we firstly provide a background to the teaching and learning of proportional reasoning from the extensive literature that has developed. We then report on part of an on-going investigation into the teaching of ratio, rate and proportion in lower secondary school. The study reported here is a preliminary investigation into the ways ratio and proportion are presented in junior secondary textbooks in an attempt to explain one factor that may contribute to the continuing difficulty middle school students experience with topics based on proportional reasoning.

Background

Proportional Reasoning

The complexity of the proportion concept appears to rest in the extent of prior knowledge required for its meaningful development. As outlined by Post, Behr and Lesh (1988), proportional reasoning:

> …requires firm grasp of various rational number concepts such as order and equivalence, the relationship between the unit and its parts, the meaning and interpretation of ratio, and issues dealing with division, especially as it relates to dividing smaller numbers by larger ones. (p. 80)

Prerequisite knowledge necessary for proportional reasoning has been suggested by others. English and Halford (1995) stated that "fractions are the building blocks of proportion" (p. 254) and Behr et al. (1992) argued that " the concept of fraction order and

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equivalence, and proportionality are one component of this very significant and global mathematical concept" (p. 316). Streefland (1985) suggested that "learning to view something 'in proportion' or 'in proportion with' precedes the acquisition of the proper concept of ratio" (p. 83). According to Behr et al., the development of an understanding of ratio and proportion is intertwined with many mathematical concepts, including multiplication, division, fractions and decimals, but the essence of proportional reasoning lies in understanding the multiplicative structure of proportional situations. Multiplicative structure is contrasted to additive structure in being able to view, for example, 4 in relation to 8 as multiplying by 2 rather than adding 4.

As previously stated, research has indicated that students' understanding of proportion is generally poor. As stated by Behr et al. (1992), "there is a great deal of agreement that learning rational number concepts remains a serious obstacle in the mathematical development of children" (p. 300). Several authors have suggested reasons for this state of play. According to Streefland (1985) "ratio is introduced too late to be connected with mathematically related ideas such as equivalence of fractions, scale, percentage" (p. 78). English and Halford (1995) suggested that proportional reasoning is taught in isolation and thus remains unrelated to other topics. Behr et al. (1992) stated that "the elementary curriculum is deficient by failing to include the basic concepts and principles relating to multiplicative structures necessary for later learning in intermediate grades (p. 300).

The Topic of Proportion in the Middle Years Curriculum

In describing the topics of proportion in the middle years mathematics curriculum, Ben-Chaim et al. (1998) outlined the general types of proportional reasoning problems as comparisons of two parts of a whole (e.g., ratio of boys to girls in a class), rate or density problems (e.g., cents per litre, kilograms per cubic metre), and scaling problems (e.g., similar triangles). In Ben-Chaim et al.'s analysis, the solution methods for such problems required either a comparison of two complete ratios (e.g., which one is faster) or the calculation of a "missing value" when the other three values in an equivalent pair of ratios is known. The latter solution method arises from representation of the given ratios as a statement of proportion.

The standard algorithm for proportional situations is the representation of equal ratios, that is $^a/_b = {}^c/_d$ (Touriniare & Pulos, 1985), or $^a/_b = {}^c/_x$ where a, b and c are given, and x is the unknown. The standard solution procedure for solving proportion equations is via algebraic means: " cross-multiply and solve for x" (Post, Behr & Lesh, 1988, p. 81) or through rule application: "multiply the two numbers across from one another and divide by the other number" (Robinson, 1981, p. 6). The teaching of either the standard algorithm or the rule, however, appears to be a controversial issue. For example, Hart (1981) stated, "Teaching an algorithm such as $^a/_b = {}^c/_d$ is of little value unless the child understands the need for it and is capable of using it. Children who are not at a suitable level to the understanding of $^a/_b = {}^c/_d$ will just forget the formula" (p. 101). Further, Cramer, Post and Currier (1992) stated, the "cross-product algorithm is efficient, [yet] it has little meaning. In fact, it is impossible to explain why one would want to find the product of contrasting elements from two different rate pairs…The cross-product rule has no physical referent and therefore lacks meaning for students and for the rest of us as well" (p. 170).

As an historical interlude, the rule application for proportion situations outlined above (multiply the two numbers across from one another and divide by the other number), is generally known as the Rule of Three. The significance of the Rule of Three is outlined by Swetz (1992):

> The 'Rule of Three', commonly known in its time as the 'Golden Rule' or the 'Merchant's rule' was highly esteemed in the fifteenth and sixteenth century as being a powerful mathematical technique applicable to solve many problem situations. Today this rule would be recognised as a statement of simple proportion involving three quantities from which a fourth must be found. (p. 373)

It is through Swetz's words that the Rule of Three is seen as an ancient, efficient and very "handy" rule for solving proportion equations. However, developing meaning for the rule is an issue in developing conceptual understanding of proportion.

One strategy for giving meaning to the cross-multiply method for solving proportion equations (Rule of Three) has been outlined by Robinson (1981) where the construction of ratio "boxes" to correspond to the information given in a ratio situation is advocated. In a manner similar to the ratio tables advocated by Streefland (1985) and English and Halford (1995), Robinson's ratio boxes are designed to reflect the multiplicative structure inherent in proportional situations. For example, the situation of John catching 2 fish to Jim's 3 would be represented as follows:

John's fish	2
Jim's fish	3

When asked to determine how many fish John would have caught if Jim caught 15, the table would be extended as follows:

John's fish	2	2	2	2	2	=	10
Jim's fish	3	3	3	3	3	=	15

In simplified terms, the table would show the situation as a proportion equation:

2	x
3	15

or even simpler: $^2/_3 = {}^x/_{15}$.

According to Robinson, exploration of proportional situations in this form will lead children to discover the cross-multiply procedure for themselves.

In the John's Fish/Jim's Fish example given above, the fractional representation of the proportion equation follows from the initial tabular representation. Yet the proportion equation ($^2/_3 = {}^x/_{15}$) is representing something quite different to a part/whole fraction situation. The fractional representation of a proportion situation clearly must be linked and connected to the proportional situation it is representing, but it must also be contrasted to the part/whole fraction meaning.

Although there remains controversy over the explicit teaching of proportion equation-solving procedures, there is general consensus in the literature that the proportion equation must be introduced to students in a meaningful manner, with students provided with experiences to enable them to develop their own solution strategies. As proportional reasoning transcends, connects with, and is based upon many other mathematical concepts, instruction must focus on "connectedness" and "structure and context" which are two of a set of basic principles for the design of teaching (Bell, 1993). As stated by Bell, in a constructivist teaching style, students should be helped to see the links between related mathematical ideas (connectedness). Further, Bell contended, most student do not recognise the common structure that underlies parts of mathematics presented as different topics, and hence do not appreciate that similar solution methods can be applied. With the centrality of the idea of proportional reasoning in the middle years mathematics, there is considerable scope for instruction to focus on helping students recognise the proportional structure of many topics (structure and context). Given that many secondary school teachers follow the prescribed textbook for planning instruction in mathematics (Lianghou & Kaeley, 2000), there is an expectation that secondary mathematics texts will assist students make connections between topics that are often presented in different chapters and units.

The Study

The extent to which popular secondary mathematics texts incorporate Bell's principles of connectedness and structure and context was the focus of this investigation. As part of a wider study into the teaching of proportional concepts in the middle years of schooling, the focus of this part of the study was to investigate the ways that proportional concepts are portrayed in the widely used mathematics textbooks that provide the foundation for much of the mathematics instruction in our schools. In this paper, we report on the contents of one chapter from each of two frequently prescribed textbooks, namely: (A) Atkinson & Ward (1996) and (B) Brodie and Swift (1989). (The two textbooks will be referred to as Text A and Text B respectively throughout this report). The selected chapters from the two texts are entitled "Ratio and Proportion" and "Ratios and Rates" respectively. Exploration of the selected chapters occurred through analysis of the given definitions, worked examples and exercises/problems. The aim was to uncover the underlying pedagogy implicit in the chapters. In particular, we were looking for evidence of explication of the structure of proportional reasoning as well as linking and consistency of approach across topics such as ratio, proportion and rate. The solution methods for the problems, particularly those involving finding the missing value in a proportion equation, were also a focus.

Results

Definition of Ratio

A common practice in mathematics textbooks is to highlight definitions of mathematical terms through the use of text boxes, coloured ink, and so on. In Text A, the presented definition of ratio is as follows:

> A **ratio** is a comparison of two or more quantities of the **same kind** and of the same unit. (p. 221) (their emphasis)

This definition is followed by some examples using the colon notation such as "3 km:5 km". The quantities presented are the same units and show a part:part comparison or a whole:whole comparison. There is no mention that the "comparison" is relative (multiplicative) rather than an absolute (additive) one. Comparison by division is implied in the following statement, with ratio represented as a fraction and a percentage:

> Ratios can be written as ***a* : *b*** (we say '*a* is to *b*') or as a fraction $^a/_b$ (we say '*a* over *b*') or as a percentage.
> Where possible, simplify ratios, so *a* and *b* are **natural** (whole) numbers.
> We can write 2:3 or $^2/_3$ or 66.6% to all show exactly the same thing. (p. 221)

It is interesting to note here that there is no specification of the quantities being compared in terms of parts and wholes. The definition given moves from real quantities (3km:5km) to mathematical symbols and manipulation where ratio is represented as a fraction and then as a percent. Implicit in the initial definition and example of ratio is the comparisons of part:part or whole:whole. However, the fraction representation provided shifts meaning from a part:part or whole:whole comparison to a part:whole representation in which students' prior experience and most likely understanding of fractions is based. Therefore, the possibility of confusion for students is apparent as no qualification of the use of fraction *notation* to represent a ratio and the use of percentage (a part out of a whole of 100) to express a ratio is given.

In Text B, no single definition of ratio is provided. Rather, the chapter opens with examples of ratios in use, including mixtures such as juice and 2-stroke fuel. The notations for expressing a ratio are defined using a pictorial example of 8 boys and 10 girls on a beach with the accompanying description, that "this can be stated as: 8 to 10, 8 is to 10, 8:10, ${}^{8}/_{10}$." In this example, the transition from a statement of ratio in words, to the use of a colon to substitute for the words "is to", to the fractional representation is given, but legitimisation of the part:whole fraction notation to represent a clearly part:part or whole:whole situation is not. Further information about ratio representation is given as follows, with ratios as parts explicitly stated:

> The **order** of a ratio is important. If the ratio of sand to cement in mortar is 3:1, an entirely different mortar would result from a mix containing 1 part sand to 3 parts cement.
>
> Ratios may involve *more than two* parts. An example of this is the mixture for concrete, where sand, gravel and cement are mixed in the ratio 2:4:1. . . .
>
> It is possible to simplify ratios in the same way as fractions.

In the last statement, permission is given for ratios to be expressed and manipulated in the same way as fractions, but no conceptual link is provided. The text then goes on to provide a worked example of how ratios and fractions can be simplified:

> Write the ratio of boys to girls in figure 7.1 in simplest form.
> *Solution* Boys : Girls = 8:10
> $= {}^{8}/_{10}$ (written as a fraction)
> $= {}^{\cancel{8}\,4}/_{\cancel{10}}{}^{5}$ (cancel)
> $= 4{:}5$ or ${}^{4}/_{5}$
> Ratios compare quantities of the same kind. If measurements are compared they must have the same units. (p. 257)

In this case the word "part" is used in the discussion but again there is no distinction drawn between the nature of comparison in a ratio (part:part) and the comparison in a fraction (part:whole). The text uses the words "written as a fraction" and "simplify ratios in the same way as fractions", but does not clarify the distinction between the two. No indication of ways to work with ratios containing more than two parts is given and the subsequent exercises do not contain any examples with more than two parts.

Sharing Quantities

The exploration of ratios in terms of parts and their relationship to the whole features in both texts through "sharing" problems. The unitary method is presented to enable students to solve sharing problems. That is, the component unit value of the ratios are explored in terms of their relation to the whole. The setting out of the examples in the two texts is virtually identical. Both texts use the term "parts" to describe the sharing. An example from Text B follows:

> Share \$42 in the ratio 3 : 4.
> Total number of parts = 3 + 4 = 7
> Each part = \$42 ÷ 7
> = \$6
> So 3 parts = 3 × \$6 = \$18
> And 4 parts = 4 × \$6 = \$24
> Total = \$42 (p. 262)

Proportion Problems

Both texts introduce problems involving two equal ratios with one number unknown. Text B first introduces the idea of equivalent ratios and demonstrates two methods to test whether two ratios are equivalent. The first method involves cross-multiplication: "The cross products are *equal*, so the ratios *are equivalent*" (p. 265). The second method involves reducing each ratio to a ratio to one and seeing if the first numbers are equal (e.g., 1.72:1 and 1.75:1 are not equivalent).

Text B provides two methods for solving a proportion problem with one unknown as in the following example.

> The ratio of boys to girls in a class is 4:5. If there are 15 girls, how many boys are in the class?
> First we write the proportion using the information given.
>
> $$\frac{\text{Number of boys}}{\text{Number of girls}} = \frac{4}{5} \Rightarrow \frac{\text{Number of boys}}{15} = \frac{4}{5}$$
>
> Now give the unknown number a variable name so that it is easier to write.
> Let the number of boys = b. So $^{b}/_{15} = {}^{4}/_{5}$
>
> To solve the proportion you could use either method below.
>
> METHOD 1
>
> $$\frac{b}{15} = \frac{4}{5} \quad \xleftarrow{\times 3}$$
>
> Numerator and denominator must be multiplied by the same number.
>
> $$\text{So } b = 4 \times 3 = 12$$
>
> METHOD 2
>
> $$\frac{b}{15} = \frac{4}{5} \quad \text{(Cross multiply)}$$
>
> $$5b = 4 \times 15$$
>
> $$b = \frac{4 \times \cancel{15}^{3}}{\cancel{5}^{1}} \quad \text{(Cancel)}$$
>
> $$b = 12$$
>
> There are 12 boys in the class.

Text A does not discuss equivalent ratios before introducing the following proportion example:

> Four car tyres cost $240 all together. How many tyres could you buy for $300?

Steps to follow	Solution 2
1. Write as ratio	4 new tyres : *n* new tyres = $240 : $300
2. Substitute pronumerals	$4 : n = 240 : 300$
3. Write as a fraction	$4 / n = 240 / 300$
4. Solve the equation	$4 / n \times 300n = 240 / 300 \times 300n$ $1200 = 240n$ $5 = n$
5. Write your answer	∴ five tyres could be purchased

Both books present solutions for this type of problem by writing the ratios in fraction notation and solving for the unknown. Text A uses an algebraic equation-solving method that amounts to multiplying both sides of the equation by a number equal to the lowest common denominator of the two fractions. Method 1 in Text B involves the multiplicative idea of proportion. The students are encouraged to think of the multiplier operating between the two known denominators and then apply that multiplier to the numerators including the unknown. Method 2 uses cross-multiplication and then equation solving, rather than a direct application of the Rule of Three.

Rate

Rate is defined similarly in both texts. The definitions presented below are from Text A and Text B respectively:

> A **rate** is a comparison of quantities of *different* kinds. (p. 268)
>
> A **rate** is a comparison of two quantities of **different kinds**, with **different units**. (p. 230)

Again there is no mention of the idea that this is a comparison by division. Both texts discuss commonly used rate types and their units. Text B then focuses mainly on speed as an example of rate. Speed is defined as distance divided by time and the examples use direct substitution into the formula. There are also a number of examples using the idea of a "ready-reckoner". Graphs are used for some situations and a link is made with direct proportion.

The worked examples on rate in Text A are set out in steps, as shown below.

Simplify each of these rates: (a) 8 kilometres in 2 hours (b) 320 tonne of wheat in eight trucks

Steps to follow	**Solution a**	**Solution b**	
1 Write the question	8 km in 2 h	320 t in 8 trucks	
2 Change to fraction form	= 8 km / 2h	= 320 t / 8 trucks	
3 Simplify if possible	= 4 km / 1 h	= 40 t / 1 truck	
4 Write your answer	= 4 km/h	= 40 t/truck	(p. 231)

The text reminds students of the definition of ratio and the difference between ratio and rate. It also notes that working with rate is similar to working with ratio:

> As with ratio, you should simplify the rate so that the quantities being compared are natural or whole numbers. (p. 231)

Discussion

The aim of this investigation was to examine the use of the concept of proportional reasoning in teaching students about ratio and rate, with the textbook being the major teaching resource. We were particularly interested in finding if explicit use was made of this concept and how explicitly the ideas in the related topics of ratio, proportion and rate were linked. While the aim was not to criticise the authors of the chosen texts, it is inevitable that criticism is implied in our comments.

Both texts introduce definitions of ratio and rate similar to those used in many other texts. Neither text distinguishes between absolute comparisons (subtraction) and relative comparisons (division) in their definitions of ratio and rate. The use of division becomes apparent through the examples. In the definition of rate, the meaning of comparing two quantities of different kinds is not pursued. All that is implied is that division is used but the meaning of "comparing" kilometres with hours is not discussed. In Text A there is no notion of rate as a measure of change, that is, the proportional idea that as one variable changes, the other changes in a related way. Text B makes use of the idea of ready-reckoners which demonstrates proportionality. However, in the text no explicit mention of this link is made and the idea is not discussed with the definition of rate.

The use of fraction notation to work with ratios, proportions and rates is a feature of both texts. However, neither makes the distinction between ratio, a part:part comparison in most of the applications discussed, and fraction and percentage which are normally part:whole comparisons. The notion of parts and wholes only arises when "sharing" problems are addressed. Also, no reasons for the writing of ratios in fraction notation are provided, being presented as the conventional way of working with these problems.

The use of proportional (multiplicative) thinking is presented in these texts as an aside, rather than the main structural idea unifying the topics. Text B has some explicit use of multiplicative thinking in the work on ready-reckoners and in one method of solving a proportion problem. However, the main solution methods presented for the problems involve solving the proportion equation by cross-multiplication and algebraic methods. The solution method for rate problems does not acknowledge the link with proportion at all, relying entirely on the use of a formula. Neither text makes explicit the proportional structure underlying all the problem types investigated, nor do they capitalise on the possible connections that could be made to help students develop a deeper understanding of the ideas.

Conclusion

This brief analysis of just two chapters within two textbooks reveals the limitations of such texts in their definitions, worked examples and suggested solution procedures for the topics of ratio, rate and proportion. The symbolic representation of proportion situations and subsequent manipulation of numbers within proportion equations provides little meaning, either to the real-context of examples presented, or to prior knowledge of other related mathematics topics (e.g., particularly the topic of fractions and percent). The treatment of the topics of ratio, rate and proportion in these two texts appears to offer little connectedness and does little to expose the predominantly part:part (or whole:whole) structure of proportion situations; a structure that can contextualise to other related topics. The definitions and worked examples taken from the texts illustrate the brevity of background information provided for textbook users, suggesting that such information would be quite unhelpful for students and possibly parents. It could be argued that the confusing nature of information given would reinforce a view of mathematics as a meaningless, rule-dominated and highly-specialised subject, accessible to few people. As a first step to exploring the complexity of the topic of proportion, this study has indicated that school texts appear to be limited in their ability to assist students (and teachers) to develop the proportional thinking skills necessary for the successful and meaningful learning of school mathematics.

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Reflections on the Middle Years Numeracy Research Project – Is It a Case of Too Much Too Soon, For Too Many?

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This paper will provide an overview of *the Middle Years Numeracy Research Project* conducted in Victoria from 1999 to 2000. Results from the final stage of the project indicates that teachers working in professional teams in a coordinated and purposeful way do make a difference to numeracy outcomes for the majority of students. However, evidence from individual interviews with a sample of 'at-risk' students suggests that schools still face a significant challenge in recognising and dealing with difference at this level. Implications for further research and current practice are discussed

Although there have been a significant number of numeracy-related projects in all Australian States and Territories in recent years, relatively few have focussed on the middle years of schooling or collected large-scale data on student numeracy performance based on the *National Numeracy Benchmarks for Years 5 and 7* (Curriculum Corporation, 2000). Possible reasons for this include uncertainty about what constitutes numeracy in the middle years, the relative lack of numeracy-specific research at this level, and higher profile given to early identification and intervention in the early years (for example, Commonwealth of Australia, 2000).

The Victorian *Middle Years Numeracy Research Project* (MYNRP) was essentially commissioned to identify and document what works and does not work in numeracy teaching in Years 5 to 9 particularly in relation to those students who 'fall behind'.

The key research questions to be addressed in this paper are listed below.

1. What does the initial data indicate about student numeracy performance in the middle years of schooling?
2. To what extent were Trial Schools successful in improving numeracy outcomes?
3. What characterises the numeracy learning experiences of 'at-risk' students?

Numeracy or mathematical literacy has been variously described in the literature and public policy domain (for example, AAMT, 1997; Dossey, 1997; Scott, 1999; Willis, 1990, 1997; and the Commonwealth of Australia, 2000). For the purposes of the MYNRP, numeracy in the middle years was seen to involve:

- core mathematical knowledge (in this case, number sense, measurement and data sense and spatial sense as elaborated in *the National Numeracy Benchmarks for Years 5 and 7*);
- the capacity to critically apply what is known in a particular context to achieve a desired purpose; and the
- actual processes and strategies needed to communicate what was done and why (Siemon, Virgona & Corneille, 2001)

Research Design and Methodology

Given the duration (18 months) and largely explorative nature of the project, it was felt that the most appropriate research design was a quasi pre-post design involving a

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representative sample and a structured sub-sample. In the first phase of the project (September-December, 1999), data was collected from just under 7000 Year 5 to 9 students in 47 schools using rich assessment tasks (see Siemon & Griffin, 2000; Siemon & Stevens, 2001). This data was collected to provide baseline information on student numeracy performance and some insights into what appeared to be impacting numeracy performance at this level. The following year, a structured sub-sample of 20 schools was selected to participate in a year-long trial phase aimed at identifying ways to improve student numeracy performance. This sample was selected on the basis of the initial student numeracy data (high and low) and the extent of evidence concerning supportive school-wide policies and practices (rich and poor). All students in Years 5 to 9 in the structured sub-sample of schools were given a parallel version of the initial assessment tasks in November 2000 (n=2899).

The student numeracy data was analysed using SPSS and Quest, a Rasch modelling tool developed by Adams & Khoo (1993). The item analysis confirmed that it is possible to measure a complex construct such as numeracy using rich assessment tasks that incorporate performance measures of content knowledge and process (general thinking skills and strategies) across a range of topic areas using teachers-as assessors. The overall item analysis provided evidence that the degree of difficulty of the tasks chosen was appropriate for the cohort tested. A detailed analysis of the distribution of items also supported the development of an eight-level scale, referred to as the *Emergent Numeracy Profile* (Siemon & Griffin, 2000). All other data was analysed qualitatively by means of content or category analysis. That is, by an examination of the response sets for patterns, relative frequencies and/or relationships to demographic and/or student numeracy performance data.

Main Findings

The following summary of the main findings is derived from the *Report on the Middle Years Numeracy Research Project* (Siemon, Virgona & Corneille, 2001). The findings were based on the analysis of the school surveys, student numeracy performance data, individual interviews of a selected sample of students, Trial School Action Plan Reports, teacher journals and surveys, and the relevant literature. Due to space limitations only those findings that relate to the key research questions listed above will be considered in this paper.

1. What Does the Initial Data Indicate about Student Numeracy Performance in the Middle Years of Schooling?

As the Phase 1 data collection represents the first large-scale attempt to evaluate numeracy not only in terms of the *National Numeracy Benchmarks for Years 5 and 7* but also students' capacity to interpret, apply and justify their mathematical thinking, it is difficult to gauge the significance of the results presented in Table 1. Given that the majority of mathematical content was representative of Victorian Curriculum Standards Framework (CSFII) Levels 2 to 4 for a Year 5 to 9 sample that might be expected to be operating at CSFII Levels 3 to 6, the overall performance could be viewed as disappointing. The results reported here need to be interpreted with some caution due to the relatively small number of tasks used. However, they suggest at the very least that there is considerable scope for improvement in student numeracy performance in the middle years of schooling, particularly in relation to multiplicative thinking and rational number (see Siemon & Griffin, 2000).

Table 1
Phase 1 Mean Scores Student Numeracy Performance by Year Level

Total Phase 1	Year 5	Year 6	Year 7	Year 8	Year 9
Mean Logit Score: 51.2 (53.9%)	47.8 (50.3%)	52.9 (55.7%)	50.1 (52.7%)	52.0 (54.7%)	53.1 (55.9%)
Std. Dev. 11.2	10.5	11.0	10.9	11.1	11.8
N = 6859	1314	1318	1467	1484	1276

All differences between means for consecutive Year levels are significant. ($p<.001$ for Years 5-8, $p<.05$ for Year 8-9). The significant 'dip' in numeracy performance between Years 6 and 7 is consistent with similar data reported in relation to literacy performance in the middle years of schooling (Hill & Russell, 1999). While there are many other contributing factors, such as the transition from primary to secondary school and a range of social and emotional issues associated with emerging adolescence, evidence from the teacher surveys suggests that at least some of the variance may be due to the relatively lower expectations of Year 7 students by their teachers.

There were some interesting differences with respect to location where it appears the overall 'dip' between Year 6 and 7 performance was largely driven by the significant difference in the performance of Year 6 and 7 students from metropolitan schools (n=4303, mean logit scores: 53.9 and 48.4 respectively). By contrast, in regional and/or rural schools (n = 2556) there is no 'dip' in performance at this level. In fact, the performance of Year 7 students (mean logit score: 53.7) is significantly better than their Year 6 counterparts (mean logit score: 51.6). Possible reasons for this cannot be explored here but it may have something to do with 'social connectedness', that is, how comfortable a Year 6 student feels about moving from their familiar, generally much smaller primary environment to a new secondary school (Siemon, 2001b).

The distribution of students across the eight *Emergent Profile Levels* in each of Years 5 to 9 supports the phenomenon observed in the First International Mathematics and Science Study (eg, Keeves & Radford, 1969) of the 'seven-year gap' in mathematics performance of students in the middle years of schooling. This suggests that in any one, 'mixed-ability' class from Year 5 to 9 there is as much variation in performance as there is in the whole of Years 5 to 9 (Siemon, 2001a). While this does not shed any light on how to optimise learning opportunities in the middle years of schooling, it does suggest that something quite radical needs to be done if the learning needs of individual students are to be adequately addressed.

2. *To What Extent Were Trial Schools Successful in Improving Student Numeracy Outcomes?*

Teachers and targeted programs make a difference to student numeracy outcomes in the middle years of schooling. There was a significant improvement in Trial School student numeracy performance means from Phase 1 to Phase 2 for 18 of the 20 schools. All of the increases in student numeracy performance from Phase 1 to Phase 2 by year level are significant ($p<.05$). However, it would appear that the 'transition dip' was 'deepened' (see Table 2). As for Phase 1, there were significant differences between Years 5 and 6, Years 6 and 7, and Years 7 and 8 in the Phase 2 data ($p<.001$). In contrast to Phase 1, there was no significant difference between Year 8 and Year 9.

Table 2

Phase 1 and 2 Mean Scores Student Numeracy Performance by Year Level for Trial Schools

Trial School Phase 1 (Nov 1999)	Year 5	Year 6	Year 7	Year 8	Year 9
Mean Logit Score: 46.5 (48.9%)	40.4 (42.5%)	46.9 (49.4%)	45.3 (47.7%)	50.0 (52.6%)	49.8 (52.4%)
Std. Dev. 12.9	12.98	11.4	12.2	11.8	13.7
N = 2899	540	513	690	603	553

Trial School Phase 2 (Nov 2000)	Year 5	Year 6	Year 7	Year 8	Year 9
Mean Logit Score: 52.6 (55.3%)	49.4 (52%)	54.6 (57.4%)	51.0 (53.7%)	53.5 (56.3%)	54.7 (57.6%)
Std. Dev. 11.8	*10.0*	*10.7*	*11.3*	*13.5*	*13.5*
N = 2899	540	513	690	603	553

The item analysis suggests that the tasks associated with the most significant improvements in numeracy performance could be summarised as tasks involving a capacity to read and interpret everyday mathematical representations. There was also considerable improvement on tasks that required students to monitor and regulate their cognitive behaviour. The ability to interpret data relevant to context, perform mental calculations and recognise, describe and use patterns were also areas where student numeracy performance improved. However, the 'hotspots' identified by the Phase 1 data (see Siemon & Griffin, 2000) remained relatively 'hot' despite the overall improvement in these aspects as well as others.

Not surprisingly, given the general increase in student numeracy performance, there was a significant shift in the relative proportions of students at each level of the *Emergent Numeracy Profile* from Phase 1 to Phase 2 (see Table 3). This is illustrated by the fact that in November 1999, just over 61% of the students in Years 5 to 9 were performing at or above Level D on the *Emergent Numeracy Profile*, while in November 2000 this proportion had risen to just under 80%. The mean shift across all Year levels was 1.52 Profile levels (Siemon, Virgona & Corneille, 2001).

Table 3

Proportion of Trial School Students by Profile Level for Phase 1 and 2

Total	A	B	C	D	E	F	G	H
Ph1 Trial N=2899	10.6%	10.3%	17.7%	15.9%	18.1%	16.7%	8.7%	2.0%
Ph2 Trial N=2899	3.7%	5.9%	10.5%	15.8%	20.0%	22.3%	15.2%	6.6%

It is clear from the improvements in student numeracy performance achieved by all Trial Schools that teachers and targeted programs make a difference and, in particular, that a whole-school approach to numeracy improvement is a key element in achieving success.

Some of what characterised the practice of those schools that made the most improvement in student numeracy performance and/or sustained relatively high levels of student numeracy performance will be referred to in the discussion below.

3. What Can We Learn from Students Identified as 'At Risk' About Their Experience of Learning School Mathematics?

To explore students' experience of learning school mathematics two students who were nominated as 'typically weak' or 'at risk' by their school were interviewed and asked to respond to an attitude survey (n = 40). The following propositions were derived from the responses to the interview and survey for the students who 'fall behind'.

- These students believe that mathematics is important and relevant.
- They generally want to learn and be able to apply mathematics.
- Mathematics is not perceived to be as 'boring' or irrelevant as is often assumed.
- They are prepared to accept some of the responsibility for learning
- Addition and measurement were nominated as the 'easiest' areas of mathematics. Multiplication, division, fractions, and decimals were nominated as the 'hardest'.
- The most critical element in their learning from the students' perspective is the quality of teacher explanations, in particular, the capacity of teachers to connect with their level of understanding and communicate effectively.
- Traditional, text-only based approaches are seen as a major impediment to engagement and successful learning.
- Success is crucial to engagement.
- Students would prefer more one-on-one assistance within the context of the whole class lesson.
- Students prefer mathematics classes to be activity-based (that is, games, manipulatives, investigations), deliver success, involve problem solving, and be conducted in a constructive and positive manner.
- 'Relevance' appears to be about connectedness, it is not necessarily about immediately applicable, 'real-world' tasks, although this is important. It is, at least in part, about being able to access what is seen to translate to further opportunities to study mathematics, 'real' maths, and access to 'good' jobs (Siemon, Virgona & Corneille, 2001).

For these students, the quality of teacher explanations was seen to be one of the most important factors affecting their learning of mathematics. However, this also depends on the students' disposition to engage in this enterprise. Students 'at risk' clearly value the use of a broader range of more inclusive practices. But, their responses to the interview questions suggest that disengagement may have as much to do with their perceptions of how they are treated by their teachers, as it does with the particular nature of the teaching practices used. In particular, it appears that the extent to which efforts are made (and seen to be made) to communicate respectfully with students in a way which recognises and accepts 'where they are at' is a key factor in whether or not middle year students are prepared to engage in the task of learning mathematics. Engagement is also very closely related to past success. Students are willing to engage in the task of learning and applying mathematics to the extent that they believe they understand what is required of them and they experience some success.

Discussion

The Action Plans and Final Reports of those schools that made the most improvement in student numeracy performance and/or sustained relatively high levels of student numeracy performance were found to be remarkably convergent. The distribution of strategies confirming the usefulness of the Hill and Crévola (1997) model for school improvement. Analysis of the Final Reports from these schools suggested that improvements in numeracy outcomes were largely achieved as a consequence of a concerted focus on recognised 'best practice' in the teaching and learning of mathematics. For instance, features reported included:

- regular and systematic use of open-ended questions, games, authentic problems, and extended investigations;
- teaching strategies focused on connections and developing students' strategies for making connections;
- active engagement of students in conversations and texts that encourage them to reflect on their learning and explain and justify their thinking; and
- learning activities chosen in relation to learners' needs and interests (Siemon, Virgona & Cornelle, 2001).

However, the reports also acknowledged, and this was strongly supported by the teacher surveys, that powerful constraints mitigate against a sustained implementation of more effective practice at this level. From the teachers' perspective, a major impediment to more effective practice was the sheer amount of perceived content that they felt obliged "to cover". From the students' perspective, particularly those that 'fall behind', it was evident that many did not have access to the key underpinning ideas and strategies needed to achieve some measure of success.

Quite clearly, more time and space needs to be found to ensure the 'big ideas' of mathematics and the connections between them are the principal focus of school mathematics at this level. There are many issues which impact the transition from primary to secondary school (for example, Cumming, 1996; Barber, 1999). However, the significant dip in student numeracy performance between Years 6 and 7 suggests, among other things, that serious and urgent consideration needs to be given to what mathematics is taught and how it is taught at this level. Traditional approaches based on linear sequences of topics may not be the most effective way to engage young learners, many of whom need additional and special assistance.

Student engagement has become a major focus of recent research (for example, Clarke, 2001; Barber, 1999; Otero, 1999; Cumming, 1996). The observations here are supported by a large-scale study by Marks (2000) who identified three factors as strong predictors of student engagement in the middle years: a "positive orientation to school, as reflected in school success" (p.173); authentic instructional work that

> involves students intellectually in a process of disciplined inquiry to solve meaningful problems, with relevance to the real world beyond the classroom and of interest to them personally (p.158);

and systems of social support. That is,

> a positive school environment … normed for respect, fairness, safety, and positive communications … in which students experience high levels of expectations and receive help from teachers and peers" (p.174).

The distinction observed by Marr (2001) in relation to talk in adult numeracy classrooms, that is, the *opportunity to speak* and the *means to speak* is also relevant to the issue of student engagement. While schools and teachers need to ensure students are given the *opportunity to engage* through the selection of appropriate content and the use of a variety of teaching approaches, this on its own is insufficient. Students also need access to the *means to engage*. That is, how to read, write and speak mathematically to participate in the conversation and text of mathematics (Pimm, 1987). So, although there is a case for focusing on key underpinning ideas, such as part-whole relationships, place-value and multiplicative thinking, teachers also need to deal directly and overtly with the ways in which mathematics is represented and communicated. From the students point of view the most important contribution teachers can make is to communicate mathematical ideas and texts effectively *with them*, on a one-to-one basis where needed, to enable them to take their place in the mathematics learning community. This message is overwhelming and cannot be ignored.

While speaking and listening are key ingredients in building shared meaning for mathematical ideas and texts, quality speaking and listening can only occur where there is sufficient trust, knowledge and confidence to share and work at what is known and how it is known. Above all, where there is sufficient time to focus on meaning as opposed to just 'doing'. Again, this has important implications for the design and delivery of school mathematics programs. It would appear that for too many students and teachers there is simply too much to do and not enough time to do it. While many students will be able to learn from the experience of doing, this depends on having access to a network of related ideas which inform and are shaped by the doing. Without the linking, connecting ideas and the means to access and elaborate those ideas, the doing becomes a boring, repetitive and unproductive exercise. Teachers and students need time to elaborate and explore ideas. This does not mean a reduction in expectations but a shift in expectations and targets from a large range of relatively disconnected ideas to a very much smaller, far more connected set of 'big ideas' supported by descriptions of the sort of conversations that teachers might be expected to have with students as they work on those ideas.

One of the clear implications that can be drawn from the student numeracy performance data and the student interviews is that early diagnosis and intervention is critical. Targeted professional development is needed to support teachers identify numeracy-related learning needs and provide the scaffolding needed to support students' learning. While existing research in mathematics education can be used to support some of this (for example, Clarke et al, 2000), further research is needed to identify and elaborate the key growth points and scaffolding needed to support the development of numeracy-specific ideas and strategies at this level.

The relative lack of stability in student numeracy performance over the course of the middle years of schooling and the suggestion that students are 'back on track' by Year 9 suggests that formally differentiating the curriculum before students achieve some sense of responsibility for their own learning can only exacerbate these differences. While further work is needed on more appropriate organisations for learning in the middle years, it would appear that the use of flexible grouping within mixed ability classrooms supported by specialist one-on-one intervention within classes is a more appropriate option in dealing with difference in relation to school mathematics and/or numeracy. Further work is also needed to explore the efficacy of vertical and/or cross-curriculum or integrated curriculum arrangements in relation to improved numeracy performance.

It is clear from the work undertaken in relation to this project that there is an urgent need to identify, describe and resource more effective ways of supporting teaching and learning for numeracy in the middle years of schooling. Structured professional development programs to support and enhance the work of teaching school mathematics at this level are a logical first step in improving numeracy outcomes. However, sustained improvement will also require a serious review of how school mathematics is represented and positioned within the context of teaching and learning at this level.

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Positioning the Personal in Mathematics Teacher Education Through Pedagogical Conversations

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This paper presents a narrative inquiry into the personal dimensions of mathematics teacher education. It describes my experiences as a teacher educator exploring the use of *personal theories* to develop and share *pedagogical conversations* with prospective teachers. *Personal theories (PTs)* can be thought of as a set of beliefs and practices that are personally meaningful, and based on experience. This narrative describes the developmental process involved in establishing personal theories, and outlines three levels of authority and ownership that were evident in final versions of prospective teachers' personal theories.

For prospective teachers, becoming a teacher of mathematics is a very personal experience that is "felt" differently by each individual, depending on his or her past and present lived experiences. As mathematics teacher educators, one of the most vital actions we can take is to put the personal upfront and explicitly address these lived mathematical experiences through the use of *personal theories* (PTs) within teacher education programs. PTs can have their origins; in the past, perhaps as a result of an "apprenticeship of observation" (Lortie, 1975) or other compelling experiences; in present contexts as prospective teachers move through their teacher education courses; or they may be related to imagining future possibilities for action. It is argued that developing PTs can provide opportunities for learning from experience by reflecting on experiences and then theorising those lived experiences.

An important aspect of learning from experience in relation to this inquiry is the use of reflective tools that allow for continuous and meaningful *interactions* between learners in a teacher education setting. These inner (personal) and outer (social) interactions, or conversations can enable and empower prospective teachers to become theory generators and well as theory users as they reflect on their own practice (Cole & Knowles, 2000). The term *personal theories* describes one such reflective tool that provides an opportunity for educative conversations that explicitly address personal beliefs. Such an approach to teacher education can be thought of as *pedagogy as conversation*.

Underlying Principles of "Pedagogy as Conversation"

There are two important underlying principles of *pedagogy as conversation*. First, there is a need to maintain a "stance of inquiry" (Nelson, 1997) within our teaching programs so that we provide opportunities for "learning from experience" (Boud, Cohen and Walker, 1993). Learning from experience needs to occur within a community of learners where all participants involved in an inquiry are viewed as learners who create and recreate *ways of coming to know*. Recognising that there are many *ways of coming to know* that coexist and support each other is essential if we are to celebrate the diversity of individuals. A "stance of inquiry" suggests a generative and purposeful view of learning through making meaning from experience. These experiences need to be explicitly articulated and shared within a

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community of learners if they are to be critiqued and validated (Belenky, Clinchy, Goldberger & Tarule, 1986; Jaworski, 1999).

Second, *pedagogy as conversation* requires a reflective orientation towards learning so that learning can be shared conversationally in written and oral forms. In this way, the dichotomy between theory and practice can be somewhat disturbed when we look at theorising practice through a more personal and relevant process of reflection in the form of PTs. Writing reflectively fixes thought on paper and externalises what is essentially internal thinking (van Manen, 1997). From this perspective, the writing of PTs can become a meaning making process for recording lived experiences and their relationship with internal feelings, thoughts and beliefs. This form of recording allows PTs to be shared, revisited and perhaps reframed. One way of explicitly articulating ways of *coming to know* emerges through the development of PTs about teaching, learning and assessing mathematics. The writing of PTs is put forward as a pedagogical practice that can be thought of as written and oral conversations between learners that are purposefully focused on learning about pedagogy. The use of *pedagogical conversations* requires a shift in practice from telling to listening.

The term "pedagogy" in *pedagogy as conversation* embodies the reciprocal relationship between the three educative processes of teaching, learning and assessing. In van Manen's (1999) terms, pedagogy can be thought of as ways of interacting as we live and work responsibly with learners. In this sense, "pedagogy" recognises that the holistic and relational nature of *coming to know* exists within the three synergistic processes of teaching, learning and assessing. The term "conversation" refers to both written and oral interactions within a community of learners. Davis' (1996) notion of conversation as a "reciprocal engagement" in a topic of mutual concern is particularly useful for this inquiry. A conversation suggests talk among friends where understanding emerges as a joint construction by participants who are seeking a deeper, shared understanding of the issue at hand. An important characteristic of conversations is their open-ended nature that allows for unexpected revelations and destinations.

The Emergence of Personal Theories (PTs)

PTs are grounded in a teacher's "appreciation system" that is based in beliefs and consists of a repertoire of values, knowledge, theories and practices (Schon, 1983). On the other hand, *public theories* represent all other forms of traditional research that are represented in books, journals and other related literature. Both theories can be thought of as the "living, intertwining tendrils of knowledge, which grow from and feed into practice" (Griffiths & Tann, 1992, p. 71).

Research on learning to teach has begun to support a focus on the need to provide opportunities to make explicit, and build on, prospective teachers' existing knowledge, beliefs and attitudes about teaching and learning because "...what has to be learned is intimately connected to what is already known" (Beattie, 2000, p. 19). By acknowledging and eliciting prospective teachers' beliefs about teaching, learning, and assessing mathematics using PTs (Bullough & Gitlin, 1995; Connelly & Clandinin, 1990; Griffiths & Tann, 1992), teacher educators can explicitly address these PTs and provide links to more public theories to help bridge the gap between theory and practice, and the knower and the known (Beattie, 2000; Feiman-Nemser & Remillard, 1995).

The writing of PTs is one technique for positioning the personal within a mathematics teacher education context so that the personal "interconnected web of ideas" developed by prospective teachers can be explored (Ball, 1988). This technique was designed to provide opportunities for prospective teachers to author their own learning and make explicit their beliefs and concerns related to mathematics education.

Using a Narrative Inquiry Method

This inquiry tells a story of experience – my own experience as a teacher educator and inquirer, and the experiences of the prospective teachers that I worked with as we explored the development and use of PTs. This paper provides a snapshot of how PTs appeared to reflect varying levels of authority and ownership as they were developed over a twelve-month period. To locate a sense of our experiences, I adopted a narrative inquiry approach that Clandinin & Connelly (2000) describe as both methodology and phenomenon. Narrative inquiry is the study of experience that involves people in relation contextually and temporally. In describing narrative inquiry, Clandinin & Connelly (2000) outline a three-dimensional inquiry space that acknowledges the importance of *context* (celebrating a diversity of settings) *conversation* (both personal and social), and *continuity* (between past, present and future experiences). These three dimensions of narrative inquiry have been used to provide a conceptual framework for this inquiry and are now described.

In relation to the *contextual* dimension, this inquiry took place within a rural university setting where final year B Ed (Primary) prospective teachers were enrolled in a yearlong subject *Assessment and Diagnosis across the Curriculum: Mathematics Lobe*. During this twelve-month period, prospective teachers developed their PTs related to teaching, learning and assessing mathematics. This inquiry traces the changes in those PTs over that period in an effort to interpret the generative nature of those changes. An important component of the *contextual* dimension is the variety of settings that the prospective teachers experienced during their involvement with this subject. While this inquiry is only reporting on personal theory development, the fifty-one prospective teachers involved in this inquiry experienced a ten-week internship in one school setting, five weeks of microteaching in a different school setting, and three hours of lectures and workshops per week over a total period of sixteen weeks.

The *conversational* dimension of this inquiry refers to the written PTs that provided occasion and opportunity for understanding the pedagogical experiences related to mathematics. As personal theories took shape throughout the subject, they were shared by prospective teachers and continually referred to as part of our pedagogical conversations. The relational aspects of sharing personal theories allowed for a conversational relationship to be formed as I read prospective teachers' PTs and responded with feedback. The conversational quality of PTs was enhanced as they were shared, critiqued and reconstructed throughout the yearlong subject related to the assessment and diagnosis of mathematics. Other elements of *pedagogy as conversation* not referred to in this paper included two case investigations (in each of the school settings mentioned earlier), and poster presentations that synthesised prospective teachers' findings from the two case investigations, which were also shared and critiqued as part of the yearlong course.

The dimension of *continuity* recognises that prospective teachers' PTs are affected by past and present experiences that will be reflected in their theories about future practices. Such a temporal dimension acknowledges the importance of making connections between experiences in different *contexts*, including various *pedagogical conversations*. As educators, emphasising the temporal aspects of learning from experience acknowledges the importance of connecting personal experiences from the past and the present with the theories of others (public theories) to produce PTs that outline a personal pedagogical stance. The three dimensions are interrelated, and when connected allowed for a closer look at the themes emerging from prospective teachers' PTs that indicated a generative growth in agency and ownership of learning.

Emerging Themes Within Personal Theories

Although this paper describes findings from the final version of prospective teachers' PTs, it is important to briefly talk about prospective teachers' initial PTs. In essence, the initial attempts at capturing pedagogical beliefs related to mathematics generally lacked any authority and ownership. Instead, the PTs were personally silent and held onto other people's theories to reflect their own beliefs. Where there was some evidence of personal beliefs, for example, sentences that began with "I believe" or "Personally, I think that", there was no elaboration of theories using personal anecdotes of experience to support their beliefs. The absence of personal commitment, elaboration and the ability to personally name beliefs became the shared growth points that framed the criteria for marking the final PTs.

One initial limitation of this inquiry can also be thought of as a positive outcome. I am committed to assessment tasks that are formative and have an educative component. I believe, along with Clarke (1997) that we flag what we value as educators by what and how we assess. While I chose to use prospective teachers' PTs as an assessment task to value the time and personal commitment required in their writing, I acknowledge that this may have influenced what they chose to write about. However, the overwhelmingly honest responses to these tasks during focus group conversations after the completion of the subject reflected the general feeling that "you treated us as teachers in our own right and valued what we thought as teachers and not just what you wanted to hear" (Maria, focus group transcript Nov 13, 2001). This excerpt was representative of the twelve prospective teachers who volunteered to attend the focus group conversations and reinforces the importance of a sense of personal authority, or agency in learning to teach mathematics.

The principles of *pedagogy as conversation* emphasise the importance of listening within dialogical relationships. As a teacher educator, reading and analysing prospective teachers' PTs provided an opportunity to "listen in" to their inner conversations to determine the conditions within PTs that made personal understanding and *coming to know* possible. As I "listened in" to what prospective teachers wrote, some of what I "heard" in their final version of their PTs can be characterised to illuminate three *ways of coming to know*. These characterisations may contribute to framing future learning experiences and describing some vital aspects related to learning from experience and assessing that learning. I have taken excerpts from prospective teachers' PTs that are representative of three levels of authority and ownership that emerged from the final analysis, which was achieved through the process of "analysis of narratives" (PTs) outlined by Polkinghorne (1988).

Silent Theories: A Lack of Authority and Ownership

A small number of prospective teachers' final PTs could be characterised as *silent theories*. As a teacher educator, a shift "from looking to listening" (Davis, 1996, p. 30) led to experiencing the *sounds of silence* in some prospective teachers' PTs. The *sounds of silence* refer to the absence of personal voice, commitment, creativity, and "internal authority" that belonged to the prospective teacher. In short, there was no ownership of what was being written. Instead, the PTs simply reiterated or restated public theories that had been explored in our subject and offered no elaboration of personal experiences or connections to other lived experiences or public theories that could support their personal theories. An excerpt from Molly's PTs illustrates this point:

> For learning to be effective, I am obliged to provide experiences that foster and promote every child's socio-emotional, cognitive, aesthetic, moral and physical development…These experiences need also be offered in a supportive, trusting and rich sensory environment that offers opportunities to engage in hands on, concrete experiences which invite children to observe, to be active, to make choices, experiment and encourage creative energy and curiosity. (PT entry, Nov 2001)

While any teacher educator would celebrate such insightful theories, this excerpt is personally silent and could have been written by anyone who had access to journals, textbooks or teacher resources (public theories). *Silent theories* reflect the absence of personal experiences that would authenticate these theories as personal understandings. This characterisation reflects the notion of a "silenced" or "received" knower described by Belenky et al. (1986) as ways of knowing knowledge, authority and truth that are reproduced from "external authority". Molly's excerpt was representative of a small number of prospective teachers whose final version of their PTs still lacked authentic and supportive evidence of *how* and *why* their personal theories were formed. As a teacher educator, I characterised these theories as *silent theories* because I was left wondering whether any personal and meaningful learning had been experienced. In contrast, many more prospective teachers were able to take personal ownership of their own theories by elaborating, to varying degrees, on *how* and *why* their theories had emerged. I believe that this next level of ownership reflected the nature of our shared conversational experiences where explicit criteria were developed collaboratively to provide clear guidelines for theory development.

Elaboration of Theories: The Emergence of Authority and Ownership

The majority of prospective teachers were able to quell the *sound of* silence by elaborating, to some extent, on their stated personal theories. This elaboration took the form of anecdotes of personal experiences, and explicit connections between past and present experiences and future possibilities. *Elaboration of theories* provided "a few more layers to give them (PTs) more depth" (Kathryn's PTs, Nov. 2001) and represented a deeper level of ownership. Prospective teachers' theories related to such things as: the importance of making personal connections with students to enhance learning; the importance of maintaining a balance between structure and flexibility when dealing with culturally diverse students; being taught by students to listen with eyes and ears; and the transferable nature of Working Mathematically outcomes into all KLAs. These theories were all elaborated and authenticated with an anecdote from personal experience that may have originated in any number of learning contexts (for example their internship setting, their microteaching experience or the learning community developed at university).

An excerpt from Sally's PTs described in detail a turnaround in her commitment to life long learning as a result of her interactions with a student who had learning difficulties in mathematics. I have omitted some phrases and sentences to enhance the flow of the one idea, but I hope I have maintained Sally's intent and purpose:

> I found through reading, dealing with the student and speaking to the support teacher that I was able to develop specific strategies for teaching maths to this particular student which I could integrate

> into the maths lessons for the rest of the class…I recognised that she loved bright, shiny objects, so I created ten frames and redesigned "Count me in too" activities so that they were made of fluoro coloured paper and used large sequins as counters…Working with this particular student helped me to realise that meeting the changing needs of my students from year to year would mean a continued process of learning… So, with some initial reluctance, I have eaten my words and acknowledged the value of the ideal of life long learning. In-services and further study here I come!!! (PT entry, Nov. 2001)

In this anecdote, Sally's use of the words "created" and "redesigned" suggested a level of ownership and creativity that exemplified the authenticity of personal theories. Her commitment to the "ideal of life long learning" was elaborated with personal reference to *how* this theory emerged from her own past experience (practice), which was then connected to a future theory of action (in-services and further study here I come).

This elaborative process represents the integration of emotional and personal knowing with knowledge learned from others. In other words, it represents the integration of personal practice with other people's theories. Belenky et al. (1986) described this as "constructed knowing", a balance between subjective and objective strategies that, once experienced, foster a passionate approach to learning. Prospective teachers' reflective responses at this level encompassed a passion for authoring their own learning and "feeling important enough to have our own ideas" (Trina, focus group transcript, Nov 13, 2001).

Noticing and Naming Theories: A Developed Sense of Authority and Ownership

Another *way of knowing* that exemplified a developed sense of authority and ownership was the ability to create your own knowledge by using a personal language of experience. I have called this the process of *noticing and naming*, which was influenced by the notion of "noticing and marking" outlined by Mason (1998). My change of phrase from *marking* to *naming* is deliberate because *naming* suggests a more creative, authentic and personal opportunity to author learning through the language we choose to name our noticing. Noticing requires a heightened awareness of *context, conversation and continuity*.

The concept of *resonance* identified by Conle (1996) in her work with prospective teachers helped to frame the construct of *noticing and naming* that was evident in prospective teachers' PTs. The term *resonance* describes an experienced awareness, or a "metaphorical correspondence" between current or past personal experiences and another new experience with self or others. To illustrate the construct of *noticing and naming* as a result of a moment of resonance, I have chosen an excerpt from Elizabeth's PTs that embodies the notion of *noticing and naming* an understanding, or "meaningful whole" that becomes part of a language of experience (Polkinghorne, 1988). It is important to note the *context* that this excerpt was written. Elizabeth wrote this reflection *during* a workshop as a result of an "Aha" experience working with isometric dot paper. While this inner conversation could not have been anticipated or planned for, it led to a metaphorical correspondence in her PTs as she formulated them later that day:

> During class today, I was the child off task. Well, this is what I was thinking as I was playing and drawing a picture on the paper (isometric paper) that you provided. However, I still learnt. In fact I learnt more than I thought. Through my play, I discovered something about 3D space…I now have a new metaphor for representing my understanding of the teaching, learning and assessing process – a 3D staircase to understanding.

The conversational tone of Elizabeth's entry reflects my goal for a relational and dialogic process of reflective writing that can become a shared conversation. Her written reflections continued to describe the process of "talking to know":

> I write as though I am talking because talking is how I make sense of things, it is how I interpret meaning of a particular concept. By writing this I become the pupil reporting to my guide about the journey I have just taken…My philosophy is that our life experiences, our "doings" are a much stronger way of learning because we have a hold of it, it's in our hands, we own it, we play with it and then we put it away only to bring it back out to show someone who might want to know how to use it. The friend will only know how to use it once they have played with it too. It's fine to read about it and see it but doing it and feeling it is how I learn. No that's not completely true I think I learn more from talking about it. My name for this is "talking to know"…in the end I work it out from talking and making sense. (Personal theory entry 21/10/01)

By writing reflectively, Elizabeth was able to name the learning process "talking to know". As a result, her personal, inner conversations could be shared with others in more collaborative ways. This process allowed me to "listen in" and then join her inner conversation through my written feedback that was both personally relevant to her beliefs, and part of a shared process of *coming to know*. I would argue that listening attentively to inner and outer conversations and noticing what resonates with you as a learner is a vital aspect of effective learning. Elizabeth's reflection-in-the-moment during a workshop exemplifies the process of naming a resonating thought using a personally meaningful language (talking to know). Prospective teachers (in fact, any learners) who have opportunities to author their own learning can become more empowered to be theory generators as well as theory users, which further disturbs the dichotomy between practice and theory. Rediscovering mathematics through reflective writing can clearly lead to a deeper level of understanding and agency for prospective teachers.

As you read this paper, you may be *noticing and naming* similar experiences if they have resonated with you as a teacher educator or inquirer. Perhaps you are shaking your head as a result of a sense of dissonance with this written text. Either way, I believe your response signals an authentic *way of coming to know* that is relevant to you personally. The opposite of *noticing and naming* would be a sense of disengagement where you have not experienced any resonating moments and you are thinking about how long this paper has to go!

A Closing Conversation…

Open-ended tasks such as writing and sharing PTs about the pedagogy of mathematics can be an enabling and enlightening experience for both prospective teachers and teacher educators. The narrative space created by PTs provides an important context for pedagogical conversations that lead to the theorising of practice. By passing the intellectual authority onto prospective teachers, they can feel more empowered to find their own intellectual voices (Cooney & Shealy, 1997). PTs illustrate one way that the affective and cognitive aspects of learning to teach can live side-by-side in relative harmony. In many ways, the development of PTs require prospective teachers to experience and theorise about mathematics with new eyes and ears so that they may see, hear and feel mathematics in new ways. PTs can then provide empirical evidence of what Jaworski (1999) described as the "plurality of knowledge growth" so that the many ways of *coming to know* mathematics teaching can begin to be captured and described.

A crucial factor related to the development and use of PTs in this inquiry is the move from writing *silent theories* that are personally silent, towards an *elaboration of theories* that reflect personal authority and ownership of coming to know. To achieve this, a culture of heightened awareness needs to be established so that learners are listening in to inner and outer conversations with the intellectual authority to author their own learning. This level of ownership requires a sense of agency on the part of the prospective teacher, a sense of "feeling important enough to have our own ideas" (Trina, focus group transcript, Nov 13, 2001).

As outer conversations (other people's theories) resonate with personal theories, a process of *noticing and naming* those theories can create a personal stance that leads to the development of personally meaningful theories that have emerged from practice. A pedagogical tool for developing personal theories is the establishment of criteria for the writing of PTs that is collaboratively developed to provide a shared purpose and direction.

The contribution that PTs can make to the mathematics research and teacher education communities can be judged by its apparency and invitational quality (Connelly & Clandinin, 2000), and the extent to which this narrative resonates with its readers and informs future action.

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Further Evidence of Conceptual Difficulties with Decimal Notation

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Over 3000 students in Grades 4 to 10 completed about 10000 tests to identify and track their conceptions about decimal notation. This paper reports the incidence of two particular error-patterns on these tests and how this incidence changes over time, from cross-sectional and longitudinal viewpoints. At some time, approximately 3% of students answered almost all test items incorrectly and over 10% were unable to compare decimal numbers with the same tenths and hundredths digits. Explanations for these error-patterns are given.

It is well recognised that many students have difficulty understanding decimal notation. The reasons for this lie both in the nature of the mathematical and psychological aspects of the task and in the teaching they receive. Understanding decimal notation is a complex challenge, which draws on previous learning and fundamental metaphors of number and direction, both to advantage and disadvantage. As a consequence, there are a wide variety of erroneous ways in which students interpret decimal numbers, often referred to as decimal *misconceptions*. This paper reports results from a study of 3204 students in Grades 4 to 10, who completed 9856 tests between 1995 and 1999. Each item on the Decimal Comparison Test required the student to identify the larger number from a pair of decimal numbers (e.g. 0.8 and 0.75). This comparison task has been widely used because patterns of errors among carefully chosen sets of comparison items can reveal a great deal of information about how students interpret decimal notation. Every test is allocated a code according to the pattern of errors on the 30 items. The study has both a cross-sectional and longitudinal component, so that the prevalence of different test codes can be determined and the *paths* that students take, between the codes, over some years can be traced. In previous papers, (eg Stacey & Steinle, 1998), we have reported on the development of the 30-item Decimal Comparison Test; how inferences about students' thinking can be made reliably from the codes allocated to the tests; the incidence of expertise and other codes, as well as the paths through the codes which students commonly follow on the way to expertise (Steinle & Stacey, 1998; Stacey & Steinle, 1999a and 1999b). We have also examined more closely particular test codes and have provided some explanations in terms of how the students may be thinking about decimal notation, drawing on data from school students and teacher education students in several countries (Stacey, Helme, Steinle, Baturo, Irwin & Bana, 2001; Steinle & Stacey, 2001; Stacey, Helme & Steinle, 2001) and examined the effectiveness of targeted teaching (Helme & Stacey, 2000). This paper focuses on two test codes that are particularly intriguing; following up in this large sample, observations reported earlier in interviews and smaller scale studies. We report the incidence of these codes by grade level from both a cross-sectional and longitudinal point of view (explained below) and investigate the behaviour of such students on later tests to obtain additional insight into how they might be thinking.

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Methodology and Definitions

Volunteer secondary schools and some of their feeder primary schools in six geographical areas of Melbourne (labelled A to F) participated in testing in volunteer classes at six monthly intervals (although not every school tested students on every occasion). Table 1 shows the number of tests by grade level and school group. The sample is not random, but the six regions included very low, medium and high socio-economic areas. For this reason, the results of the school groups are reported separately, to provide an estimate the range of variation that might be in the whole population. Small differences between the basic numbers of the large sample reported in this paper and elsewhere are due to ongoing checking of the database. Larger differences from earlier published work are due to further data collection. Full details of the methodology can be found in Steinle et al (1998) and Stacey et al (1999b). Students took about 10 minutes to complete the test. Tests were coded by the researchers and results promptly returned to the teacher.

Table 1
Number of Tests by Grade and School Group

Grade	School Group						Total number of tests
	A	B	C	D	E	F	
4	39	29	178	90	0	0	336
5	158	66	355	263	121	0	963
6	232	140	454	415	206	0	1447
7	329	356	615	478	314	205	2297
8	345	327	459	293	222	456	2102
9	290	336	196	109	180	534	1645
10	278	249	0	0	50	489	1066
Total	1671	1503	2257	1648	1093	1684	9856

Definitions and Hypotheses

Twelve codes based on error-patterns have been identified from the 30-item test and these have been associated with different *ways of thinking* or *misconceptions* about decimal notation. Some of these ways of thinking are well established in other research (e.g., Hiebert & Wearne, 1986; Irwin, 1996; Putt, 1995; Sackur-Grisvard and Leonard, 1985) while others have been recently identified (e.g., Stacey et al, 1998). Space does not permit a full description of all of the misconceptions, but Figure 1 gives a brief description. For a particular test, the error-patterns on ten *core* test items results in a coarse code of A, L, S or U. These core test items are especially created to avoid particular difficulties for students- e.g. the digit zero is not used in the decimal columns. The codes are then further subdivided (e.g. into L1, L2 and L4) according to performance on a range of specialised items. The code A1 refers to expert behaviour on the comparison task, although this does not imply that the student has full understanding of decimals beyond this "simple" comparison task (e.g. they may not be able to place numbers on a number line).

This paper explores codes A2 and U2. Students testing as A2 (being one of the A codes) seem to be expert from all the core items, such as comparison of 0.8 with 0.75 (which all L students get wrong) and of 0.7 with 0.85 (which all S students get wrong). However, they are unable to select the larger number from items such as 8.41 and 8.41253. In Steinle et al (2001), we reported results of 315 students given a different version of the test that allowed them to select the larger number from each pair or to state that the numbers are equal. About 4% of students stated that numbers like 8.41 and 8.41253 are equal, indicating that they believe the decimal number system is discrete. Like dollars and cents, they believe the amounts after the hundredths (say) are not real but perhaps some sort of rounding error. Other students do not believe numbers such as 8.41 and 8.41253 are equal, but they make errors specifically on this item type. We hypothesise that they have no strategy for deciding which is larger, and so need to guess because they do not know (how) to add zeros and therefore their left-to-right digit-by-digit comparison strategy (which works in all other comparisons) fails when confronted with comparing the 2 in 8.41253 with a blank space in 8.41. In this paper we investigate the incidence of A2 in this larger and younger sample (noting that students who guess correctly on such items will be coded as A1 and hence not detected) and the later paths that these A2 students follow.

U2 is an intriguing code. These students select the smaller decimal to be the larger (nearly) completely consistently, so that they must be using a rule that is of comparable complexity to the correct rule. There are several possible explanations: they may be expert students who misread the instructions (in which case, we might expect A1 codes on their earlier and subsequent tests); they may be mischievous expert students having fun puzzling the researchers; alternatively, they may be correctly comparing digits according to place value, but then mysteriously reverse their otherwise correct answers. Amongst 103 Australian student teachers, Steinle et al (2001) found that 18% think that decimals which start as 0.xxx are less than zero (which they often insist is *not* the same as being negative), so it is possible that they also believe that just as -3 is greater than -4, so 0.3 is greater than 0.4 (although 1.3 is greater than less than 1.4). All of these explanations lead to the prediction that students who test as U2 are likely to have adjacent tests as A1, rather than other codes. This paper investigates this hypothesis.

Code	Description
A1	Task expert – successful on all comparisons (small allowance for carelessness)
A2, A3	Successful on core comparisons: errors in "unusual" comparisons
L1, L2, L4	Different ways of thinking which result in the general behaviour of choosing the longer decimal as the larger (e.g. may interpret decimal part as a whole number)
S1, S2, S3, S4	Different ways of thinking which result in the general behaviour of choosing the shorter decimal as the larger (e.g. may interpret decimal part as fraction)
U2	A maximum of 5 correct answers on entire test (i.e. at least 25 errors)
U1	Unclassified – not fitting into any other code

Figure 1. The classification system for decimal comparison test.

Results

Cross-sectional Incidence of U2 and A2

The incidence of any code can be considered from two points of view, which we call cross-sectional incidence and longitudinal incidence. The cross-sectional incidence measures the percentage of a given population who exhibit the behaviour at a particular point in time, whereas the longitudinal incidence measures the percentage of the population who exhibit the behaviour *at some time*. To illustrate with an analogy, if we consider the population of people at a shopping centre on one day, only a small percentage, say 25%, will be teenagers, but all the people in the shopping centre will be (or have been) teenagers at some time of their lives. So the cross-sectional incidence may be 25%, but the longitudinal incidence is nearly 100%. Both these measures are important. For decimal misconceptions, it is useful to know how many of a given school population (say the students in Grade 5) may be thinking in a certain way, but it is also useful to know what percentage of students think that way at some stage during their school career. Note that, typically, the longitudinal incidence will be greater than the cross-sectional incidence.

Table 2 presents the cross-sectional incidence for codes U2 and A2 by grade of the student at the time of testing. At this stage, no attempt is made to follow individual students; thus many students contribute several times to the calculations in this table as they get older. All of the cells of the table are percentages based on large numbers of tests. Overall 0.9% of the tests are in category U2, including an average of 0.2% up to Grade 7 and about 1.5% thereafter. Thus there is a general increasing percentage of U2 students with age, and Table 2 demonstrates that this is generally evident in the results of each school group. Paradoxically, the increase with age is not surprising, because a student needs to follow a complex incorrect thinking pattern very consistently to make so many errors. The probability of making at least 25 errors from 30 multiple-choice items by random guessing of all items is less than 0.5%, so the primary school results are consistent with this, but the secondary school incidence indicates a real effect.

Table 2 also shows the cross-sectional incidence of code A2. This has a higher incidence than U2, with an overall average of 4.2% of tests. Once again, incidence generally increases with age. In every school group, the primary school incidence is less than the secondary school incidence.

Table 2
Cross-Sectional Incidence (%) of U2 and A2 Tests by Grade and School Group

Percentage incidence of U2 by school group						Grade	Percentage incidence of A2 by school group					
A	B	C	D	E	F		A	B	C	D	E	F
0.0	0.0	0.0	0.0	-	-	4	0	0	0.6	2.2	-	-
0.6	0.0	0.0	0.0	0.0	-	5	0.6	3.0	2.8	3.8	0.0	-
0.0	0.0	0.4	0.0	0.0	-	6	3.9	4.3	2.9	3.4	3.4	-
0.0	1.1	0.3	0.0	0.3	0.5	7	6.1	4.5	3.4	5.2	4.1	4.4
2.6	1.5	1.1	1.7	1.8	1.1	8	3.2	4.3	4.6	3.8	4.5	6.8
0.7	1.5	2.6	0.9	0.5	1.7	9	4.1	3.6	4.6	2.8	2.8	8.2
2.5	1.6	-	-	0.0	2.0	10	4.7	3.6	-	-	2.0	6.7
1.1	1.2	0.6	0.4	0.5	1.5	Total	3.9	3.9	3.3	3.9	3.3	6.9

-indicates no testing in this cell

Longitudinal Incidence of U2 and A2

The longitudinal incidence is an analysis by student rather than by test. As noted above, although there are 9856 tests in this study, there are only 3204 students so that each student completed an average of 3.1 tests each. In particular, the number of tests per student varied from only one (772 students or 24% of the sample) to 7 tests for the 49 students present every time. Table 3 shows the percentage of students in each school group who at some stage tested as U2 or A2. This table demonstrates that U2 is relatively rare behaviour, exhibited at some stage by only about one student in forty (2.3%), but about one student in ten (10.9%) exhibits A2 behaviour at some stage. School group F, has the highest longitudinal incidence of both categories. School groups A and B are relatively high in U2 and relatively low in A2, whilst school groups D and E are the reverse.

Note that the longitudinal incidence, as with every measure from a sample, is affected by the structure of the sample. When one relies upon volunteer teachers and volunteer students in volunteer schools, the sampling will always be less than ideal. The longitudinal estimates here are probably an underestimate of the "real" incidence, because 24% of students have been tested only once. By testing them on only one occasion, the chance of them displaying the U2 or A2 behaviour is less than if they had been tested more times. Had there been more testing of the same students, then the number of U2 or A2 tests would have increased (or at least stayed constant), whilst the denominator (the number of students) would have been the same. This is just one instance of the difficulties of dealing with the influence of structure of the sample on results. In this case, however, it does not explain the higher longitudinal incidences of School Group F, which had a below average number of tests per student (2.8 compared to 3.1 for the whole sample). Both cross-sectional and longitudinal incidences are high, indicating greater prevalence of misconceptions.

Table 3

Longitudinal Incidences of U2 and A2 by School Group

School Group	Number of students	Average number of tests per student	Percentage of students with one test	Longitudinal incidence (percentage of students)	
				U2	A2
A	709	2.4	33	2.3	8.5
B	679	2.2	38	2.4	8.4
C	549	4.1	10	2.4	11.1
D	397	4.2	10	1.3	12.8
E	258	4.2	9	1.9	13.2
F	612	2.8	28	3.3	14.2
Overall	3204	3.1	24	2.3	10.9

What Happens to U2 and A2 Students on Their Next Tests?

Students who have a test which was coded as either U2 or A2 have been tracked to their next test. This subsequent test may be allocated the same code as before or a different code and this can be used to give some idea of the way in which students' conceptions of decimals develop. The numbers of students with exactly 1, 2, 3 or 4 occurrences of U2 are 64, 9, 2, 0, respectively, and of A2 are 300, 34, 14, 2, respectively. No student had more than 4 occurrences of either of these categories. The numbers of students drop rapidly down the list, but it is important to remember that once again this is due in part to the structure of the sample, as fewer students have undertaken the larger numbers of tests. It does not necessarily mean students have stopped this behaviour. This section examines the behaviour at the next test of students testing as U2 or A2. Both Tables 5 and 7 give the conditional probability of the code of the next test; sometimes adjacent grades have been combined, so that the number of students is reasonably large in each cell.

The Paths of U2 Students

There were 51 tests occasions where a student with a U2 test completed another test later. Note that this does not refer to 51 *students* as the 11 students with several occurrences of U2 contribute several times to this count. The probabilities of an A1 or U2 test following a U2 test, split by the grade of the U2 test, are given in Table 5. The probability of a younger U2 student next testing as A1 is 50%. This probability decreases with age while the probability of retesting as U2 increases. We expected "truly expert" mis-reading or mischievous U2 students to retest as A1 or U2. In fact, the U2 students are approximately equally likely to retest as A1, U2 or another code. The high probability of retesting in another code indicates that there is a significant group of U2 students who are not mischievous experts or mis-reading but instead have a misconception. Most likely they are reversing the order of the decimals because they think they are "less than zero". The high incidence of this amongst the teacher education students (Steinle et al, 2001) provides evidence to support the latter explanation.

Table 5
Conditional Probabilities of The Test Following U2

Grade of U2 test	Pr(A1/U2)	Pr(U2/U2)	Pr(other/U2)
Grades 5-8 (n=32)	0.50	0.16	0.34
Grades 9-10 (n=19)	0.32	0.37	0.31
Overall (n=51)	0.43	0.24	0.33

Looking at the test histories of students who often tested as U2 provides other detail. Table 6 gives the test histories of the eleven students in the whole sample who tested as U2 on more than one occasion. They come from all school groups, but are in the older grades. These students are very persistent in their U2 behaviour. On only 2 of the 14 possible occasions is a U2 test followed by any other code (on both occasions A1, expert). While Student 600705121 oscillates between U2 and A1 (suggesting misreading), the other students are either consistently mischievous or consistently using a wrong way of thinking. It is also very interesting that the last test of every one of these students (except Student 600803016) is U2. This suggests that they may well be stuck in the way of thinking that produces this U2 code.

Table 6
Test Histories of the Eleven Students with More Than One Occurrence of U2.

School Group	ID Number	Grade 5		Grade 6		Grade 7		Grade 8		Grade 9		Grade 10	
A	100705035						A1	U2	U2				
A	100905043										U2	U2	U2
B	210503018		A2				U2	U2					
B	200703081						A2				U2	U2	
C	310501006		L2		S2	S4	U2			U2			
D	400704093					A1	U1	U1	U2	U2			
E	500703045						U2	U2					
F	600703055						L4		A3	U2	U2		
F	600705121						U2	A1	U2				
F	600803007								A1		U2	U2	U2
F	600803016								A1		U2	U2	A1

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The Paths of A2 Students

The conditional probabilities in Table 7 show that, following an A2 test, a student is most likely to become an expert (A1), although this probability decreases with age. This is a general phenomenon in this domain – the chance of becoming an expert decreases as non-expert students become older (see also Stacey et al, 1999b). The chance that they will retest with the same code (A2) increases with age, however, so that by Grades 9 and 10 it is as high as the chance of becoming an expert. The other columns of Table 7 demonstrate that there is some movement to other codes, and this is similar to what has been found for the sample as a whole: older students are unlikely to move to an L code, but the chance of becoming an S remains at about 10% (Stacey et al, 1999a and 1999b).

Table 7

Conditional Probabilities for the Test Following A2

Grade of A2 test	Test following the A2 test					
	A1	A2	A3	U1	Any L	Any S
Grades 4/5 (n=24)	0.67	0.04	0.00	0.13	0.13	0.04
Grade 6 (n=33)	0.67	0.09	0.03	0.06	0.06	0.09
Grade 7 (n=88)	0.55	0.16	0.02	0.08	0.03	0.16
Grade 8 (n=79)	0.54	0.24	0.06	0.09	0.01	0.05
Grades 9/10 (n=56)	0.32	0.29	0.16	0.14	0.00	0.09
Overall (n=280)	0.53	0.19	0.06	0.10	0.03	0.10

Tracing individual students with several A2 tests is also interesting, but there are too many to show all the individual paths; recall above that 50 students have two or more A2 tests. Thus, Table 8 shows the test histories of the students with at least 2 A2 tests, from the school group and cohort with the largest percentage of A2 tests. Only one student has become an expert at their final test, and 6 have remained A2. There are 2 students with 4 A2 tests. The A3 tests may indicate a student guessing inconsistently with the failing left-to-right digit-by-digit comparison strategy and the A1 tests may even arise from correct guessing. In this extreme example, it is likely that teaching which always emphasizes rounding answers to a fixed number of decimal places is responsible.

Table 8

Details of All Students from Cohort 1989 School Group F with At Least Two A2 Tests

ID Number	Total number of tests	Number of A2 tests	Grade 7	Grade 8	Grade 9		Grade 10
600703014	4	4	A2	A2	A2	A2	
600703027	5	2	S1	A2	U1	A2	U1
600703040	5	2	L4	A1	A1	A2	A2
600805003	4	3		L4	A2	A2	A2
600805016	3	3		A2	A2	A2	
600805020	3	3		A2	A2	A2	
600805047	4	3		A2	A2	A2	A1
600805061	4	4		A2	A2	A2	A2
600805072	4	2		S1	A2	A2	A3
600805085	3	3		A2	A2	A2	A3

Discussion and Conclusion

Examining the two case studies of A2 and U2 codes has demonstrated the value of the longitudinal data. The U2 group is small, comparable with chance in the primary school, but increasing through the secondary school. Whilst some of the U2 tests may come from mischievous experts, tracking their next tests does not indicate that most of them have mastered the decimal comparison task. It is likely that they are still struggling to integrate various ideas about negative numbers, decimals, fractions and place value. The A2 group is much larger, with 10% of students testing this way at some time. The existence of this group points to the need for teachers to treat the rounding process carefully and to continue to attend to basic place value issues (such as when zeros can be inserted into a numeral without changing its value) in the junior secondary years. Future studies will report other aspects of the longitudinal data.

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Exploring Teachers' Knowledge for Teaching Mathematics

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This paper reports some initial results from a research project[1] aimed at identifying and describing teaching approaches which are effective in improving numeracy learning in primary classrooms. To ascertain the influences on the teachers' intentions we posed a task that sought some indication of what aspects of a topic the teachers might consider when planning a set of experiences for their students. It seems that the teachers had a limited understanding of the ways that the concepts develop and the types of experiences that they might choose for their students. We argue that teacher education could focus more on key aspects of teaching the important topics of the mathematics curriculum.

Knowledge for Teaching

An old Latin cliché goes something like "if you don't know what harbour you are going to, then no wind is a good wind". We believe this applies to teaching mathematics. Assuming that the components of teaching mathematics include researching, planning, teaching (including explaining, interacting, listening), assessing, reporting and evaluating, it seems that each of these elements are informed by what the teacher knows about the subject matter. This knowledge includes the concepts themselves, as well as understandings about the way the concepts develop for students individually and collectively, and knowledge of the types of experiences that are likely to foster student learning.

We are focusing here on teacher knowledge rather than on teacher attitudes (see for example, Bell, Costello, & Kuchemann, 1983) or beliefs (see, for example, Perry, Howard, & Conroy, 1996). Clearly effective teaching is a mix of appropriate knowledge, attitudes and beliefs, but we argue that even with positive attitudes and beliefs, without the necessary knowledge it is unlikely that teachers can optimise the effectiveness of their teaching.

The importance of the knowledge and intention of the teacher is now widely acknowledged. Cobb and McClain (1999), for example, argued that teachers should have a clear impression of the direction that the learning of the individuals and the class will take. They proposed that the teacher should form an "instructional sequence (that) takes the form of a conjectured learning trajectory that culminates with the mathematical ideas that constitute our overall instructional intent" (p. 24). Likewise, the Early Numeracy Research

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B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 641-648). Sydney: MERGA. ©2002 MERGA Inc.

Project (ENRP) (see Clarke & Cheeseman, 2000) was based on a view that learning mathematics is about forming, linking and extending a few key ideas, with these key ideas providing the basis of planning, teaching and assessment. Teaching was seen as active, structured and explicit, while maximising the engagement of the students in carefully selected tasks and experiences. It is interesting to note that the case studies in that project identified teacher knowledge of both the content and the processes for teaching the content as key characteristics of effective teachers.

Teacher knowledge has been categorised into content knowledge, pedagogical content knowledge and pedagogical knowledge (Shulman, 1986). Lappan and Theule-Lubienski (1992) also proposed three key kinds of teacher knowledge: knowledge of mathematics, knowledge of students, and knowledge of the pedagogy of mathematics. Our project focuses mainly on pedagogical processes, and we are interested in gaining some insights into types of knowledge that influences the pedagogical processes teachers choose to use. We also consider that there are three phases for each teaching event: first is the preliminary phase of researching and planning, next is the actual teaching, and the third phase is assessing, reporting and evaluating. Each of these phases are informed by both teachers' content knowledge, and their pedagogical content knowledge.

Mewborn (2001) summarised and critiqued research of teacher content knowledge. She identified particular stands in that research including studies that sought to examine the relationship between teacher knowledge and student achievement, those that sought to characterise strengths and weaknesses in teachers' knowledge, those that compared the knowledge of different groups, such as primary and secondary teachers, and qualitative studies that sought to explicate relationships and to clarify the obvious complexity of the issues. Mewborn argued that, on balance it seemed that teachers' conceptions of mathematics did influence the way it is taught, but that while knowing the mathematics is important it is not the same as knowing how to teaching it. Mewborn called for research in a broader range of topics, for longitudinal research, and research into the conceptions of teachers who do have a firm understanding of the discipline.

Pedagogical content knowledge, according to Shulman (1986), is "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (p. 9). It includes knowledge of the most useful forms of representation, the most useful metaphors, and the most powerful ways of explaining or illustrating particular aspects of the content domain to make it comprehensible to others. It also includes an understanding of what makes the learning of specific topics easy or difficult and the common misconceptions and likely preconceptions that students may bring to the learning of specific content areas.

Ma (1999) captures the importance and interdependence of these two forms of knowledge powerfully in her comparative study of Chinese and American teachers of mathematics. Ma found that Chinese teachers were more likely to have a "profound understanding of fundamental mathematics" (p. xxiii) than their American counterparts and that this difference parallelled the comparative difference in student achievement. This understanding included not only a deeper knowledge of subject matter but also a knowledge of *how* to teach it – Ma refers to this as "*teachers*' subject matter knowledge" (p.145, Ma's emphasis).

This subject matter knowledge does not only refer to lists of strategies for teaching mathematics (e.g. Sullivan, 1999), or to the topics that are commonly prominent in mathematics teacher education, both pre- and in-service, such as co-operative groups, using concrete materials, problem solving strategies, and metacognition. More important is whether teachers can delineate the key phases through which students' learning develops, the types of experiences that are necessary at those phases, and the ways such experiences can be differentiated for particular students.

Seeking Insights into Teachers' Knowledge for Teaching

Clearly, there are methodological and ethical difficulties in researching teacher knowledge that are exacerbated in the case of mathematics teaching by lack of access to a well-developed, meta-language to talk about the teaching of mathematics. To overcome these, and with a view to providing potential resources for teacher professional development, an instrument was developed that would allow participant involvement in a non-threatening and professionally respectful way.

All teachers of mathematics at the 16 research schools[2] were asked to complete the task at the first research schools meeting held in October, 2001. Teachers working at similar levels, P-2, 3-4 or 5-6, were asked to work in threes to prepare a concept map for a particular topic. Teachers were also asked to list the key pre-requisite ideas, describe an illustrative teaching task, indicate connections to other key learning areas, and suggest an open-ended question or investigation, an assessment task, and a real world or practical activity. This exercise was located in the context of a joint planning activity and supported by a common format.

Year 5-6 teachers considered the teaching of place value, Year 3-4 teachers considered multiplication, and Year P-2 teachers looked at teaching length. Coded scoring rubrics were developed for each aspect of the task where 0 represented the lowest rating and 3 the highest. Most scores in the rubrics refer to two aspects: usually one relates to the legitimacy of the particular example; and the other refers to the adequacy of the elements of the concept being considered. Both were evaluated by reference to research relevant to the particular concept considered.

We focus here on only four of these components. The instructions for these components were for the teachers to record:

- A concept map for the set of activities. This should contain, at least, identification of the key experiences or concepts associated with the particular topics, and lines to illustrate connections. Use as few words as possible.
- The key ideas that the teacher should be thinking about.
- An illustrative task or activity
- An open-ended questions or investigation that could be used

We based our scoring of the concept of length on the framework developed by the ENRP. The key elements of the framework for length are comparing (including conservation and transitivity), quantifying (including the idea of a unit iteration), using standard units, and applying (see Sullivan & McDonough, 2002 for elaboration of this). The teachers completed the instrument in groups of three, all teaching at the same level, but not necessarily from the same school. The responses were scored by two members of the project team, in collaboration, and then scored again by another member of the team. Where there were discrepancies, these were checked and debated. Basically we expected that the teachers, in their response, would indicated their understanding of the particular strategy and of the key elements in the learning of length. For example, the ENRP noted that three quarters of students at the start of schooling can already compare the length of two objects. While we did not expect the teachers to be aware of this result, we assumed they should be aware what their students can do. So tasks at Grade 1 level that focus on simple comparisons, for example, received a low rating.

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[2] A structured, representative sample of Victorian primary schools including 4 Catholic schools, 1 Independent school and 1 special school.

The percentage of responses assigned a particular score for the concept mapping task for length is shown in Table 1. Note that concept maps have been used elsewhere as a way of accessing teacher conceptual understanding in mathematics (e.g., Williams 1998), and this particular prompt was the focus of the task.

Table 1

Percentage Ratings for Teaching Length in Year 1 in Concept Maps (N=83)

Rating	Rating descriptor	Percent
0	Lines not illustrative of conceptual connections	57
1	Some lines illustrative of connections, others not	41
2	For concepts shown, lines are relevant, indicative of connections	2
3	Lines illustrative of links and sequences, clearly shown	0

Even though the teachers worked in groups (or perhaps because of that) the concepts maps did not seem to indicate a firm understanding of the links and connections associated with the teaching of length.

Figure 1 presents an example of a concept map that was rated as "lines not illustrative of conceptual connections":

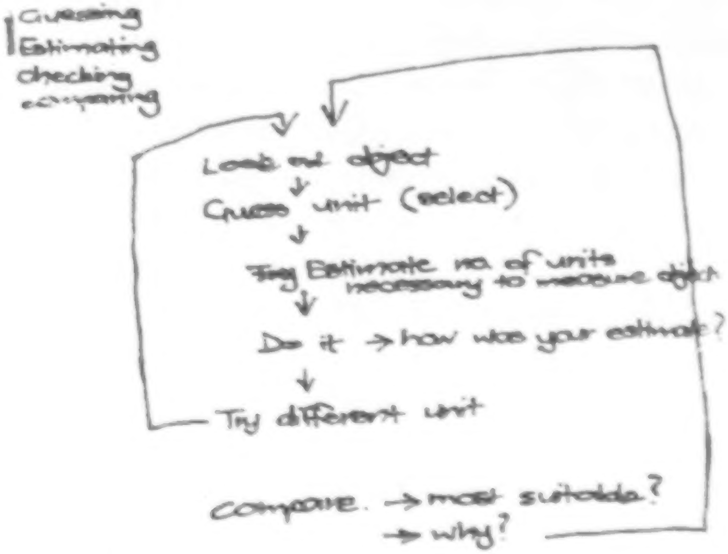
Guessing
Estimating
checking
comparing
Look at object
Guess unit (select)
~~Try~~ Estimate no. of units necessary to measure object
Do it → how was your estimate?
Try different unit
compare. → most suitable?
→ why?

Figure 1. A concept map "not illustrative of conceptual connections".

While there are some relevant elements present in the map, there is no real sense of connections and logical links. To the extent that this map is a representation of how these teachers see the concept of length, it is difficult to anticipate that their teaching of this topic would be focused or coherent.

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We also sought responses to prompts that would provide different insights into their perspectives on the teaching of length. Table 2 presents the scoring of responses to the prompt that asked for the key pre-requisite/subsequent ideas for length to be listed.

Table 2

Percentage Ratings for the Key Ideas for Teaching Length In Year 1 (N=83)

Rating	Rating descriptor	Percent
0	Lists ideas, no clear links	45
1	Some prior and subsequent ideas	43
2	Most key prior and subsequent ideas	10
3	Key prior and subsequent ideas	2

Most responses were scored in the lowest two categories. Again it seems that the teachers overall have not articulated an understanding of the way these key ideas develop. To illustrate the way the responses were interpreted, the following is an example rated as "lists ideas, no clear links":

> Discover children's prior knowledge of language in relation to length

It is possible that this group saw prior knowledge as a key idea. However, given the context of the prompt, we required the teachers to incorporate some the elements of learning length into their responses. Generally the responses at this level were similarly lacking in detail.

The following is an example that was rated at "some prior and subsequent ideas":

> Prior: knowledge of long/short/same as. Placing objects end to end. Ordering of length of objects

In this case, some of the relevant elements are present, although it is clearly only a limited subset of the possible concepts in length at this level. This response is representative of the other responses scored at this level. We expected a broader range of the key ideas to be presented. Note that nearly 90% of the teachers' responses were rated at or below this level.

The following is an example that was rated at "most key prior and subsequent ideas":

> Basic vocabulary of length e.g., long and short. Things can be measured using a variety of equipment – formal/informal. The need for appropriate units of measure – using relevant equipment

While this level of description was presented by comparatively few teachers, we can infer that these teachers, at least as a group, could plan effective experiences for teaching length.

The following is an example that was rated at "key prior and subsequent ideas":

> Concepts of long and short. Length and width. Informal tools to measure (making appropriate choices). Estimating and comparing lengths. Need for a standard unit of measure

These teachers have touched the relevant bases, and the response is indicative of a firm understanding of the key elements of the teaching of length.

It is possible that teachers know more than they recorded on the response sheets. However there results are still cause for concern.

A further perspective was sought through the teachers' suggestions of an illustrative teaching task. The rating of responses is presented in Table 3.

Table 3

Percentage Ratings for the Illustrative Task for Teaching Length in Year 1 (N=83)

Rating	Rating descriptor	Percent
0	Focus or purpose of task not clear	20
1	Task emphasizes skill or application only	45
2	Useful task but loosely connected to concept	33
3	Task emphasizes concept or its understanding	2

The responses received higher ratings for this task. An example of an illustrative task that was rated at "focus or purpose of task not clear" was:

> Make a paper chain the length of your ruler using common units

While the task may seem reasonable, it was scored at the lowest level because it seems a mix between a chain with presumably variable loops and some other task. Either it does not make sense or it is trivial.

An example of an illustrative task that was rated as "task emphasizes skill or application only" was:

> Handspans- create cut paste student handspans. Establish the need for a common language to share information about distance.

While at least in this case, the units for an individual would be the same, it is not clear what the students would be required to measure.

An example of an illustrative task that was rated as "useful task, but loosely connected to concept" was:

> Who has the longest name. Graph results.

One way, this is just a counting and graphing task. There is though considerable potential in the task and so it was scored more highly than the previous example.

An example of an illustrative task activity that was rated as "task emphasizes concept or its understanding" was:

> Bring your soft toy from home. Measure its length ("how long it is") using a strip of paper. Find another toy that's the same length.

Even though it is a low level and easy task at this level, it does have some openness and some possibility for exploring problematic aspects of measuring the length of toys.

Overall the teachers seemed slightly better at specifying illustrative tasks, although it should be noted that this does not provide evidence of their ability to plan sets of coherent experiences.

We also asked the teachers to give an example of a relevant open-ended question. The profile of the ratings of their responses is presented in Table 4.

Table 4
Percentage Ratings for the Open-Ended Questions for Teaching Length in Year 1 (N=83)

Rating	Rating descriptor	Percent
0	Closed or inappropriate activity	26
1	Some openness but only loosely linked to the concept	67
2	Open and loosely linked to one of the key ideas	5
3	Open, illustrates or elaborates key ideas	2

An example of a response that was rated as "closed or inappropriate activity" was:

> How do we measure? Why do we measure?

This response was scored at the lowest level because it is inappropriate in that it does not necessarily focus on length, and in any case it is hard to know what the students would be expected to say.

An example of an activity that was rated as "some openness but only loosely linked to the concept" was:

> Measure your own arm, leg etc. Find someone else in the room with the same measurements. Measure other body parts. Find total length of bones in your body

This task has some potential but it was written in a way that does not convey an appreciation of the type of investigation needed by students at this level.

An example of an activity that was rated as "open and loosely linked to one of the key ideas" was:

> Find things in the room that are longer than your hand but shorter than your arm

This is both clear and open in that it prompts and allows a variety of responses. It does, though, focus on just the comparing aspect of length.

An example of an activity that was open, and illustrated or elaborated key idea was:

> What objects that you can see in the classroom could you use to measure the netball court; picture of the blue whale; the oval?

This also allows a variety of response, and has the advantage that it addresses a key aspects of measuring for students at this level, that of appropriate units of measure.

Overall, the responses to the concept mapping, the key ideas, and the open-ended question prompts were sufficiently low to be of concern. The responses to the illustrative task were a little better. While there are a number of reasons to suspect that these responses under-represent the actual pedagogical content knowledge of the teachers, they do indicate some potential concerns about the depth of teachers' knowledge of key aspects of teaching length, and subject to further exploration, may suggest some potential alternate emphases in mathematics teacher education.

Conclusion

Teachers' knowledge is recognised as a key component of their capacity to plan, implement and evaluate mathematics learning experiences. *The Primary Numeracy Research Project* is concerned with identifying and describing a range of numeracy-specific teaching approaches. To the extent that teachers' knowledge shapes their intentions and subsequent actions in the classroom, the project is interested in exploring its nature and role in planning and implementing classroom interactions aimed at scaffolding student numeracy learning.

The evidence presented suggests that these teachers did not include in their responses the type of knowledge of the teaching of length that would make us confident in the adequacy of their conceptualisation of the key elements of its teaching and learning. The responses of the middle primary teachers on multiplication, and of the upper primary teaching on decimal place value were scored lower.

It is possible that the context of the data collection did not allow the teachers to give full responses. They were working in groups, possibly including some unfamiliar colleagues. They were perhaps satisfied with a general response when they may have produced more details had they actually been planning classroom experiences. Nevertheless, that they chose these particular responses rather than others is possibly evidence of a lack of awareness of the nature of the learning of length.

This was a preliminary instrument and was developed more as the basis of some subsequent professional development. Nevertheless, taken on face value, the data raise concerns about the level of teachers' pedagogical content knowledge. We suspect that it may mean that teacher education programs should focus more on key aspects of teaching and learning the important concepts in mathematics.

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Contexts in Mathematics Teaching: Snakes or Ladders?

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This paper reports analysis of discussions between mathematics educators about ways in which the contexts in which classroom tasks are embedded can have the effect of alienating some students, especially those who may have less appreciation of the purposes of schooling. The results suggest that some care could be taken in the choice of and introduction of contexts of mathematical problems, in order to ensure that all students relate appropriately to the contexts as well as the mathematics.

Introduction

> The *Melbourne Age* of 12th of January reported that some members of the Chinese Army are overweight. They suggested that for soldiers between the ages of 20 and 29 their weight should be their height less 105 with a tolerance of 10%. For soldiers between the ages of 30 and 34 their weight should be their height less 100 with a tolerance of 10%, and with a 12% tolerance for 35 to 39 year olds.

This is an example of the context that could form the basis of a mathematical experience. Before using such a context teachers would need to make judgments about its mathematical suitability, the interest or relevance to the students, the potential motivational impact, and the possibility of negative effects or tendency to exclude some students.

This paper is a report of some data related to the use of contexts from the first phase of a project, entitled *Overcoming Barriers to Mathematics Learning*, that is exploring a range of such aspects of mathematics pedagogy. Our basic contention is that there may be some aspects of currently recommended approaches to mathematics teaching that improve the learning of most students but that may be alienating for some.

Theoretical Bases of the Project

We base the project on the use of open-ended questions since it seems that they incorporate many aspects of current approaches to mathematics teaching, and have been demonstrated to be effective generally with students at a variety of levels (see Sullivan, Bourke, & Scott, 1997; Sullivan, 1999). What has not been researched is whether such tasks, and the associated pedagogies, advantage *all* students. It is recognised that students from socially and culturally divergent backgrounds are the most at risk of mathematical failure. It has been suggested that the very socio-cultural nature of mathematics and mathematics education has led to a differentiation amongst its learners between those who can engage with the presentations of the subject and those who are unable to do so (Burton, 1996; Dengate & Lerman, 1995; Salomon & Perkins, 1998). Thus we are concerned that the very characteristics that make open-ended tasks effective for many students may create

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difficulties for others. For example, at least part of the way that open-ended tasks function is by removing some of the detailed task demands, making the requirements less explicit; yet this may have the effect of creating insecurity in some students.

A further theoretical basis of this research is a construct proposed by Bernstein's (1990, 1996) pedagogic discourse. Bernstein claimed that through "visible pedagogy" and "invisible pedagogy", students receive messages about both the overt and the hidden curriculum of schools. He suggested that middle-class students are able to make sense of invisible pedagogy more effectively than their working class peers, due to their familiarity with the embedded social values and norms; and hence they have more chance of success.

A third body of work that we draw on is Cobb and McClain's (1999) use of a socio-mathematical framework where two complementary norms of activity in mathematics classrooms are delineated: mathematical and socio-cultural norms. Wood (2001) elaborated these two dimensions, emphasising the social nature of children's learning and the way that rich social situations contribute to this, noting that a key element is the way that children try to make sense of the classroom culture. Wood described the other dimension as the interplay of children's developing cognition and the structure that underlies mathematics. Based on this work, we define "mathematical norms" to be the principles, generalisations, processes, and products that form the basis of the mathematics curriculum and serve as the tools for the teaching and learning of mathematics itself. The "socio-cultural norms" are the usual practices, organisational routines, and modes of communication that impact on approaches to learning, types of responses valued, views about legitimacy of knowledge produced, and responsibilities of individual learners. These two dimensions underpin our project, and we argue that it is important to make social norms comprehensible and communicable. Of course, as our goal is to explore mathematics, then this also needs to be explicit. Thus in this project processes and ideas typically used with open tasks (mathematical inquiry, creation of multiple solutions, etc.) are one focus, while teachers' and classrooms' pedagogical activities (mode of introduction and explanation of the tasks, and patterns of interaction during discussion of the varied responses, etc.) are another. We do not see these as discrete phenomena; and in fact we are investigating, in the context of open-ended tasks, the *intersection* of the mathematical and socio-cultural norms and their joint impact on learning outcomes.

Contexts of Tasks

In this paper we are focussing on the use of relevant contexts to make problems seem realistic and relevant. This strategy has both mathematical and socio-cultural aspects, and is an example of a commonly accepted aspect of mathematics teaching that, if not used carefully, has potential to restrict the mathematical development of some students.

Borasi (1986) defined context as a situation in which a problem is embedded, and proposed that the role of context is to provide problem solvers with information that may enable the solution of the problem. Meyer, Dekker and Querelle (2001) discussed the use of contexts in mathematics curriculum, drawing on examples from five recent curriculum documents developed in the United States, all incorporating "pervasive use of context" (p. 522). They suggested that contexts can be used to motivate, to illustrate potential applications, as a source of opportunities for mathematical reasoning and thinking, and to anchor student understanding. Meyer, Dekker and Querelle argued that high quality contexts should support the mathematics and not overwhelm it; be real or at least imaginable; be varied; relate to real problems to solve; be sensitive to cultural, gender and racial norms and not exclude any group of students; and allow the making of models.

We accept a somewhat broad definition of a relevant context, including everyday and imaginary situations that may appeal to children. Wiest (2001) studied the responses of 273 children, in six Year 4 classrooms and six Year 6 classrooms in the USA, to various contexts. Equivalent problems had the same mathematical structure and demands on problem-solving skills. The sets were categorised as low fantasy, high fantasy, children's real-world, and adults' real-world. Wiest found that the context of problems affected a range of variables including the children's interest in, attentiveness to, and willingness to engage with problems; the strategies they used; the effort they expended; their perception of and their actual success; and the extent to which measurable learning outcomes were attained. Wiest suggested that while the general type of context did not appear to affect children's actual performance, fantasy responses evoked stronger affective responses than the other types of problems.

While the general consensus among teachers is that the practice of embedding school mathematics into some "real" context supports learning, this consensus requires some further exploration. For example, in their comprehensive review of the national testing system in the UK, Cooper and Dunne (1999) argued that contextualising mathematics creates another layer of difficulty for students. They undertook a sociological analysis of learning tasks and children's performance, concluding that contextualising tasks creates particular difficulties for working-class students, so much so that they performed significantly poorer than their middle-class peers on these tasks whereas performance on decontextualised tasks was equivalent. Drawing on a large sample, these findings are important as they pose a new dimension to understanding how some practices in school mathematics—and in this case contextualising tasks—create further barriers to success for some students. Cooper and Dunne argued that the process of recontextualisation whereby school mathematics is connected to another field—in this case the everyday experience and knowledge of students—creates a new set of demands previously not recognized. They proposed that students need to identify the recontextualisation process so that they recognise the demands of the task as being school mathematics in spite of its immediate appearance of being an everyday task.

To examine further the problematic nature of using contexts, consider the following examples drawn from authoritative sources of information about mathematics teaching. The National Council of Teachers of Mathematics (1989) argued that problems using contexts enrich the experience of learning mathematics. They gave an example of a contextual problem, supposedly arising from a social studies lesson about commerce between North America and Hong Kong. The problem is about a pilot for a major airline transport company being curious about the shortest route between New York and Hong Kong. The problem is clearly unrealistic in that no airline pilot would sit down to work out the shortest route. (It is more likely to be the airline accountant!). The context neither provides a rationale for exploring the mathematical problem nor in any way contributes towards clarifying the problem or making it more accessible.

Another example is from Stern (2000), who used problems with multiple entry points as a way of teaching all students without using ability groups. She presented, as an example, a problem related to an iced cake which, when sliced, had pieces with different numbers of faces with icing. The task was to infer the number of slices of cake. Again, the task is unreal in that no one ever works backward from the appearance of slices of a cake to work out how many slices there were originally. No doubt there would be children who would not appreciate the significance of the icing; as in no way did it contribute to an understanding of the mathematics involved, make the problem more accessible, or give the children a sense of real purpose.

A further example is from Brinker Kent (2000) who used contexts in a culturally diverse elementary school and concluded that all students are capable of learning significant concepts when they have the opportunity to explore the ideas in meaningful contexts. The example she provided is about teaching integers, using the process of ships moving from a canal filled with water at one level to a river that has a different water level. This context is also problematic. If students are unfamiliar with river shipping then it is no more meaningful than frogs jumping backwards on lily ponds. Even for students familiar with the context, the link to the mathematical concepts is somewhat doubtful. It is not really a suitable model of negative integers, since the negative heights are only negative in a relative sense.

We stress that these problems are drawn from rigorously reviewed sources, indicating the complexity of selecting appropriate contexts. We hasten to add that we are not being critical of teachers using such contexts, or even these authors. Indeed we have used similar contexts ourselves, both in our teaching and in our research. We are suggesting that there has not been enough critical examination of this aspect of the debate.

We are *not* arguing that contexts should not be used, indeed we believe that contexts have much to offer. The issue for us is that teachers need to be fully aware of the purpose and implications of using a particular context at a given time with particular students; to choose contexts that are relevant to both the content of problems and children's experience; and to develop strategies for making uses of contexts clear and explicit to students.

Overcoming Barriers to Mathematics Learning

The data reported in this paper arose from the first phase of a larger project exploring barriers to learning mathematics. The intention of this phase was to collect information from a range of educators on what they saw as the challenges and opportunities of open-ended approaches to teaching, with the aim of articulating some specific teaching strategies that could contribute to the second phase of the project.

In this first phase of the project we conducted three discussion groups: one mostly teachers, another all academics, and a mixed group (see Mousley, Zevenbergen, & Sullivan, in press). Each of the groups included people with expertise in aspects of minority-group pedagogy (e.g. Aboriginal education, or literacy for children with non-English-speaking backgrounds). Some of the teacher educators involved were from mathematics education while some were from other fields.

Each of the groups was shown some short excerpts from videotapes of mathematics lessons. The focus groups viewed the videotapes and responded to the prompt "Did you see any aspects of the classroom interaction or activity that may not suit some children?" Participants' contributions were audio recorded and the tapes were transcribed then entered into NVIVO software, coded and categorised. That is, using a grounded theory approach (outlined by Richards & Richards, 1989, and described in Mousley, Sullivan & Waywood, 1998), a list of nodes and sub-nodes was created and refined progressively until the data were grouped under useful headings.

We focus here on one particular excerpt that all three groups watched. The point of the mathematics lesson shown in this extract was to teach about the notion of mathematical mean. The core activity was an open-ended task posed to the class:

> *The mean height of three people in the room is about 155 cm. You are one of those people. Who might be the other two?*

The main purpose of the learning experience was to focus on different ways of examining the meaning of "mean" and ways of arriving at it, although estimation of height was a dimension of the lesson.

The basic task has two features that are relevant for our project overall. One is that the task was open-ended and so allowed a range of possible approaches, procedures and answers for any student. It was intended that the openness of the question would provide an environment for the learning of mathematics that emphasised the possibility of multiple responses, making explicit to the students that it was their own exploration that was required, and both valuing and learning from the range of responses produced. The second relevant feature of the task is that it had a personal dimension. That is, the students were working in a practical context that was directly relevant for them—the estimation or measurement of their heights.

In order to establish a realistic context for this problem, the teacher on the video extract had chosen a poster that projected five people being presented as part of a police lineup. In using this initial context, the teacher attempted to give some rationale or meaning to the context in which the estimation of height and the calculation of means would be enacted. However, it is this context that was the focus of much of the discussions by the groups who viewed the videos.

Data

Our intention was to identify issues related to any aspects of the high-quality teaching shown on the videotapes that practitioner and expert panels would raise in a group discussion. Of course we had prior conceptions of necessary considerations for such teaching, yet by creating forums for discussion among educators we hoped to hear alternate perspectives. The focus groups covered many aspects of mathematics teaching, but we restrict the discussions here to the comments on the context in which the particular problem above was embedded.

The main focus of the lesson was the problem for each student of finding two other students so that together the group had a given mean height. Perhaps thinking that this problem would be unrealistic by itself, the teacher used a police line-up poster as a preliminary context, and it was this element of the lesson's introduction that provoked the most critical comment.

Many of the participants affirmed the strategy of setting problems in realistic contexts:

> The kids get a certain message about the nature of mathematics [that] it's cut off from the world; it's no use to anybody.

Some were concerned, however, about the appropriateness of this context. There were comments about the importance of awareness and sensitivity. For example:

> Dad's off in gaol or someone else has been arrested in the community, so they know about line-ups. They know about the police, they know about the justice system by what's happening to them. So you could use it if it wasn't that sensitive. It comes down to knowing their backgrounds and the sensitivities for the children that you're teaching.

In the same vein, one participant raised the issue of teachers' knowledge of the familiarity of particular situations:

> Should one always use a familiar context? It would depend on the relationship between the school, the community, the teacher and those children. If the teacher knows what's going on at home and …

Others noted the potential ambiguity of the student responses to the prompt:

> What I was concerned about (is) engagement in the task, because one of students said, "Yes, we want to cooperate with the police, we want to find out the clues." The other type of students: it could be their dad!

Some participants also discussed how children's responses to the context might be related to socio-economic status. For example:

> It's a middle class thing—the criminal is more threatening and you have vested interest in identifying them to police. If you are a student from working class, chances are you are on the operative side of things.

Clearly an important issue is that of choosing appropriate contexts. The tenor of discussion was to suggest that the context used in this case was inappropriate and different ones would be preferred. For example, when we asked participants to suggest characteristics of a context that might be culturally positive as distinct from culturally negative, the responses included:

> Australian netball, and a football team. There you have gender equity and the netball teams are sort of a star team. Or the cricket team—cricket mightn't be very relevant—maybe soccer [would be okay], and most kids have a knowledge of an AFL team.
>
> It would be interesting to choose a context like several sports because the heights are so different. What do we think is the average height of a basketballer? How high might that be on the wall? And what about footballers and what about soccer players?
>
> Drag in a few Year 12 girls. You'll find none of them are six feet tall and yet every netballer is six feet tall. Who won gold medals for Australia at the Olympics? The Tai Kwon Do person was this big, and the rowers were that big.
>
> The context needs to be close to the kids but not so close that it's an emotive issue.

In summary, it was suggested by members of all three of the focus groups that before being used a context needs to be considered appropriate in terms of familiarity to the students, culturally sensitivity, and appropriateness in terms of their socio economic background.

We recognise that it is difficult, especially within the usual time constraints on planning teaching, for an individual teacher to consider fully the ramifications of a variety of specific contexts, or to allow time for children to create their own relevant contexts especially with new mathematical concepts. The alternative may be to avoid some potentially interesting contexts and to use somewhat less exciting ones.

This raises, in turn, the related issue of whether mathematics teaching is politically or culturally neutral. On one hand there were arguments to the effect:

> What would be the purpose of teaching? If it's mathematics then your aim really isn't to teach about the justice system or educate in that sense. It may come up but [the] aim is to teach about mathematics. The poster was just an introduction to the point of the lesson, which was estimation.

Another participant recognised the relevance of politically sensitive topics, but suggested a cautious approach to their use:

> Politics and issues of race and culture could enter maths classrooms, but you've got to recognise that it's very dangerous stuff. A lot of teachers stay away from it because they think they're going to get into trouble if they introduce those [issues] into their teaching.

We suspect that this is not a strong rationale for a choice to use or not use a particular context. Consider again the context posed for the middle-secondary problem at the start of this paper—optimum weight for members of the Chinese army. It is reasonable to argue that using such a context with teenagers could be alienating for some. Yet, body image and health and employment opportunities are important issues for teenagers. There is also potential for looking at how cultural groups vary on typical physiques and how these are changing over time. Perhaps such issues may well be amenable to some mathematically oriented investigation, and in that way, explicate the context in a way that illustrates mathematically based analysis of a relevant, socio-culturally situated and practical context.

As an aside, the misconception that the lesson was about estimation of height was common. It is reasonable to assume that the children experiencing the same introduction would also see it as a lesson on estimation, although the focus was the notion of "mean". This illustrates how contexts can add layers of complexity.

Conclusion

With respect to the choice of context in which a task is set, the focus group participants were concerned about (a) appropriateness of the context, particularly considering sensitivity to the background of the students; (b) whether there is an inherent class bias, either positive or negative in the context; (c) whether the focus and purpose of the mathematics learning remain clear; and (d) whether and in what ways relevant but sensitive topics could be used.

Our project is working with teachers in four schools who will use open ended questions as the basis of units of work as part of their regular curriculum. One of the outcomes of the project will be the provision of written advice for teachers—and we will be providing particular advice. With this objective in mind, we have identified pedagogies that are important for learning and in particular those which may not be explicit to all students in the class, with the deliberate intention to find ways of making those pedagogies explicit. Our intention is also to measure the way that this affects learning. The determination of this advice is part of the focus of the second phase of this project.

To differentiate our work from Bernstein's invisible pedagogy, we are using the term "implicit pedagogy", because we intend to make explicit particular aspects of invisible pedagogy that have been identified. This "making explicit" needs to take two forms. First, teachers need to become more aware of specific, common aspects of teaching that may not be optimal for certain groups of pupils, and then address these when working at improving their typical patterns of interaction in mathematics classrooms. Second, aspects or approaches to teaching that they decide to use purposefully need to be made more explicit to the children so that potential for confusion is reduced and reasons for using particular strategies are well understood.

If you, the reader, have already worked out whether you would be accepted into the Chinese army then that context has excited your interest. You may have asked yourself questions, such as, "What are the rules for other age groups?" or "Is it different for women?" If so, you have seen the mathematical potential. You may be reluctant to use such a context with teenagers because of the potentially alienating effect on some students although teachers of other subjects presumably have ways of coping with such issues. Clearly the use of such a contextual prompt is potentially ambiguous, yet even that ambiguity is contestable, and we suspect that avoiding the use of interesting but potentially difficult contexts is not the solution.

The difficulties in more commonly-used contexts are often not as obvious, yet they may exist. Hence while we agree with the mainstream belief that contexts can be useful, it is clear that teachers need to develop sensitivities and take appropriate steps to avoid the potential for contexts selected to be alienating, excluding or exacerbating of disadvantage.

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Professional Development of Teachers of Mathematics Through Extended Collegial Dialogue: The ACT Mathematics Quality Teacher Program

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Perceived pressure to cover content in limited time often results in secondary mathematics teachers valuing "quick-fix" professional development activities that focus on classroom materials over extended programs that promote professional growth through discussion and reflection. The Australian Capital Territory Quality Teacher Program (Mathematics) commenced as a collaborative exercise in developing curriculum materials, but evolved into a process of reflection. This evolution, and the resulting change in teachers' beliefs, is described and discussed from two perspectives: that of the facilitator and a participant.

The QTP

The Quality Teacher Program (QTP) is an Australian Commonwealth Government initiative designed to develop and integrate new pedagogies, diversify teaching roles, take teachers to the global community, enhance teachers' power to direct students to broader knowledge bases and, as a result, raise the status of teaching as a profession. It focuses on the areas of literacy, numeracy, information and communication technology, mathematics, vocational education and indigenous education. The major project in the Australian capital Territory is the Curriculum Integration Model (CIM), which focuses on infusing information and communication technology into pedagogy across Key Learning Areas, by involving groups of teachers in designing curriculum materials that will embed ICT into everyday practice. Two smaller projects in science and mathematics were also conducted. The QTP Science project developed support materials to promote science-based excursions in the community, and offered workshops and short courses. The QTP Mathematics project targeted teachers in the transition years from high school (years 7 to 10) to college (years 11 and 12), who were to collaborate in developing a package of materials that would assist teachers to review their teaching practice in line with contemporary teaching approaches, including the integration of ICT.

Mathematics QTP in the ACT

In June 2000 an initial steering committee, including project personnel, teachers and an external consultant, met to discuss possible directions and strategies for the mathematics QTP in the ACT. The transition from high school to college was identified as a critical issue (Coutts-Smith 1995), particularly in an era of increasing use of technologies such as computers and graphics calculators. This meeting decided that functions and coordinate geometry provided a suitable content focus for the development of curriculum materials that would address these transition issues, and that could incorporate the effective use of new technologies. Groups of teachers from two senior secondary colleges and their feeder

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high schools were invited to express interest in collaborating to develop innovative units of work in this area.

The QTP was designed as a professional development program, rather than as a research project. Hence the observations and narratives in this paper have been written with the benefit of hindsight. They describe how the project evolved from one with a focus on materials development to one involving active reflection and extended collegial dialogue.

The Facilitator's Story (Steve Thornton)

An initial full day meeting of the teachers selected to take part in the project was held in September 2000. It was attended by only four teachers, representing two ACT colleges and two high schools. Apologies were received from two other teachers who felt that they were unable to take time from their classes, despite teacher release being provided. Together with the ACT DECS Numeracy Officer, I acted as facilitator for the day.

QTP 1

After a brief introduction to the aims of the project, teachers were given an opportunity to raise issues concerning the transition from high school to college that they felt should be addressed in the project. Five issues were identified as inhibiting students' progress in mathematics, and preventing the introduction of investigative or problem-centred pedagogy, or the use of technology. These were:

- The loss of time for mathematics, often including longer periods with less frequent contact, with no corresponding reduction in content;
- The range of abilities with which students enter high schools or college;
- The unwise choices that students often make at college, with the resultant high proportion of students attempting courses in which they are unlikely to succeed;
- The lack of access to computer facilities within the school; and
- The continuing need for students to learn basic facts and formulas.

The discussion was essentially negative, focusing on problems rather than possibilities. As facilitator I attempted to draw together the threads of the conversation, and to draw the participants' attention to readings that may be relevant to the issues identified. The group discussed the readings described below during the second session.

Hadamard (1996) described the stages of the thinking process through which professional mathematicians such as Poincaré and Einstein commonly progressed as preparation, incubation, illumination and verification. Drawing on this study, the group discussed what was really meant by the lack of time in the mathematics curriculum, and whether the issue was really time to cover content or time to think deeply.

The group also discussed what it really meant to understand mathematics. We looked at the meaning of mathematical understanding as described by writers such as Skemp (1976) and Carpenter & Lehrer (1999). The group agreed that much of the focus of school mathematics was developing instrumental understanding, in which mathematics was seen as a set of isolated skills or processes. We looked at what "ability" really meant in this context.

The group discussed whether the seemingly frequent making of unwise course choices in year 11 and 12 was actually a function of students' unrealistic assessment of their own mathematical capacity, or whether it really highlighted a deeper issue of students' failure to reflect on their own learning. The importance of students developing effective learning behaviours (Corkill, 1999) and productive habits of mind, such as flexibility, curtailment, logical thought and metacognition (Krutetskii, 1976) was highlighted.

Although lack of access to appropriate technology was identified as a problem, we tried to focus on how technology could be used to create a community of learners (Galbraith, Goos, Renshaw & Geiger, 1999) in which deep, shared understanding of mathematics was valued. Participants suggested that their current experience was that of technology being seen frequently as master, less frequently as servant, and seldom as a partner in learning or an extension of self.

While participants agreed that facts and formulae were important in developing students' mathematical facility, there was some discussion about how much students needed to remember, and about which facts or formulae were really important. The group developed a long list of concepts associated with functions and coordinate geometry, and attempted to link them in a concept map. This concept map was used to set priorities for the development of suitable curriculum materials.

One further full day meeting was held during 2000, involving the original four participants and some teachers who had not previously been part of the project team. The influx of new participants, some of whom brought their own agenda to the discussion, resulted in a lack of shared understandings, and consequently created some disunity in the group.

Throughout these discussions, the agenda of developing curriculum materials remained, and the team felt pressured to commence this task. Two sub-groups of teachers were formed, each with a brief of writing a unit of work. These sub-groups rapidly became isolated individuals, each developing his or her own materials.

As facilitator I found the process immensely frustrating. It seemed to me that the project brief had been developed hurriedly, with a focus on deliverables, an emphasis which flew in the face of the literature on effective professional development (Loucks-Horsley, Hewson, Love, & Stiles, 1998). Loucks-Horsley (1999) stresses that providing time, contexts, and support for teachers to think and work at resolving dissonances is essential for professional growth. Askew, Brown, Rhodes, William and Johnson (1997) found that participation in extended courses of professional development in mathematics was strongly related to belief orientation, and with pupil gains. Thus effective professional development must be an extended process involving deep reflection on teachers' beliefs, knowledge and practices. A learning environment in which mathematical knowledge is fixed and owned by experts, in which mathematical learning is a process of memorisation and recitation of procedures, and in which timed tests determine what is valued by both teachers and students, seems so firmly entrenched in teachers' everyday practice that new materials or activities become subservient to the traditional culture of the school mathematics classroom. In reflecting on the process during 2000, I suggested that the culture of the school mathematics classroom, as described by most of the participants, was so robust that meaningful change was unlikely to occur through the existing aim of developing classroom materials.

The lack of obvious collaboration and progress, and the difficulties associated with developing a coherent theme for the materials, reinforced the belief that the project, as initially conceived, was unlikely to bring about the new pedagogies or focus on student learning described in the aims of the QTP. In consultation with the manager of the ACT QTP, we therefore decided to refocus the mathematics project away from materials development, to attempt to make teacher reflection the central activity of the project, and to introduce teacher narratives as one of the principal outcomes of the project. Participating teachers would be asked to conduct observations in their and others' classrooms, to reflect upon the learning taking place, and to engage in dialogue with teachers in their own and other schools.

QTP 2

The refocused aim of the QTP then became to "promote quality teaching and learning in mathematics, through cross-sectoral discourse involving reflective practice and professional conversation focused on teachers' work and student outcomes". The principles underpinning this aim were that teachers needed to engage in professional reflection and discussion, that quality input was vital to professional growth, and that collaboration with other teachers supported quality professional development. These aims and principles stood in stark contrast to the initial aim of designing "a package of materials to assist teachers to incorporate ICT and current research on student learning into their teaching". It had a clear focus on teachers as creators, rather than end-users, of knowledge.

The QTP meetings that followed this refocusing took a very different turn. The emphasis moved from product to process, from generating materials to writing stories. While the conversation still touched on many of the commonly expressed problems faced by teachers in their daily work, the need to find solutions diminished. By reflecting on videos of their own and other classrooms, such as those in the Third International Mathematics and Science Study (US Dept of Education, 1997), the participants gained a sense of solidarity with each other and with teachers around the world. The value of sharing experiences and of seeing teaching as part of life-long learning became paramount in the participants' discussion.

Comments such as "Thank you for listening" and "The discussion allows me to question things that bug me and perhaps re-evaluate my feeling about what I teach" were now common in participants' evaluations of these sessions.

Not all the participants appreciated the change in focus. One commented that "as time progressed and the group decreased the objectives seemed to change and these changes were being imposed from above … I felt there was an agenda of which we hadn't been made aware." Another noted that "much of the later discussions and focus… were not really appropriate to my situation".

Yet the conversations suggested that the participating teachers' beliefs about mathematics and learning had changed in a very fundamental way. The problems identified by teachers during the first discussion in 2000 were now seen as opportunities rather than hurdles. The participating teachers appeared to see themselves as learners, rather than imparters of knowledge. The changed tenor of the discussion is exemplified by an extract from a paper paraphrasing a spontaneous conversation that arose at one of the meetings.

> Sometimes it's hard to get started with technology. One of the big hurdles is the time it takes for students to learn the technology. This might seem to take time away from the intended curriculum. But one of the things we have found is that the time taken to learn the technology pays off in the end. Students are then able to do far more, in far greater depth, than would be possible with pen, paper and book alone. We have learnt that you have to be prepared to invest the time to let the students play, and to just become familiar with what the technology can do.
>
> It also takes time for students to learn to talk to each other. That seems to be one of the more surprising outcomes of using technology – students talk to each other more, and learn from each other more. And the quality of the conversation is often different. Instead of talking about if their

answer is right or wrong, the students talk about ideas and concepts. The conversation adds a new depth and dimension to the classroom learning. No longer is the conversation a monologue by the teacher, nor even a dialogue between teacher and students. We are all contributors to the knowledge and debate in the classroom. The conversation is about mathematics as something that is discovered and owned by the community.

Perhaps the most important message we want to emphasise is that teaching is really about learning. Classrooms are learning places, not work places. And technology can help to shift the focus away from teaching, although that's still important, and onto learning.

A Participant's Story (Kerrie Blain)

In September 2000 I was invited to participate in the research circle of the Quality Teacher Program (Mathematics) in the ACT. I teach mathematics at a high school, and accepted the challenge as this was of vital interest to me, as a practising teacher, for a number of reasons.

Firstly some background information – I had spent several years as a relief and part-time teacher in ACT high schools and colleges while having, and caring for, a large family. I then found it necessary to be away from teaching for 14 years. This placed me in the unusual situation of being able to objectively compare and contrast classrooms over the years. During this time, however, I was still very involved in the education and growth of teenagers. I realised that the teenagers of 1984 were vastly different to those of 1998. To my dismay, however, when arriving back in the classroom in 1998, I found that teaching methods in mathematics had changed very little during the intervening years. So my initial aim by being involved in this project was to encourage teachers to adapt to the new climate in education. Teenagers are leading the way. It is obvious that, as educators, we *must* speak their language. This means the efficient use of information and communication technology in the classroom.

The initial challenge delivered by the QTP team was to develop a professional development program designed to incorporate information and communication technologies into the teaching of mathematics in years 9-12, specifically in the topic of coordinate geometry. In so doing it was hoped to encourage all teachers to consider contemporary teaching practices.

I entered this program with the hope of engaging my students more enthusiastically in their mathematics lessons. This was to be across all ability levels. I am motivated to change teaching strategies by students who ask: "When will we ever need this?", or state "This is boring". Our culture is changing, so it is vital that we, as educators, adapt our approach to classroom teaching.

My aims in this project were therefore:

- To engage and stimulate students – making mathematics more relevant using ICT (Aspin 1997) and practical examples;
- To tap into their multiple intelligences, keeping in mind that differently students have their own preferred learning styles; and
- To improve the transition from high school to college.

Students have changed dramatically over the past 20 years; teachers have not. As educators therefore we need to be in tune with our students and present the curriculum in a more exciting fashion to give them a basis and desire for future learning. The school at which I teach was also part of the Pilot Year 9 Exhibitions of Student Learning in High

Schools Project. This project aimed to introduce Roundtable Exhibitions as a form of assessment of student work (ANSN, 1998). I hoped that the mathematics QTP might support the Exhibitions Project to achieve this goal of presenting more exciting and enduring learning.

I joined the Mathematics QTP in October 2000. I found the discussions lively and informative. It was obvious before long that the real benefits of forming such a group were that the discussions not only stimulated all teachers to continue their pursuit of excellence, but also provided a medium for sharing teaching practices, especially in the use of relevant ICT sources.

I was part of a group of teachers from a high school and college, looking particularly at how to make effective use of graphing software in the mathematics classroom. Having previously read and reflected on the potential of graphics calculators for enhancing student learning (AAMT, 2000), I invited a colleague from a nearby college to visit my classroom to give instruction in the use of graphics calculators, and to observe my students using computer programs in the learning of coordinate geometry. I also visited a college class to observe their students using technology. The teachers involved provided verbal and written feedback to each other and videotaped lessons to enable us to observe our own teaching methods. As experienced teachers, this was something we had not done for a long time.

As teachers involved in the mathematics QTP we identified the following outcomes. The QTP:

- Provided an opportunity for professional interaction through professional conversations;
- Promoted transfer of ideas to others within the school community, thus effecting change in colleagues' practices;
- Affirmed our own innovative classroom practices;
- Provided valuable classroom strategies and methods;
- Updated our educational thinking – challenged our pedagogical mindset;
- Promoted self-evaluation;
- Exposed us to broader educational issues through a critical friend who was an educator in a field other than mathematics;
- Helped us to reflect on our own practices and ideas;
- Provided an environment beyond school, time away from the classroom which facilitated valuable reflection and discussion without interruption;
- Provided specific professional development for mathematics teachers;
- Provided motivation to build on what we know and showcase what we do – a collegial activity with both personal and communal dimensions;
- Facilitated cross-sectoral conversations between high schools and colleges; and
- Encouraged teachers as learners who adopted an intellectual commitment to change.

My Conclusions

- Every student can learn mathematics better. We need to continually evaluate our classroom methods realising that different students learn in different ways.
- All teachers can improve their classroom practices by keeping up to date with technology. Teachers must accept change – students are changing, therefore teachers must be able to adapt to this change. Teachers must be learners too.

- It is important to maintain collegial conversation. All teachers face similar problems. Sharing innovative classroom practices could help solve these. This has been invaluable to me in this project. Time – to talk, and therefore to share ideas.
- There is a gap between high school and college which needs to be addressed. Students will benefit if teachers have time to reflect on, and converse, about these problems.
- It is time to change the curriculum. It can and must be improved to suit today's conditions. We must not be restricted by existing curricula. They must be adapted to incorporate modern technology.

Translating the Process into Broader Professional Development

One of the major goals of the QTP was to translate the process through which the participants had been led to reflect upon their own teaching into a broader professional development program for teachers throughout the ACT. The challenge of condensing some eighteen months of discussion into two half-day sessions with an intervening two-hour workshop, particularly given the focus on reflection and narrative, was extremely daunting. In developing the program for the two half-day sessions we attempted to mirror the process of extended professional dialogue, and to develop a blueprint which could be used by other teachers in their own school situations.

By using excerpts from classroom videos, discussion and feedback protocols such as those developed by the Australian National Schools Network (ANSN 2000), and providing suitable discussion stimuli, the teacher and facilitator team was able to establish an environment that promoted active reflection on classroom practice. Teachers participating in the public phase of the QTP stated that they valued the experience of others, the insights into classroom practice afforded by the videos, and the opportunity to discuss their classroom.

> I think that one can never underestimate the importance of reflecting on one's practice – the positive as well as the negative.
>
> I feel that we all have something to learn from our colleagues. Hopefully this is taken away and used or at least discussed.
>
> I will closely examine my teaching practices and identify traits viewed in the videos. I would hope to reduce the amount of mechanical formula-based work and increase discovery learning.
>
> I think it would be good to conduct these discussions on a faculty-wide basis.

Among the outcomes identified by other teachers who attended the three session were:

- The opportunity to experience learning similar to that of the teacher participants;
- Introduction to a framework by which participants could engage in professional discourse;
- Reflection on their personal and collegial teaching beliefs and practices;
- Sharing of resources which modelled quality teaching in mathematics; and
- Introduction to strategies for integrating information and communication technology into classroom practice.

Discussion and Conclusions

Much of the professional development provided for teachers of mathematics can be characterised as "quick fix" remedies, providing good ideas and materials that can be used immediately in teachers' classrooms. While there is always some value in new ideas, such measures are unlikely to make a long-term impact on the beliefs and practices so firmly imbedded in the traditional practice of secondary mathematics classrooms. These beliefs and practices are, in turn, adopted as the dominant culture of the secondary mathematics classroom, a culture which is extremely robust.

The failures and successes of the ACT Quality Teacher Program pointed to the importance of active and structured reflection as a key factor in promoting changes in those beliefs. The long-term impact of the program is impossible to evaluate, at this stage. Whether or not the day to day pressures of the classroom prevent participants from implementing the reflective practices introduced through the QTP remains to be seen. However, for many of the teachers, such as Kerrie, who took part in the extended professional dialogue, the process brought about a questioning of beliefs and a renewal of professional commitment. Their personal narratives make clear that reflection and dialogue are crucial in their on-going development as professional practitioners.

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Preservice Work Within Schools: Teaching Knowledge in Production

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The principal purpose of this paper is to critique concepts of 'content knowledge' and its 'pedagogical application' as they are currently fashioned in the field of mathematics education. Drawing upon work which is primarily philosophical in character, I propose a view of teaching based on my empirical study with preservice teachers in schools. In the first instance I introduce theoretical ideas and suggest how they might function in the process of data analysis. Second, I use these ideas to analyse instances of teaching knowledge in production. Finally, I argue for a conceptualisation of pedagogical content knowledge which recognises the reciprocal constitution of knowledge and subjective experience.

Preservice mathematics teacher education has always been a pivotal agent in the transformation of the mathematics knowledge base of contemporary society. The notion of the preservice mathematics teacher as instrumental to national development and security is premised on the rational actor who will teach better when he or she knows more. This commonsense idea is so embedded in past and contemporary forms of mathematics education that numerous efforts to improve education through policy, research, and practice have all been constructed around a "deficit model" (Askew *et al.*, 1997; Brown *et al.*, 1998) of teachers' knowledge. In particular, the influential work of Shulman and his associates at Stanford University (e.g., Shulman, 1986, 1987; Grossman, *et al.*, 1989) is organised around "improving teachers' own subject knowledge" (Poulson, 2001, p. 43) as a means for better pedagogical practice and, ultimately, for improved social conditions.

Brown and her colleagues (e.g., Brown *et al.*, 1998; Askew *et al.*, 1997) have serious reservations regarding the applicability of the deficit model of pedagogical content knowledge for primary school mathematics teachers. They argue: "[S]ome teachers of younger children have real problems over subject knowledge, but it is not clear how much this affects their effectiveness" (Askew *et al.*, 1997, p. 59). In their recent studies of 90 numeracy teachers these researchers recommend that a reasonable sense of what needs to be taught, together with a skill at communicating and linking this to other aspects for children, might be preferential to a deep understanding of content knowledge. Similarly, Poulson (2001, p. 460) reports that a study by Calderhead (1998) found that content knowledge factored in less significantly on preservice teachers' planning and teaching than did their observations of supervising teachers. Poulson (*ibid.*) notes: "even when students had sound knowledge, they did not draw upon it in their planning and teaching, but preferred to copy and adapt ideas suggested by their supervising teacher/mentor" (p. 46) or which they had sourced elsewhere.

In this paper I look at aspects of classroom practice which contribute to the creation of pedagogical knowledge in the primary school mathematics classroom. My particular focus is centred on the experiences of preservice teachers and is directed at the specific times during the preservice programme when the course offered within the tertiary institution is given a context in schools. My analysis begins with a critical reconsideration of the continuity between knowledge and effective practice. In response to seemingly conflicting

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evidence and unresolved dilemmas which presented in a similar study undertaken by the author the previous year (Walshaw & Savell, 2001), in this replication study I have found it more useful to understand the construction of teacher identity as continually mobile and never taken-for-granted. Approaching teacher identity in this way demanded an acknowledgment of the tentative and shifting balance between theory and classroom experience, and the recurring tension between curriculum and the emergent personal relationships and meanings within the classroom.

In the first section I introduce poststructuralist ideas on knowledge and subjectivity. I draw on these to explore their significance for the construction of an alternative conceptual model for assessing the role of content knowledge in the identity formation of preservice mathematics teachers. In the second section I explore conditions and circumstances for preservice teachers within the context of school work. I focus on one question: what does it mean to engage in pedagogical work during the teaching practicum? Finally I draw upon the empirical data to reveal the discursive practices at play in the process of becoming a mathematics teacher. From the analysis I argue that, rather than exalting content knowledge, we need to look closely at embodied relations and the unconscious to see how these relations exhibit a power differential in teacher/student relations and how they inform, shape and interrupt the possibility of a theory of pedagogy based on linear models of content knowledge application.

A Framework for the Formation of Teacher Identity

A preoccupation with conditions and circumstances in the formation of identity is given theoretical expression in poststructuralist theory. Central to the poststructuralist argument is an understanding that knowledge is a production, created by cognitive agents within practices and positions of differing power and privilege. In this theorising questions about knowers, their identities and their relative locations of knowing are all integral to analyses of the production of knowledge. These ideas allow one to move from a discussion of identity in the context of the classroom as a cultural phenomenon of the psychological self, towards a conceptualisation of multiple conceptions of subjectivity located within the discourses and practices of the classroom. The starting point is that subjectivities are produced within a range of organisational and institutional discourses and practices. Integral to the construction of subjectivity and identity formation, is an *a priori* set of rules of formation which enables certain thoughts and practices of preservice teachers to be entertained at one time and constrained at another time. As structuring principles, discursive practices govern beliefs and practices in such a way as to produce a certain network of material and embodied relations; they are both determined by and constitutive of the power relations that permeate institutional and organisational processes and structures.

When we apply poststructuralist ideas to a study of the 'making' of teachers during the practicum and to an investigation of how pedagogical knowledge is produced by institutional processes and practices we first need to understand the teacher as an historical and culturally specific entity who does not establish his or her materiality with a fixed prediscursive essence. Britzman (1997) and Davies (2000) both speak of this presupposition as the discursive constitution of pedagogical subjectivity. Their poststructuralist language sits uneasily for those of us trained in a tradition where an ontology of sameness casts individuals as autonomous, sovereign and interchangeable. Nevertheless, this move from autonomous individual to discursive subjectivity is a productive means to talk about and investigate the practices and construction of the identity of preservice teachers in school sites, since, firstly, it opens up new possibilities for teacher education. Secondly, it offers a more complex and layered notion of the teacher. Thirdly, such a move allows us to engage the historically specific relationship between preservice pedagogical practice and forms of social control and possibility. And lastly, an understanding of shifting subjectivity suggests an understanding of pedagogical practice and agency as an active yet never-completed process of enculturation.

Methodological Issues

Fundamental to this study is the discursive constitution of subjectivity. More specifically the subjective is regarded as constituting the meaning of student teacher experience – both in terms of where the experience comes from and how it relates to concrete social practices. Relocating the project of preservice teaching within the context of schools for this way of thinking shifts the focus from empirical observations to a broader form of inquiry – one informed by the close relationship between identity, power and knowledge. The issue is less of making linear connections between content knowledge and practice than of showing whose experiences and what knowledges come to count or are dismissed during the process of establishing pedagogic authority.

All 72 students enrolled in the second year compulsory mathematics course at the university in attendance on the day the questionnaire was administered were invited to participate in the study. All of the students agreed to participate. During the three weeks in which they were out in schools during the third (of four) school terms, they worked closely with one classroom teacher (the associate), endeavouring to build a professional partnership together within a supportive environment formed by links with the university and the school.

The Discursive Production of Teaching Practice

Preservice teachers in this study had engaged with a complex knot of course-related practices and knowledges which together produced the possibility and effectivity of classroom teaching in New Zealand primary schools. Ensor (2001) refers to the symbolic and material resources mapped out in the university course as a "privileged teaching repertoire" (p. 299). Through explicit engagements with the official curriculum statement and its theoretical representations of development, cognition, pedagogy, assessment and the learner, they had also learned what counts as evidential content and process and the kinds of questions central to the field. They knew what particular pedagogic modes are legitimated and the types of resources and classroom arrangements deemed central to knowledge facilitation. By producing the terms of school mathematics and thus the parameters of school mathematical practice, the course had created an identification for these preservice teachers. However, like practitioners in other domains of professional practice, prospective teachers are confronted with learning 'discursive codes of practice'. As they move from the university course into the school they enter a different network of political and social discursive practices. The identity positions and politics which these discourses offer provide access to a differential engagement and positioning in relation to the regime of 'knowledgeable' practice operating in schools.

The student teachers in this study soon learned that, irrespective of school size, mathematics was routinely taught in the morning (82% of the schools). A third (30%) of the students saw mathematics taught on a daily basis, and only a relatively small number (15%) reported that mathematics took place less than four times per week. Students quickly learned how long each lesson would be programmed, given that the scheduled length of time was consistent from one day to the next. However the expected duration varied considerably from one classroom to another. Whilst the median time spent on mathematics during the school week was three hours twelve minutes, one student came to expect five hours regularly each week. Another student soon came to appreciate that mathematics would be sacrificed on school concert practice days. Mathematics in this particular classroom lasted only ten minutes on both of two days of the possible fifteen.

In practices of administration like these the production of mathematical knowledge is normalised. Precisely because these practices fix limits, controlling the 'time' around which pedagogical reality might take place, they determine to some extent what is able to count as mathematics. Not only is the status and production of mathematical knowledge implicated in school administration practices but so too are specific forms of knowledge production. Mathematics teachers in schools invest in particular discursive codes of mathematics pedagogy which foreground particular processes and practices, shaping how teachers should plan and enact practices in the classroom. These discursive codes based on theoretical decisions are not always made explicit to the novice, yet to the associate teacher they constitute a closely scripted first strategy of how the teacher's work is to be constructed.

Theoretical decisions about learning have important implications for the ways in which pedagogical relations can be conceptualised and enacted. In creating particular modes of activity, ways of being and interpersonal relationships, such decision making governs the production of knowledge, legitimising and excluding particular forms and events. This is an important point because it suggests that knowledge, including practitioners' knowledge, is implicated not only in the practices of administration and normalisation, but also in the production of forms of sociality.

Elaborating further on this epistemological vantage point, student teachers believed that their associate teachers used "Mathematics in the New Zealand Curriculum", the official mathematics curriculum statement for New Zealand schools, more often for planning (78%) than any other resource. They volunteered that planning was also guided by commercially produced teachers' guides supporting the New Zealand curriculum (60%), worksheets and notes from previous years (55%) and textbooks (50%). The Internet (6%) was perceived to be the least used planning resource.

Students were asked to consider what particular pedagogical relations and practice were actualised in the classroom. Classes were routinely grouped according to ability. Intraclass ability grouping took place in 64% of all classes seen. Not only was ability grouping observed within the class; inter-class streaming/setting arrangements typified an additional 11%. The remaining 25% of classes were ungrouped. In both these latter two arrangements the children were invariably older, and were taught as a whole class, with teacher exposition and children's practice of skills and concepts scheduled for the bulk of the lesson time. In normalising 'who shares the space' in which pedagogy is to take place, these arrangements determine what qualifies in the classroom as learning reality.

Student teachers' observations of a typical lesson for the classes which were organised according to ability grouping reveal that a relationship is orchestrated between mathematics, the teacher, and the child in which the teacher is to maintain prior knowledge, introduce new concepts for the day, make links with prior knowledge, provide explanations, model, pose questions for the children, supply work and activities to enable practice of these ideas, and finally reflect on the work. In this logic the teacher moves reflexively from writing on the board, to observing, to talk, and to questioning, grounding understanding through the process of children's activity and written work. Like many educational texts, the typical lesson structures pedagogical arrangements for school work, establishing a set of institutional and social relations for the teacher and learner in the classroom. In this account the teacher is situated with a particular view of pedagogy, predisposed to reading the classroom world of mathematics from a particular epistemological vantage point.

Students were asked to choose from a list of classroom activities the types of approaches associate teachers engaged in their classes. The three most commonly identified activities (seen by 82% of the respondents) were: teacher talk and exposition; children listening; and children doing worksheets. Students were also asked to apportion time to each teaching approach. Teacher talk and exposition with children listening consumed 29% of classroom time, whilst students spent 37% of classroom time working on prepared worksheets. Constituting around 50% of class time, cooperative and group activities were the most lengthy activities. Children using equipment also took up a high proportion (44%) of class time in those classes where observed. There was wide variation in the feedback which teachers give their pupils. 20% of the students reported that they never saw the teacher give any formal feedback to the children. Most commonly seen was an oral assessment or correction of an individual (seen by 58% of the students), followed by an oral assessment or correction given to the whole class (seen by 50% of the students). Teachers written comments provided more formal assessment in 43% of classrooms, with ticks and crosses from the teacher a common practice in 41%. Peer assessment or correction was seen in 28% of the classes. Children marked and corrected their own work from written answers in 32% of the classes.

Affirmative Practice and Conflicting Meanings

The preservice teachers in the study, like all teachers, have an understanding of the kind of place they would like their own classrooms to be: the rules and norms they would like to establish for their own conduct and that of the children; the placement of furniture and children's access to resources; and the type of 'feeling' evident in the classroom. In this section we look at making connections between the students' impressions of how teaching practice met their expectations. To avoid repetition only a selection of responses is given.

Well structured programme. Timed clearly. Well planned in advance.

[I was interested to see] the use of concrete materials to indicate how to solve the problem.

The programme was very well run and children progressed throughout the unit.

[I was interested to see] that concepts were put into real, relevant contexts and that children were able to experience these.

HANDS ON! Children did enjoy the practical activities.

I enjoyed teaching maths with the equipment.

Teacher integrated maths into the morning roll call, as children counted how many children were at school, how many boys/girls, the difference between the number of *boys/girls etc. Teacher always asked 'how did you work that out?' and got children to explain their working out.*

Maths was taught well, with enthusiasm and interest.

I found the school had great resources.

Math was very much made relevant and hands on for the children who experienced a lot of different activities, e.g., popcorn (mass/weight), cooking recipes, different food containers.

Group work went well as children are closer in ability.

Of interest was the way many children in my class supported each other in their work, or were willing to tutor each other.

The students' observations here reveal that the practices and positionings made possible for teachers through the university course cohere with actual classroom practice. The world of school mathematics which classroom practice offers, sits comfortably with personal meanings. Such alliances make for smooth partnerships for second-year students unsure of their own ability in producing effective classroom learning environments. 47% of all the students in the study voiced their uncertainties and concerns at the beginning of the practicum. A sense of their tentativeness can be gleaned from the following:

Worried about doing maths in the classroom.

Bit uncomfortable – Wanted more in-depth knowledge of how to get it across.

Nervous. Very nervous about teaching mathematics. Often find I get confused myself.

Scared of teaching maths!

A bit unsure.

Unfamiliar with routines.

I was familiar with the resources and activities but I had no idea how to run the maths lessons.

Knowledge about teaching mathematics is produced by social interaction and as such, is steeped with power and social positioning, and subject to negotiation, consent and circumstance. The working of power in the preservice/associate teacher relation is delicate and seemingly intangible, yet its networks can determine the very texture of teaching and its possibilities. More importantly, subtleties within the networks of power can shape and misshape loves of and passions for teaching mathematics, particularly those feeling vulnerable and lacking in confidence.

My associate was very well organised and supportive. She shared all her plans and resources with me. She provided quality feedback with positive ideas for me to improve on.

It was amazing to watch her in action. She definitely loves to teach and maths has a very high profile in her class.

What I am at pains to do here is *not* to suggest that there is any straightforward casual link between the determining structures (associate's practice) and the action of individuals (preservice teaching practice). Rather, teaching mathematics is *relational* for the preservice teacher. By that I mean that the preservice teacher is simultaneously inscribed within and refashions classroom existence in relation to others, engaging with, negotiating and contesting the cultural logics of the associate's practice.

It was a bit difficult to get my Associate to help me with my Maths planning. She LOVED Maths – it was really her thing. She motivated the kids and they all seemed quite happy about it – which was strange. I suppose she naturally assumed I had the same knowledge.

I was forced to follow her methods of teaching in maths as that is what she had planned and wanted maintained. I am confident in maths but was given little opportunity to express my confidence. Could not go outside the square.

In the teaching practicum, each move that the novice and associate teacher make in the relationship is situated within institutional arrangements that in many ways push them both to engage in particular discursive practices. Discursive practices in schools are powerful and it is precisely because they shape the construction of teacher's work that any attempt at practice which is not compliant must contend with a set of discursive impediments. In the above excerpts we engage with the observations of students in the process of trying to construct their teacher identity when the discursive practices in schools are at odds with their own personal theories. The identity papers the students had acquired through the course are challenged by and can scarcely be validated by the discursive practices upheld by the associate teacher. In those examples it is not possible to separate the preservice teachers' identity from the contradictory material and discursive practices which seek to regulate their pedagogical practice.

I have found that my strength in Maths increased with teaching it. The [course] had a lot more consequences with the way I viewed maths and viewed how children would receive my teaching. I now know that maths teaching requires many different approaches to find the answer to any problem. I basically had to follow my Associate's methods and then adjust them to my own approach.

Teaching mathematics in primary school comprises close and complex relationships of power and knowledge, in which the spoken and the unspoken becomes intricately linked both to the production of knowledge and to the subjectivity of teachers. This calls for an understanding of the subjectivity of the student teacher as itself continually reshaped by the changing designations students employ to understand themselves - those designations which significant others assign to them and the spaces they all share.

Conclusion

This paper has explored the question of what it means to engage in pedagogical work in New Zealand primary schools. Taking a different approach to the project of mapping preservice teaching practice in schools, it seeks to draw attention to a set of issues which commonly have remained outside the scope of standard epistemological analyses of teacher education. These taken-for-granted epistemologies have framed investigations linking classroom practice with content knowledge, and the deficiencies which these studies have reported have spawned new proposals and policies in teacher education. Within this conventional paradigm there is no place to consider the preservice teacher in any terms other than in a model of normality/pathology.

Drawing on the insights of poststructuralism, I have looked closely at discourse in action in the classroom for an examination and a critical reading of engagements of what is best named *teaching knowledge in production*. In this paper I have pointed to a theoretical and empirical direction for an analysis of mathematics teacher education which begins from a recognition of the politics of knowledge and their reciprocally constituted effects on subjectivity. In this theorising preservice teacher identity is fractured and fragmented, and the classroom is a site of political struggle over the real and its meanings. The concept of teacher identity then is best thought of as complex and multiple, developed in response to other identities which are sometimes held in opposition. Teaching experience then becomes much more than an issue of content knowledge and technical skills; it is, above all, a source of (micro) political engagement. Developing a sense of pedagogical and professional norms grows out of a history of response to local discursive classroom codes and wider educational discourses and practices, all of which interrupt, derail and elide the best intentions of the student teacher; all of which signal changes in the cognitive structures which constitute the preservice teacher's desire for the pedagogical.

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Arithmetic and Quasi-variables: A Year 2 Lesson to Introduce Algebra in the Early Years

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This paper investigates instruction that assists young children generalise their mathematical thinking in terms of quasi-variables (Fujii & Stephens, 2001). Sixty-five year 2 children participated in a lesson on investigating the patterns in multiples of 5 and 3. From the results it seems that young children are capable of understanding generalising and that specific features of instruction assist this process. The materials, the types of activities and the questions asked by the teacher all play an important role in assisting young children abstract underlying mathematical relationships.

Underpinning all higher levels of mathematics is the ability to operate algebraically (NCTM, 2000). However, in spite of the wealth of international and national research with high school and tertiary students, many students still exhibit a vast array of misconceptions in algebra (TIMSS, 1998) and algebraic thinking still persists as a barrier for participation in high levels of mathematics (NCTM, 2000). Current research is therefore beginning to turn to arithmetic as a key for access to algebra (Carpenter, & Levi, 2000; Carraher, Schliemann, & Brizuela, 2001; Warren & Cooper, 2001; Kaput & Blanton, 2001).

It is believed that the most pressing factor for algebraic reform is the ability of primary teachers to "algebrafy" arithmetic (Kaput & Blanton, 2001), that is, to develop in their students the arithmetic underpinnings of algebra (Warren & Cooper, 2001) and extend these to the beginnings of algebraic reasoning (Carpenter & Franke, 2001). As was argued by Carpenter and Levi (2000), the artificial separation of arithmetic and algebra "deprives children of powerful schemes of thinking in the early grades and makes it more difficult to learn algebra in the later years" (p. 1). The aim of the research reported in this paper is to begin to investigate instruction that helps children take the next steps in generalising and formalising their informal thinking into powerful mathematical ideas that support algebra, transforming arithmetic thinking into algebraic thinking.

Quasi-variable in the early years. Carpenter and Levi (2000) argued that young children could think about arithmetic in ways that not only enhance arithmetic but also provide a foundation for algebra. In particular, they contended that first and second grade children's knowledge of number represented a capability of making and justifying generalisations about the underlying structure and properties of arithmetic. For example, as well as intuitively knowing that 3+7 was the same as 7+3, Carpenter and Levi found that some of the young children showed they understood that this relationship held for all numbers. This ability of children to comprehend general properties of arithmetic in terms of number has been termed by Fuji and Stephens (2001) as "quasi-variable" understanding.

Fuji and Stephens (2001) defined quasi-variables as the use of numbers to indicate an underlying mathematical relationship that remains true no matter what the value of the numbers are. This definition evolved from another part of the study of Carpenter and Levi (2000) in which children were asked whether number sentences such as 78-49+49=78 were

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true or not true. Carpenter and Levi argued that children, who were able to generalise that such number sentences remained true no matter what the first number is as long as you take away and add the same number had an understanding of quasi-variable. Such quasi-variable understanding was evident in answers such as "I believe it's true because you took away the 49 and it's just like getting it back" (Carpenter and Levi, 2000, p 2).

Quasi-variable, therefore, appears to offer a way of expressing generality in arithmetic that can be extended to variable. Of course, care must be taken in interpreting the existence of quasi-variable in children's behaviour because, as Warren (2001) found, children have a tendency to overgeneralise many of the patterns they see in arithmetic, some believing that 2-3 is the same as 3-2.

The focus of this paper. This paper reports on the initial stages of a five-year longitudinal study investigating young children's development of algebraic reasoning. The study focuses on three components of this development: first, the arithmetic underpinnings of algebra (equals, operations and arithmetic properties - Boulton Lewis, Cooper, Atweh, Pillay & Wills, 1998); second, generalisation of arithmetic (leading to quasi-variable and variable understandings); and, third, initial ideas of function (in terms of pattern, relationship and operational change - Warren & Cooper, 2001).

In this paper, we describe a lesson given to three Year 2 classrooms. The lesson was designed to generalise a patterning activity based on multiples of numbers to 10, with the goal of evidencing quasi-variable understanding.

Method

In the first year of the longitudinal study, we are working with three Year 2 classes from an upper middle class state primary school from an inner city suburb of a major city. The sample, therefore, comprised 65 children from three year 2 classes, the three classroom teachers and 2 researchers. These children had only been in Year 2 for a few weeks. They knew about numbers to 10 and had experienced counting to 100, but had not been taught two-digit place value. They had also been introduced to addition and subtraction but not to multiplication.

We worked collaboratively with the teachers to trial teaching ideas and document student learning in relation to teaching actions. The lesson trialled in this paper was an initial activity designed to demonstrate how to generalise arithmetic activities.

Lesson on multiples of 5s and 3s. The aim of the lesson was to identify patterns in multiples of 5 and 3. To do this, the children were provided with a large 100s board, a collection of unifix cubes, small copies of the 100s board on paper, and a coloured pencil.

The lesson began with the large 100s board and the unifix. The children were asked to put their finger beside the 1 on the 100s board and count 5 (1, 2, 3, 4 & 5) and place a unifix cube on the 5. They were then directed to count on another 5 to 10 and place a second unifix cube on the 10. After this, they were directed to keep counting on by 5 and placing unifix until they saw a visual pattern in the placement of cubes. They could then use this cue to complete the pattern.

When their pattern had been checked, the children copied the unifix cube pattern onto a small copy of the 100s board using a coloured pencil. After this, there was a class discussion of particular characteristics that the children had noticed about the 5s pattern. These findings were recorded on the board and used in a class discussion to identify a pattern that would enable numbers to 100 to be in or out of the 5s pattern. Then, the children were asked if they could think of numbers larger than 100 that would be in the 5s pattern. The lesson concluded by repeating the above for 3 – making and discussing the 3s pattern.

Data gathering techniques and procedure. The lesson was taught to each class by one of the researchers. During the teaching phases, the other researcher and classroom teacher acted as participant observers. The lessons occurred sequentially, starting with Classroom 1, Lily's class and finishing with Classroom 3. In each instance the other researcher and classroom teacher recorded field notes of significant events including student-researcher/teacher interactions. The basis of rigour in participant observation is "the careful and conscious linking of the social process of engagement in the field with the technical aspects of data collection and decisions which that linking involves" (Ball, 1997, p. 311). Thus both observers acknowledged the interplay between them as classroom participants and their role in the research process. At the completion of the teaching phase, the researcher and teacher reflected on their field notes, endeavouring to minimise the distortions inherent in this form of data collection and to come to some common perspectives of the instruction that occurred and the thinking exhibited by the children participating in the classroom discussions.

Results

Classroom 1

Due to limited classroom materials, the children in classroom 1 undertook the activity in pairs. They shared a large 100s board and jointly counted and placed the unifix cubes. However, they each made their own copy of the pattern onto a small board. They appeared not to be used to working in pairs and this may have affected their performance.

The children appeared to be reasonably successful at counting and putting on cubes, and most quickly saw the following vertical lines that are the 5s pattern on a 100s board.

1	2	3	4	☐	6	7	8	9	☐
11	12	13	14	☐	16	17	18	19	☐
21	22	23	24	☐	26	27	28	29	☐
31	32	33	34	☐	36	37	38	39	☐
41	42	43	44	☐	46	47	48	49	☐

and so on

They had little trouble copying the pattern onto the paper. However, while some children coloured their pattern using one colour, many copied the pattern using two coloured pencils (a colour for the left-hand column and another colour for the right hand column). When the children were asked to describe the characteristics of the 5s pattern, there was a spread of responses. Many tended to see two columns with each column as a separate pattern, 5 at end of the numbers in the first column and 0 at end of the other. This could explain why some insisted on using two coloured pencils to colour their pattern on the paper cutouts. However, some children did see the 5s pattern as a whole pattern as exemplified by one child's response - "the 5s pattern has numbers ending with 5 or 0.

In order to check the children's understanding of the 5s pattern, they were asked to place their paper copies face down on their desks while the researcher/teacher proposed numbers for the children to indicate if they belonged or not belonged to the 5s pattern. The teacher/researcher began with numbers ending in 0. Most children agreed that numbers ending in 0s belonged to the 5s pattern. The discussion then moved to numbers ending in 5, and finally, to numbers not in the patterns (e.g., 46, 72). To ascertain the children's ability to generalise the pattern, they were asked to give examples of numbers they believed were in the 5s patterns. Most successfully participated in this phase of the lesson.

In order to establish if they could 'see' the pattern in general terms, the children were asked to provide examples of the largest number they felt were in the 5s pattern ("tell us the largest number you can think of that would be in the 5s pattern?"). No one initially gave numbers above 100, so they were specifically asked if there were any numbers greater than 100 in the 5s pattern. This produced the desired result but the numbers were not that large; most gave examples such as 105, 1 005, 1 010. With encouragement, one child gave 1 000 005.

The children were then asked to construct the 3s pattern. They were told to do the same as they did for the 5s pattern but to count in threes until they saw the pattern. They were told to be careful but most children only counted for the first row and then immediately built vertical columns, giving the visual pattern as follows:

1	2	☐	4	5	☐	7	8	☐	10
11	12	☐	14	15	☐	17	18	☐	20
21	22	☐	24	25	☐	27	28	☐	30
31	32	☐	34	35	☐	37	38	☐	40

and so on

The researcher/teacher had to continually reinforce the need to keep counting to get most children to where they had the diagonal pattern as follows:

1	2	☐	4	5	☐	7	8	☐	10
11	☐	13	14	☐	16	17	☐	19	20
☐	22	23	☐	25	26	☐	28	29	☐
31	32	☐	34	35	☐	37	38	☐	40
41	☐	43	44	☐	46	47	☐	49	59

and so on

It seemed that for some, the visual of the 5s pattern was so strong that even as they were making the diagonal pattern they tended to regress to a vertical pattern, with their patterns tending to have 'dogleg' shapes.

1	2	☐	4	5	☐	7	8	☐	10
11	☐	13	14	☐	16	17	☐	19	20
☐	22	23	☐	25	26	☐	28	29	☐
☐	32	33	☐	35	36	☐	38	☐	40

and so on

The lesson ended with a very brief discussion about what numbers were in the 3's pattern, but no characteristics other than "goes up by 3" and "it's a diagonal" were identified.

Classroom 2

Due to more material being available, each child was given his/her own material and worked alone. Of the three classes, this class appeared to have a greater proportion of low achieving children. From the observations it seemed that overall, this group needed the most assistance in counting, in seeing patterns and in copying patterns.

Compared to the children from classroom 1, the children in this class tended to be slower but more accurate at placing the cubes on the board for the 5s pattern. They also seemed to identify the visual pattern at an earlier stage. However, when they had copied the pattern to the smaller paper board and were asked what they saw, they could not identify any characteristics of the 5s pattern other than it "went up in 5s".

In order to assist them to ascertain the pattern, the researcher/teacher teased out the discussion as follows. First, the children were asked to look at one column at a time, starting with the zero column. They were then instructed to run their fingers down this column and to read out the numbers aloud. However, many children still seemed unable to see the pattern. The process of running a finger down the column and stating numbers aloud was repeated with the focus being placed on the right hand digit (the ones). At this point, most were able to express the pattern as ending in 0s.

In order to test their ability to generalise this component of the pattern, the children were asked to turn their copies of the pattern over while the researcher/teacher proposed 40, 70 and 30 for them to decide if they felt the numbers belonged in the 5s patterns. Nearly all children agreed that they did. They then checked their responses by turning the board back over so that they could see if the numbers were coloured on their paper copy of the 100s board. The same steps were repeated for the second column. Again this process was successful in getting most students to identify the pattern.

After this, the children were asked if they could think of a rule for the 5s pattern with their answers incorporating both columns. The aim of this was to get them to combine the two conclusions above and say that if a number had 0 or 5 for the right digit then it was in the 5s pattern. Only a few children appeared to be able to articulate this response. The combined pattern was reinforced by again asking the children to turn over their paper copy and ascertain if a new set of numbers proposed by the researcher/teacher were in or out of the5s pattern.

In proposing these numbers, it became evident that in order to assist young children generalise the 5s pattern, instruction needed to start with numbers such as 60, 30, 20 and then 35, 65, 75. Care had to be taken with 50 and 55 and, with numbers like 54 where the 5 was at the left hand side rather than the right hand side. It also became apparent when they were asked to identify whether 23 or 32 belonged to the pattern that some children added the numbers and thus believed they did belong as the sum was 5.

When asked for larger numbers, this class mainly gave numbers less than 100 with the exception of a the few examples of larger numbers which consistently had many 5s (e.g., 555).

Compared with classroom 1, the difficulties the classroom 2 children had with the 5s pattern task was mirrored in their inability to generate the 3's pattern. Even after they were told that the 3's pattern was not a vertical pattern, they tended to return to vertical placement of cubes as soon as they were left on their own. Most children followed this sequence:

(1) They counted two numbers and then went vertical, as follows:

1 2 ☐ 4 5 ☐ 7 8 9 10
11 12 ☐ 14 15 ☐ 17 18 19 20
21 22 ☐ 24 25 ☐ 27 28 29 30
and so on

(2) When told to count more, they placed one more unifix and then went

vertical again, as follows:

1 2 ☐ 4 5 ☐ 7 8 ☐ 10
11 12 ☐ 14 15 ☐ 17 18 ☐ 20
21 22 ☐ 24 25 ☐ 27 28 ☐ 30
and so on

(3) When told the vertical was wrong and they had to continue counting, they tended only to do one more row and revert to vertical, as follows:

1 2 ☐ 4 5 ☐ 7 8 ☐ 10
11 ☐ 13 14 ☐ 16 17 ☐ 19 20
21 ☐ 23 24 ☐ 26 27 ☐ 29 30
31 ☐ 33 34 ☐ 36 37 ☐ 39 40
and so on

They only began to see the diagonal pattern with continual reinforcement to "keep counting". The time taken for this meant that there was no time to discuss the pattern.

Classroom 3

This class was a composite class comprising of Year 1 and Year 2 children. We worked with the Year 2 component, consisting of 14 children. They seemed much more able than the preceding two classes. This was confirmed in a later discussion with the classroom teacher.

The children quickly identified the visual pattern in the 5s. They proffered many ideas with respect to the characteristics of the pattern, for example, "2 columns", "5s at end of the first one" and "0s at end of other". They also proposed other patterns for example, "tens going up by 1 as zeros and 5s stay the same". Because of their lack of the place value vocabulary, they tended to talk about the "right hand number" going up in 1's while the left hand number stayed as 0 or 5. At the conclusion to the discussion, the pattern was summarised by more than one student as "any number with 0 or 5 at end".

When asked for larger numbers, the children gave a huge variety of numbers, including unusual ones (e.g., 765) and ones in the millions (e.g., 2 000 0005) and billions ("a billion and 5"). Finally, one child proposed "infinity five", which was immediately followed by another child's proposal "two infinity five".

The classroom 3 children also completed the 3s pattern much more easily than the previous two classrooms; they exhibited few difficulties with pattern completion. Whilst most children tended to move quickly to the pattern as diagonal after one hint to keep counting in threes, a minority exhibited the dog-leg error as follows:

1 2 ☐ 4 5 ☐ 7 8 ☐ 10
11 ☐ 13 14 ☐ 16 17 ☐ 19 20
☐ 22 23 ☐ 25 26 ☐ 28 29 ☐
31 32 ☐ 34 35 ☐ 37 38 ☐ 40
41 ☐ 43 44 ☐ 46 47 ☐ 49 59
51 ☐ 53 54 ☐ 56 ☐ 58 59 ☐
and so on

For some children, the 100s board had to be turned through 45 degrees so that the lines of unifix were vertical before they were comfortable.

In discussing the characteristics of the 3s pattern, there was an interesting collection of proposals. Some children stated that there were 2 numbers between members of 3s pattern, both vertically and horizontally, other children saw the pattern as "going back one" and spoke in terms of changes in the left-hand and right-hand digits (e.g., "for each number in a diagonal, the tens go up by 1 and the ones go down by 1"). One student even proposed the very complex expert pattern, that the digits added to 3, 6 and 9.

Discussion and Conclusions

The role of quasi-variable. The aim of this research was to see if Year 2 children were able to generalise the 5s and 3s patterns and to ascertain if their generalisations was in quasi-variable form. We believe that an understanding which can articulate that any number ending in a 0 or 5 is in the 5s pattern represents an example of using quasi-variables to indicate the underlying mathematical structure of the pattern. However, not all children achieved this level as the responses given by the children within and between the three classes varied. Children in classroom 1 mostly gave examples that contained 10s or 5s in the response (e.g., 105, 1005, 1010 and 1 000 005), whereas the responses in classroom 2 all contained 5s (e.g., 555, 55555). By contrast, Classroom 3 gave a wide variety of examples, clearly indicating an understanding that the most important feature of the 5s pattern is the number in the ones place. We contend that quasi-variable understanding was exhibited in Classroom 3 and not in Classrooms 1 and 2 because even though children in Classrooms 1 & 2 articulated that the 5s pattern has numbers ending in 5 and 0 and the examples they gave of numbers belonging to the 5s pattern were limited. We argue that expressing a generalisation in everyday language is not enough to show quasi-variable understanding; children need to then give evidence of this understanding by offering a wide array of examples, including large numbers beyond the domain that they are exploring. Thus responses such as ∞5 and 2∞5 are crucial for evidence of 'real' generalisation. What is the role of the teacher in assisting children to see generalisations and how much spontaneity is necessary for a child to exhibit quasi-variable understanding? Both these questions need further research.

Classroom instruction. This research has began to delineate some important aspects of instruction that appear to assist children to generalise and formalise their mathematical thinking. First, the overall instruction consisted of a number of phases. The children were ask to complete an activity, reflect on the patterns in the activity and then check their reflections by considering and then posing examples. Then, their thinking was challenged by asking them to give examples beyond the domain of numbers under consideration. This extension to the larger numbers seemed to offer opportunities for insightful discussions about the generalisation. It is conjectured that this discussion offered opportunities for understanding of the quasi-variable to emerge. Second, the detailed instruction in Classroom 2 appeared to assist children to begin to see the 5s pattern. While the instruction did not culminate in generalising the pattern to the extent hoped, the series of small steps helped them to focus on the main features of the task. We contend that such instruction could be classified as 'getting ready' for quasi-variables, teaching strategies that focus children's thinking on the salient features of mathematical structure of the problem.

Third, the integration of visual, auditory and kinaesthetic activities seemed effective. Stating the numbers in the pattern aloud assisted many to focus on the numbers in the pattern rather than on all the numbers on the 100s board. Fourth, breaking the pattern into parts also assisted children to see the pattern as a whole. This was especially effective in

Classroom 2. It was only when they explored the 2 columns separately that they appeared to understand the aim of the activity. Fifth, continual reinforcement such as asking children to turn over their boards and then ascertain whether numbers were in or out of the pattern by putting their hands up seemed to support understanding. The discussion that ensued from questions such as "why is the number in the pattern?" or "why is the number not in the pattern?" also appeared to play an important role. This was especially evident when the classroom discussion focused on whether numbers such as 23 and 32 were examples of the pattern: "If the numbers add to 5 does that mean they are in the pattern? Why? Why not?"

Thinking that supports understanding of generalisation. From these preliminary results, we believe that young children are able to begin to comprehend the structure of mathematics. The use of quasi-variable certainly seems to assist in this comprehension by providing a way for young children to articulate the generalisation. Of course, the performance of children varied; some of the children were quite amazing in the patterns that they could see while others required specific instruction and reinforcement to assist them. As well, the performance of the children posed questions about the nature of generalisation. By its definition, the quasi-variable is numerical, but this activity also contained a strong visual component. The relationship between these two components needs further investigation. How does the visual pattern support or mitigate against the articulation of the numerical pattern? It seems that the visual pattern certainly did not help in some instances, especially with the transition from the 5s pattern to the 3s pattern. Many children experienced little difficulty in integrating the visual with the numerical. In fact the visual seemed to assist them in quickly identifying the pattern and expressing the pattern in a numerical form. However, the vertical pattern of the 5s seemed to make the diagonal pattern of the 3s difficult to see. In fact, in order to complete the diagonal pattern of the 3s some children needed to turn their board through 45^0, thus converting the pattern to the vertical. The role of kineasethic reinforcement also needs further investigation. In this instance running their fingers down the numbers and saying the numbers seemed to assist in breaching the gap between the visual and number pattern.

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Variation as Part of Chance and Data in Grades 7 and 9

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As part of a larger project studying school students' understanding of statistical variation, 92 students in grade 7 and 90 students in grade 9 participated in a unit of work related to the chance and data curriculum, emphasising variation with respect to the topics covered. Students completed pre- and post-tests devised to assess understanding before and after the lessons in the unit. This paper reports on the teaching arrangements for the classes taking part and the change in performance after the unit.

It appears to have taken a long time for educators to catch up with statisticians in recognising the importance of variation within the realms of statistical enquiry. As statistics was emerging as part of the school mathematics curriculum (National Council of Teachers of Mathematics [NCTM], 1989; Australian Education Council [AEC], 1991), it was David Moore (1990) who stressed the omnipresence of variability. Unfortunately explicit mention in relation to variation in the Australian *National Statement* was limited to "measures of spread" (e.g., AEC, p. 178). Although followed by calls from Green (1993) and Shaughnessy (1997) for research into students' understanding of variability, there is still little reported research on school students' understanding and learning of the topic. Following the initial research of Shaughnessy, Watson, Moritz, and Reading (1999), based on drawing lollies from a container of 100 with 50 red, Reading and Shaughnessy (2000) and Torok and Watson (2000) suggested developmental sequences for the understanding of variation throughout the years of schooling.

The data reported here were collected as part of a larger follow-up study after the work of Shaughnessy et al. (1999), in particular to document change, if it occurred, resulting from a planned classroom intervention program focussing on variation within chance and data. Students in grades 3, 5, 7, and 9 participated, with different teaching plans employed at the primary (grades 3 and 5) and secondary (grades 7 and 9) levels. This paper will report on interventions planned and outcomes achieved at the secondary level.

Instructional Design

Plans for teaching units for the chance and data curriculum were devised based on an analysis of the content as suggested by organisations such as the NCTM (1989) and the AEC (1991), and respected statistics educators such as Holmes (1980) and Moore (1990). Taking into account the process of the statistical analysis – data collection and sampling, data representation, data reduction, inference, and probability – aspects of variation associated with each were identified for focus lessons.

Although for the primary grades the teacher was provided from the project team and all lessons were nearly identical in content and length, at the secondary level it was necessary to fit in with the schools' timetables and rely on the usual mathematics teachers to deliver the units of work. In light of this an extensive package was prepared covering six possible units of work emphasising variation within the topics covered in the chance and data curriculum. Some units were based upon published materials, whereas others were designed by the first author. They included variation with spinners, with dice, in sampling,

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with associations, in comparing two groups, and in numbers of chocolate chips in cookies. Each unit could be adapted for between one and three lessons. The basic topics were chosen from those in the chance and data curriculum for grade 7 or 9. The unit on variation in associations, for example, dealt with the relationship of hand span to foot length, and how variation in the data affects the conclusions reached. Teachers received 21 pages of plans and 33 pages of associated documents (e.g., work sheets, copies from relevant books).

The authors met with the teachers involved in the project to explain the purpose and distribute the pages of plans and associated documents for the units. Suggestions were made as to the order in which units might be taught but the decisions on which to choose were left to the professional judgement of the teachers. It was realised that the units might take more time to cover comprehensively than was available in the teachers' programs. None of the teachers, however, had covered chance and data yet for the year in question.

As it turned out, the greatest variation to occur in the project overall was the variation in the amount of time spent on chance and data by the secondary teachers. Teachers could not be held to an early commitment that would have made the number of lessons in the secondary schools the same as in the primary schools, or even the same as each other. It was therefore necessary to include another variable in the study: "class".

Methodology

Sample. The students who participated in the project were from two high schools, one suburban and one semi-rural in the Australian state of Tasmania. In one school, two grade 7 classes ($n_1 = 16$ and $n_2 = 21$) were taught by the same teacher, and two grade 9 classes ($n_3 = 11$ and $n_4 = 15$) were taught by different teachers. In the other school, all classes were taught by different teachers, three in grade 7 ($n_5 = 14$, $n_6 = 14$, and $n_7 = 27$) and three in grade 9 ($n_8 = 31$, $n_9 = 16$, and $n_{10} = 17$). Not all students completed both pre- and post-tests and the sample sizes reported here reflect the number of students with complete data, which may be a smaller number than the number present when teaching occurred. In grade 7 there were 34 girls and 58 boys and in grade 9 there were 39 girls and 51 boys.

Survey instruments. The survey instruments for the two grades were the same except for one additional item used with grade 9 students. All items are presented in Watson, Kelly, Callingham, and Shaughnessy (2001) and were clustered into four subscales measuring understanding of Basic Chance and Data (BCD), Chance Variation (CV), Data Variation (DV), and Sampling Variation (SV). Items in the BCD subscale included evaluating probabilities associated with random generators and drawing objects from boxes, and reading information from a pictograph and a two-way table. Items on CV asked for predictions of 60 outcomes of tossing a die, predictions of repeated trials with a 50-50 spinner, and a definition of the term random. DV questions involved inferences from graphs of different visual appearance, identifying false or unusual data from different graphs, defining variation, and calculating the average from a data set containing an outlier. Items on SV were based on a classroom scenario where a survey is framed and a list of possible sampling methods presented for evaluation, the definition of a sample, two fair ways of selecting a sample, and a news article stating evidence based on a biased sample. It was hence possible to measure change after the teaching, on the four subscales as well as the entire test. The post-test in each grade was identical to the pre-test. The tests were administered by the teachers at convenient class times before and after instruction within time windows requested by the research team.

Analysis. Responses were entered into a spreadsheet and after agreement of the authors on expected levels of response, coding was carried out by the second author. A clustering of similar answers took place on the spreadsheet and both authors checked the consistency of coding, with discrepancies resolved by discussion. Paired *t*-tests were performed for each class and for each grade overall, for the four subscales and the total of all items. Within each grade, a one-way ANOVA was performed for "post-test score – pre-test score", with class as a factor. The gender imbalance of the two grades (37% girls in grade 7 and 43% girls in grade 9) suggested a further comparison of scores by gender.

Results

The results will be discussed in four parts: outcomes for individual classes and grades, differences across classes, examples of responses to particular items, and gender differences. The pre- and post-test means, as well as standard errors of the differences, for each of the five grade 7 classes on the four subscales and the total scale are given in Table 1. Similar results for the five grade 9 classes are given in Table 2. The total possible scores for each of the common subscales for the two grades were 22 for BCD, 19 for CV, and 32 for DV. For SV, in grade 7 the total was 30 and in grade 9 it was 33. The Total score in grade 7 was 100 and in grade 9 it was 103. The item asking for a definition and example of "variation" was used in both the CV and DV subscales, but only counted once in the Total.

Table 1

Pre- and Post-test Means and Standard Errors of the Differences for Grade 7

Class	7A	7B	7C	7D	7E	Total
Scale	(n = 16)	(n = 21)	(n = 14)	(n = 14)	(n = 27)	(n = 92)
BCD	15.3 / 17.1 (0.70)*	12.9 / 17.3 (0.86)***	10.9 / 12.7 (0.65)**	11.4 / 9.6 (1.34)	12.6 / 16.4 (0.73)***	12.7 / 15.1 (0.44)***
CV	9.1 / 11.1 (0.73)**	8.4 / 12.2 (0.39)***	6.3 / 8.8 (0.42)***	7.0 / 6.9 (0.96)	7.6 / 10.2 (0.61)***	7.8 / 10.1 (0.31)***
DV	11.9 / 15.4 (1.31)**	11.8 / 17.1 (0.90)***	9.1 / 12.9 (1.06)**	10.1 / 8.1 (0.99)*	9.9 / 15.1 (0.72)***	10.6 / 14.2 (0.50)***
SV	9.6 / 14.7 (1.09)***	10.0 / 16.1 (1.09)***	4.6 / 7.7 (1.12)**	7.5 / 3.7 (1.26)**	7.1 / 11.5 (1.15)***	7.9 / 11.3 (0.62)***
Total	44.9 / 56.7 (2.80)***	42.3 / 61.3 (1.96)***	30.5 / 41.7 (1.42)***	35.8 / 28.3 (2.63)**	36.5 / 51.9 (2.06)***	38.3 / 49.7 (1.33)***

*$p < 0.05$. **$p < 0.01$. ***$p < 0.001$.

Outcomes for individual classes and grades. Paired *t*-tests for the combined samples for each of grade 7 and grade 9 indicated that for each grade, there was a significant improvement in the Total score ($p < 0.001$). For the classes in this study, the pre-test mean for grade 7 was significantly less ($p < 0.01$) than that for grade 9 but improvement was greater resulting in a post-test mean for grade 7 very close to that for grade 9. For the five grade 7 classes, differences represented significant improvement for four classes ($p < 0.001$) and a diminished performance for one class ($p < 0.05$). For grade 9, improvement occurred for four classes (two at the 0.001-level, one at the 0.05-level, and one non-significantly), whereas one class showed a diminished performance ($p < 0.001$). Classes 7D and 9I showed a decrease in performance in nearly all subscales, as well as overall.

Table 2

Pre- and Post-test Means and Standard Errors of the Differences for Grade 9

Class Scale	9F (n = 11)	9G (n = 15)	9H (n = 31)	9I (n = 16)	9J (n = 17)	Total (n = 90)
BCD	14.5 / 14.6 (0.92)	9.3 / 15.3 (1.10)***	14.3 / 17.2 (0.78)***	14.2 / 10.8 (1.21)**	11.1 / 11.1 (1.12)	12.9 / 14.3 (0.55)**
CV	10.5 / 10.5 (0.86)	7.2 / 11.1 (0.85)***	9.4 / 11.7 (0.63)***	9.2 / 6.9 (0.96)*	7.1 / 8.8 (0.70)*	8.7 / 10.1 (0.40)***
DV	13.8 / 15.9 (1.34)	7.7 / 14.6 (0.98)***	14.4 / 17.4 (0.68)***	12.3 / 8.4 (1.20)**	7.6 / 9.8 (1.13)*	11.6 / 13.7 (0.55)***
SV	14.0 / 16.7 (1.10)*	6.5 / 14.4 (1.59)***	13.5 / 17.3 (1.05)***	12.6 / 4.9 (1.24)***	8.5 / 10.2 (1.78)	11.3 / 13.2 (0.79)**
Total	51.0 / 56.3 (2.42)*	30.1 / 54.1 (2.72)***	50.8 / 62.1 (2.04)***	47.1 / 30.9 (3.19)***	34.1 / 39.4 (3.45)	43.6 / 50.2 (1.78)***

*$p < 0.05$. **$p < 0.01$. ***$p < 0.001$.

In each grade the BCD scale post-test mean was the greatest of the subscales in relation to the total possible score (69% for grade 7 and 65% for grade 9). All other subscale post-means were similar relative to the totals possible and 53% of the total or less.

For all four subscales and the Total, the average improvement for grade 9 on common items was less than for grade 7. Although in each case the grade 9 pre-test average was higher, in two cases the grade 7 post-test average was slightly higher. With the highest class average 62.1 out of 103, it is unlikely a ceiling effect was operating for the grade 9 classes. The greatest variation in subscale scores occurred for the SV subscale for both grades.

Differences across classes. Although it is clear from the tables that the improvement in Total scores varies greatly between classes at each grade, one-way ANOVAs on "post-test score – pre-test score" confirms that differences are significant at each grade level ($p < 0.001$). The "class" variable hence was associated with significant differences in "post-test – pre-test" change within the two grades.

Based on the journals provided by the teachers, estimates were made for the number of lessons taught in each of the classes. Table 3 summarises the "post – pre" differences in Total means for the classes and the number of lessons taught, ordered by the number of lessons taught. It is clear that the association of the number of lessons taught and change in Total mean score from pre- to post-test is not linear, or even particularly strong. With $r = 0.428$ and 18.3% of variance explained, the correlation is not significant ($p = 0.10$). The conclusion is reached that other factors besides the number of lessons taught are associated with the differences in "post – pre" Total means.

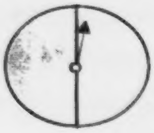
Examples of responses to particular items. Paired t-tests for each item by grade level revealed many items in which the grade 7 students improved significantly from the pre- to post-test for which the grade 9 students did not. On particular items, like the first three parts of the Spinner item in Figure 1, the grade 9 students' mean pre-test score was higher than the grade 7 mean pre-test score. This indicates that there was more room for improvement for the grade 7 students. On other items, like Part 4 of the Spinner item in Figure 1, grade 7 students had a slightly higher pre-test mean and improved more than the grade 9 students.

Table 3

Number of Lessons Taught per Class with Post – Pre Difference Means

Class	7C	9H	7D	9I	9F	7A	7B	9G	7E	9J
Number of Lessons	3	3	5	5	6	9	9	12	14	14
Post – Pre Difference Mean	10.9	11.1	-6.3	-15.5	5.2	10.5	19.1	23.8	14.8	5.2

A class used this spinner.



1. Out of 50 spins, how many times do you think the spinner will land on the shaded part? Why do you think this?
2. If you were to spin it 50 times again, would you expect to get the same number out of 50 to land on the shaded part next time? Why do you think this?
3. How many times out of 50 spins, landing on the shaded part would surprise you?
4. Suppose that you were to do 6 sets of 50 spins. Write a list that would describe what might happen for the number of times the spinner would land on the shaded part?

______, ______, ______, ______, ______, ______

Figure 1. Spinner item used on pre-and post-test.

Figure 2 shows several responses from grades 7 and 9 to the four parts of the Spinner task in Figure 1. The ranges of codes for these items were 0-3, 0-3, 0-1, and 0-2, respectively, and the pre- and post-means are given for each grade. Responses are typical of the coding levels that contributed to the means and to some degree to the improvement or lack of it in the post-test scores. To obtain the highest code (3) on Parts 1 and 2 some notion of variability or uncertainty was required. Students who acknowledged strict probability with no variation received a code of 2, whereas students who gave imaginative answers were given a code of 1. For Part 4, criteria were set for the variation displayed. Responses that did not involve numbers or were outside the 0-50 range were coded 0, those with no variation or too much variation were coded 1, and those with reasonable variation were coded 2. It was difficult for many students to suggest reasonable variation.

Gender differences. For grade 7, although the pre-test average for girls was higher on four of the five scales ($p < 0.03$), the average "post – pre" score for girls was not significantly different from the boys on any scale. The mean improvement for girls was 10.8 and for boys it was 11.8. Only for the DV scale did the girls continue to have a higher average score than boys on the post-test ($p < 0.04$). In most cases, for subscale scores and differences, the variation displayed in boys' scores was greater than in the girls' scores.

The relationship of scores for girls and boys in grade 9 was somewhat different from grade 7. The girls' average pre-test scores were significantly higher on only two subscales ($p < 0.05$) but were significantly higher on all post-test scores ($p < 0.03$ to $p < 0.001$). Although the "post – pre" score average was higher for girls on each scale, the difference was significant on only the BCD scale ($p < 0.01$) where the boys' average did not change. Again the variances in boys' difference scores were generally greater than for girls. Despite indications of better performance in various subscales on the pre- or post-tests by girls, overall the improvement was not significantly different in either grade.

1. Out of 50 spins, how many times do you think the spinner will land on the shaded part? Why?			
Grade 7 pre-mean = 1.91, post-mean = 2.16		Grade 9 pre-mean = 2.00, post-mean = 1.96	
PRE: 40, Because the arrow needs energy to come up [code 1]	POST: 25, Because the chance to land on the shaded and white part are the same [code 2]	PRE: 25, Because there is an equal chance [code 2]	POST: 25, Because there's equal chances of it landing on each part [code 2]
PRE: 29, Because that one's better [code 1]	POST: 20-25, there's only 2 parts and there's still a 50/50 chance [code 3]	PRE: 23, Because there is a 50-50 chance but some sides get hit a bit more [code 3]	POST: 19, Because I like white [code 1]
2. If you were to spin it 50 times again, would you expect to get the same number out of 50 to land on the shaded part next time? Why do you think this?			
Grade 7 pre-mean = 1.42, post-mean = 1.70		Grade 9 pre-mean = 1.67, post-mean = 1.90	
PRE: No, because it might have been pushed harder … it might land a couple of numbers different [code 1]	POST: No, because the force of your spin might change the side it goes on [code 1]	PRE: Probably not, because the saying goes – lightening only strikes once [code 1]	POST: Probably not, there are many different things that could occur, like how hard you spin it … [code 1]
PRE: No, … you might spin it harder or softer at any time [code 1]	POST: No, it might be but it would be a tiny difference [code 3]	PRE: No, because it matters how hard you spin it [code 1]	POST: No, because it's all up to chance [code 2]
3. How many times out of 50 spins, landing on the shaded part would surprise you?			
Grade 7 pre-mean = 0.65, post-mean = 0.84		Grade 9 pre-mean = 0.77, post-mean = 0.83	
PRE: Half [code 0]	POST: 50 [code 1]	PRE: Luck [code 0]	POST: None [code 1]
PRE: All of them [code 1]	POST: 50 [code 1]	PRE: Below 10 or above 45 [code 1]	POST: Anything above 40; 50 would really surprise me [code 1]
4. Suppose you were to do 6 sets of 50 spins. Write a list that would describe what might happen for the number of times the spinner would land on the shaded part?			
Grade 7 pre-mean = 1.04, post-mean = 1.43		Grade 9 pre-mean = 0.92, post-mean = 1.22	
PRE: 25, 25, 25, 25, 25, 25 [code 1]	POST: 25, 25, 25, 25, 25, 25 [code 1]	PRE: ?, ?, ?, ?, ?, ? [code 0]	POST: 25, 15, 30, 39, 28, 42 [code 1]
PRE: Shade, shade, black, shade, shade, black [code 0]	POST: 23, 21, 22, 26, 20, 24 [code 2]	PRE: 25, 25, 25, 25, 25, 25 [code 1]	POST: 45, 10, 5, 20, 50, 0 [code 1]
PRE: 24, 26, 25, 24, 23, 24 [code 1]	POST: 27, 23, 25, 26, 22, 24 [code 2]	PRE: 24, 25, 25, 26, 25, 25 [code 1]	POST: 25, 28, 30, 24, 20, 18 [code 2]

Figure 2. Typical pre-and post-test responses to the Spinner item in Figure 1.

Discussion

In terms of overall performance of students, the significant gains give support to the view that it is possible to teach chance and data with an emphasis on variation and expect positive results. The fact that the post-mean was a higher percentage in relation to possible score for the BCD subscale compared to the others, could be explained in several ways. It may be that BCD ideas, being prerequisites for dealing with variation, were covered more thoroughly in previous years, or indeed by the teachers of the classes in this study. It could also be that, having less variable or uncertain answers, BCD questions are easier for students who are used to unique answers in their study of mathematics. The other three subscales, however, all showed significant improvement overall for each grade, indicating that progress was made on integrating variation concepts with topics in chance and data.

The setting for the study in two schools that represented typical state school learning environments introduced several factors deserving attention in considering the outcomes. These will be considered in turn, in relation to the grades, classes, and number of lessons taught; optimal possible performance; potential gender differences; and teacher participation. The differences in pre-test averages for the grade 7 classes reflect school differences in that no streaming took place in either school at this grade. Differences in pre-test averages for the grade 9 classes do reflect streaming of students in the two schools, with classes 9F, 9G, and 9J considered to cater for lower ability students. This study was interested in change relative to starting point and not the starting point per se.

The number of lessons taught was in itself not an accurate predictor of the average improvement in test scores. Teacher 7E achieved excellent results from 14 lessons, whereas teacher 9J achieved positive but non-significant improvement from the same number of lessons. Noting, however, that the pre-test average for class 9J was lower than that for class 7E, it might be speculated that the concepts were more difficult for the students in class 9J and hence were not retained as well. It could also be that teacher 9J had to spend more time on ideas and that less progress was made overall. The fact that two classes (7D and 9I) had lower average scores in the post-test than in the pre-test is likely to reflect student attitude to completing the post-test, rather than an actual decrease in understanding. Students may have felt it unnecessary to provide details that they had earlier, or they may have had a negative reaction to being asked to complete the test again. The range of scores on the post-test Total was 1 to 81. This is yet another indication that students may not have taken the task seriously. The second post-test comment on Part 1 of the Spinner item from a grade 9 student ("I like white," see Figure 2), in the light of the high level pre-test comment, may in fact be intended to be facetious. If this happened often, it could have contributed to the lower post-test means for some classes.

Classes 7A and 7B with similar pre-test means, were taught virtually the same lessons, illustrating the variation in change for two similar conditions. The teacher reported anecdotally that class 7B was an eager class, willing to learn, whereas 7A was a struggle to teach. This appears to be reflected in the improvement shown.

Although significant improvements occurred for many classes, for many items, and for most subscales, performance on the post-test was still not optimal in terms of the expectations of the curriculum. In comparing change in grade 7 to change in grade 9 the question of a ceiling effect for grade 9 may arise. Although for some items (e.g., Part 3 of the Spinner item) the grade 9 pre-test mean was significantly higher than the grade 7 pre-test mean, it is difficult to claim that a ceiling effect was occurring for grade 9 students; in fact many times the mean for grade 9 did not reach 2/3 of the highest possible score for the item. If a ceiling effect operated, it reflected, on average, a less than optimal performance as judged by the authors in light of the desired outcomes.

Although the girls performed better on all pre- and post-scales, they did not appear to gain more from the lessons as judged by the "post – pre" difference in scores. In fact the boys' improvement was marginally better in grade 7. Hence it cannot be said that the lessons favoured one gender of student as gauged by this measure.

Finally, the teaching arrangements made in this study have an affinity with those of Helme and Stacey (2000) who provided resources to four teachers in a primary school with the aim of improving student understanding of decimals. Despite expressing willingness to be involved, two teachers did not use the resources at all, one teacher used them "at least once", and the fourth teacher used seven of the activities. Helme and Stacey found that student improvement was strongly related to the teachers' use of the resources, a result more clear-cut than in the current study. It does confirm, however, the difficulty of holding teachers to commitments made before studies begin. Conducting research in actual school environments is fraught with such difficulties unless the project provides the teachers (as was done elsewhere in the larger project of which this study was a part). This is not to say that research such as the current work with grade 7 and 9 teachers or Helme and Stacey's project should not take place. The actual "real-world" school setting is where curriculum implementation will take place after research is finished and it is useful to have an expectation of how successful it is likely to be.

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Research into Teacher Beliefs: Can the Past Stop Endless Repetition?

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Research into teacher beliefs has a long and troubled history and there is evidence that it is ignoring the lessons of the past. Why do researchers need to continually report the mismatch between espoused beliefs and enacted beliefs? Behaviour is driven by action theories that are different from espoused theories and both are difficult to unravel, but attempts have been made with varying degrees of success. What can history tell us about the definition of belief that focuses upon a global statement or upon a specific behavioural belief statement? Is it still useful to attempt to place teachers upon a continuum based upon their espoused beliefs? Research into the planning, thinking and decision making of teachers should provide insight into classroom events and be worth understanding because of the connection with student learning and performance. This paper discusses these issues and their implications for teacher education.

Research into teacher beliefs has a long and troubled history and there is evidence that it is ignoring the lessons of the past. There is a growing body of literature that claims mathematics teacher beliefs affect teacher classroom practice although the nature of the relationship is highly complex and dialectical (Pajares, 1992; Thompson, 1992). Mathematics teacher beliefs can usually be thought of as individual perspectives of engagement with mathematical tasks and pedagogical practices (Schoenfeld, 1985). Studies of teacher beliefs suggest that there is a causal relationship with action although it is difficult to uncover and explain. So while some studies strongly suggest that teacher beliefs influence instructional behaviour, other cases suggest that instructional practice influences teacher beliefs (Buzeika, 1996; McGalliard, 1983). There is a long tradition of research that has reported mismatches between teachers' theories and behaviour (Thompson, 1992; Cooney, 1985, Shield, 1999), where there are obvious inconsistencies between teachers' espoused beliefs and the resulting classroom practice. Other studies of teacher beliefs collect a pool of teacher beliefs while making little or no effort to uncover a link with classroom behaviour. There are some studies that do not investigate the relationship at all while leaving the reader to infer that the particular pool of teacher beliefs just collected has something to do with action. How then are teachers and researchers meant to navigate this maze of conflicting and confusing material? This paper will attempt to provide some guidelines that have their basis in past research.

Sadly the past is often ignored when it comes to the study of beliefs and to the problems afflicting this area of research. For example, why do researchers need to continually report the mismatch (Thompson, 1992) between espoused beliefs and enacted beliefs? Early researchers such as Argyris and Schon (1974) reported that behaviour was influenced by individual action theories and they identified and described espoused theories and theories-in-use (action). They expected that individuals would use an action theory that was in harmony with their espoused theory, yet what they discovered was quite

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the opposite. Not only were there significant differences between the two types of theories but often the individuals themselves were unaware of the mismatch (Argyris, 1993). They also reported that individuals developed a number of strategies that contributed to this lack of awareness. This was true for both individuals and groups, thus:

> whenever undiscussibles exist, their existence is also undiscussible... These cover-ups, and their cover-up, are indications of organisational *defensive routines*, which may be defined as any policy or practice that prevents organizations (and their agents) from experiencing embarrassment or threat *and* at the same time prevents them from identifying and reducing the causes of embarrassment or threat (Argyris, 1993, p.621).

This helps to explain how professionals avoid learning and perhaps why researchers continue to repeat belief studies reporting inconsistencies between espoused and enacted beliefs, for it shows how defensive reasoning blocks learning in spite of a high individual commitment to learn. Argyris and Schon presented alternatives for effective change that presented a theory of action to enhance human activity, responsibility, learning, effectiveness and self-actualization. People are taught to recognise the reasoning they use in designing and implementing certain behaviour and they are helped to identify the inconsistencies between their espoused and actual theories of action. This method has been successfully adapted to helping successful people learn more effectively (Argyris, 1991), and promoting effective organisational change (Argyris, 1993). The key to the process lies in the discovery, examination and discussion of a range of teacher action theories.

There has been considerable work done to uncover both types of theories (Argyris & Schon, 1974; Ajzen & Fishbein, 1980; O'Hare, 1987, Ajzen, 1988; Argyris 1993). White (1999; 2000 a, b) conducted a series of investigations into the action theories of teachers. These studies uncovered and reported theories associated with specific teacher classroom behaviours by collecting sets of behavioural beliefs through the use of phenomenography (Marton, 1986) and then applying a modification to the model titled the Theory of Planned Behaviour (TPB; Ajzen, 1985). The results were called teacher action theories to distinguish them from Argyris and Schon's earlier work. Using a different methodology, Keast (2001) conducted a study of three secondary teachers. His study was able to identify what were called the 'testable beliefs' of the teachers, in order to enable the teachers to reflect upon and ultimately alter their practice.

What becomes clear is that espoused beliefs or theories should be also regarded as being linked to the behaviour of the teacher, but not the classroom behaviour. Rather espoused theories are the beliefs teachers are comfortable exposing to public scrutiny. Thus espoused and theories-in-use, are really different types of action theories. They differ because their contexts differ. The specific situations, the teacher's behaviour and the intended consequences are not the same. Espoused theories contain an individual's beliefs, attitudes, and values that are used to describe and justify behaviour. Whereas theories-in-use or action theories are based upon actual human behaviour. Barkatsas (2001) examined the espoused beliefs of 345 Greek secondary mathematics teachers regarding assessment. While the cross-cultural comparisons were interesting, is it necessary to do a follow-up qualitative study "in an attempt to identify inconsistencies between teachers espoused beliefs and beliefs in practice" (p.369)? Research into the planning, thinking and decision making of teachers should provide insight into classroom events and be worth understanding because of the connection with student learning and performance. History has repeatedly demonstrated the folly of concentrating upon only one member of this pair of beliefs and action. For example, a concentration upon action:

> It has now been well documented in several studies that teachers asked to change features of their teaching often modify the features to fit within their pre-existing systems instead of changing the system itself. The system assimilates individual changes and swallows them up. Thus. although surface features appear to change, the fundamental nature of the instruction does not. When this happens, anticipated improvements in student learning fail to materialize and everyone wonders why (Stigler & Hiebert, 1999, p. 98).

Cobb, Yackel and Wood (1988) argue that neither beliefs nor practice have primacy, but that "beliefs and practices are dialectically related" (p.24). This suggests that teacher professional development is likely to be a long-term process where the teacher may need to examine and develop their long-term belief systems as well as work at changing practices. Stigler and Hiebert (1999) regard teaching as a cultural activity where teachers share a mental picture of what teaching is like. They call this mental picture a script. They assert that these cultural scripts are learned implicitly, through observation and participation, and not by deliberate study.

> The scripts for teaching in each country appear to rest on a relatively small and tacit set of core beliefs about the nature of the subject, about how students learn, and about the role that a teacher should play in the classroom. These beliefs, often implicit, serve to maintain the stability of cultural systems over time (Stigler & Hiebert, 1999, pp. 87-8).

The implicit nature of these theories has other implications Berliner (1994) who reported that:

> Experts have an intuitive grasp of the situation and seem to sense in nonanalytic and nondeliberative ways the appropriate response to be made. They show fluid performance, as most people do when they no longer have to choose their words when speaking, or think about where to place their feet when walking... When things are going smoothly, however, experts rarely appear to be reflective about their performance (p. 6021).

Lack of reflection and the inability to articulate practice by classroom teachers carries into other areas such as values (Bishop 1999). There are serious implications associated with the observation that expert teachers lack reflective theories and are unable to clearly articulate their classroom practice. 'Until these underlying beliefs, attitudes and knowledge bases are meaningfully challenged, the change effort is in real danger of becoming yet another 'bandwagon' ' (Siemon, 1989, p. 254). Any successful professional development process requires that a teacher constructs a coherent personal teaching theory that will guide behaviour and reflection is fundamental in this transformation from a novice to an expert.

Another historical lesson that should be considered involves the definition of beliefs, an area so confused that Pajares (1992) was able to list over twenty aliases that have been used by researchers for discussing belief within their research. However it is possible to categorise research studies according to whether the definition of belief focuses upon a global statement or a specific behavioural belief statement. To appreciate the importance of this division it is necessary to consider Ajzen (1985, 1988) who provided an overview of the historical development of two research traditions dealing with attitudes and personality traits. The personality research tradition tried to explain behaviour in terms of underlying dispositions. For example people caught cheating and lying were regarded as being dishonest. "Personality research has revealed five general response tendencies that represent robust personality characteristics: sociability, agreeableness, conscientiousness, emotional stability, and culturedness" (p. 1). Now the trait concept was an attempt to explain these dispositions in greater detail using a multitude of personality traits that had been identified. Ajzen defined a personality trait as "a characteristic of an individual that exerts pervasive influence on a broad range of trait-relevant responses" (p.2). Thus a subject's response to a stimulus was assumed to be a behavioural manifestation of an underlying trait. Early research failed to provide empirical evidence to support two important research implications. The first was that if traits were enduring behavioural dispositions then they should affect behaviour across a variety of situations. Thus a dishonest person could be expected to cheat in other situations such as tax claims, or to lie to friends, or to steal. However, findings failed to provide support for cross-situational consistency in behaviour.

The second implication was that the degree of correlation between global personality characteristics and the corresponding narrowly defined behaviours relevant to the trait in question would be high. An example would be to compare the results from an instrument such as a questionnaire designed to measure hostility with the aggressive and cooperative behaviour observed within a laboratory setting. Again the results from this type of research (Mann, 1959; Gibb, 1969) were poor. So poor that Mischel (1968) after reviewing the literature coined a term 'personality coefficient' to describe "the correlation between 0.20 and 0.30 which is found persistently when virtually any personality dimension inferred from a questionnaire is related to almost any conceivable external criterion involving responses sampled in a *different* medium - that is, not by another questionnaire" (p. 78). He pointed to the problem that behaviours used as trait indicators were highly specific to the particular context. Now these findings raise a number of important issues in the study of teacher beliefs.

Are beliefs highly specific to the particular context? This impinges on another area often associated with beliefs and that is the area of values. Values have been described as the deeply held beliefs that dispose a person to act in a certain way (Hill, 1991), and as deeper and more stable than beliefs (Seah, 1999). It is possible to see the attempt to explain behaviour in terms of underlying value system sharing similarities with the earlier attempts of using dispositions or traits. So if beliefs are highly specific to the particular context then it would seem prudent that the beliefs should be linked to the specific behaviour being examined in opposition to a more global definition.

In contrast to global belief studies, there are studies where beliefs are carefully linked to specific behaviours. In some studies, where attitude is a consideration, both belief strength and outcome evaluations are measured (Ajzen & Fishbein, 1980; Ajzen, 1988). Thus researchers gain a measure of both the teacher's beliefs about the consequences of performing an action (behavioural beliefs) and an evaluation of the consequences (outcome evaluation). Examples can be found in such diverse areas as voting behaviour (Fishbein, Ajzen & Hinkle, 1980), family planning (Fishbein, Jaccard, Davidson, Ajzen & Loken, 1980), consumer behaviour (Fishbein & Ajzen, 1980), weight reduction (Sejwacz, Fishbein & Ajzen, 1980), changing alcoholic behaviour (Fishbein, Ajzen & McArdle, 1980), smoking behaviour (Fishbein, 1982), infant feeding (Manstead, Proffitt & Smart, 1983), seatbelt use (Budd, North, & Spencer, 1984; Trafimow, & Fishbein, 1994), continuing education (Prior, 1990), driving violations (Parker, Manstead, Stradling, Reason, & Baxter, 1992), condom use (Chan, & Fishbein, 1993), work after childbirth (Granrose & Kaplan, 1994), teacher use of group work in secondary science (Lumpe, Czerniak, & Haney, 1998), teacher use of group work in primary mathematics (White, 2000a), teacher use of stencils in primary mathematics (White, 2000b), and teacher use of calculators in the primary mathematics classroom (White, 1999; 2000c). These studies report medium to strong links between behaviour and belief and can demonstrate a causal relationship

Now consider the area of research into teaching patterns often designated as orientations, styles or modes of teaching that was very popular in the 1970s and 1980s and the historical resonance with teacher beliefs. Earlier studies into these teaching styles argued that teachers developed characteristic stable or fixed patterns of behaviour that may have been conscious and deliberately chosen, or have arisen from experience and may have been largely unconscious. Many researchers tended to represent these teaching styles upon a continuum with their theoretical opposites as end points (Kerlinger, 1967; Mosston, 1972; Lapp, Bender, Ellenwood, & John, 1975; Stallings, 1977), and some researchers used more than one continuum (Green, 1971; Brady, 1985; Ernest, 1989). An extreme case is presented by Joyce and Weil (1980) who conducted a very comprehensive study of teaching styles and set up 23 distinct models, which could also be grouped into four main classes. The aim of this type of research was either to present an overall structure for current teaching practice according to the researcher's theoretical presentation, or to categorize teachers according to a particular theoretical orientation. The assumptions were that all teachers could be located somewhere along the continuum and that knowing a teachers style would provide information concerning their specific classroom practices. It was hoped that certain styles would lead to greater student learning gains and thus the promotion of effective teaching styles would be the catalyst for teacher professional development.

This area of research failed to fulfil the hopes of supporters. There were many critics who challenged the assumptions and claims of the research. Bennett (1989) who comprehensively investigated the 1970s research upon learning styles was extremely critical of its inability to identify individual teacher activities or behaviours, which related to higher mathematics achievement. He stated that it was of little value in initiating improvements in teaching and the differences in scores between styles were often relatively slight, which severely limited the use of style as an explanatory variable. The classification of teaching behaviour by assigning a position upon the continuum did little to explain the basis of teacher classroom behaviour. What was lacking was a theory that considered how and why teachers behaved as they did, and the effects of that behaviour.

Some researchers are using beliefs as a basis for placing teachers upon a continuum. For example, Howard, Perry and Lindsay (1997) surveyed 249 secondary mathematics teachers in Sydney and constructed two different patterns of beliefs. The first was identified with the "transmission" profile where the transmission and verification of information was central and where memorization of rules and procedures were fundamental. The researchers reported that this group was larger in number than the opposing group called the "constructivist" profile. Here teacher beliefs indicated that students were capable of constructing their own mathematical knowledge in an atmosphere of negotiation and relevance. This study makes no attempt to study or examine the link between beliefs and action but does say that "further investigation comparing espoused beliefs of teachers and their enacted beliefs is required" (p. 237). A later study, Perry, Howard and Tracey (1999) studied the espoused beliefs of secondary mathematics coordinators from the perspective of a two-factor model of 'transmission' and 'child centred'. Now how applicable are Bennett's (1989) criticisms to these studies? For example, can these studies be used to identify individual teacher activities or behaviours, which relate to higher mathematics achievement? Consider the possibility of a teacher who reports having more behaviourist beliefs than constructivist beliefs. How do we know which beliefs are linked to behaviour? It is possible that a teacher could use behavourist beliefs as a basis for action and constructivist beliefs for articulating practice or vice versa?. With this type of research we do not know the answers to these questions and while interesting, the value is questionable in terms of informing teacher classroom practice. "Improving teaching must be front and centre in efforts to improve students' learning" (Stigler & Hiebert, 1999, p. 3).

Conclusion

This paper attempted to use some lessons from the history of research to inform the current direction of research into teacher beliefs. It briefly identified some 'undiscussibles" in order to expose them to scrutiny. Ignoring the past and the lessons of history could be seen as a defensive routine that hinders progress and results in endless replication. The inability of research to assist classroom teachers to articulate their current classroom practices and values is detrimental to both individual growth and the mentoring of others, particularly pre-service teachers. Teachers would benefit from self-examination and a process that encourages reflection of their action theories and beliefs. Teacher educators must prepare their students for the classroom theories and behaviours that they are likely to meet in their professional experience periods and to cope with a range of teacher action theories and defensive routines. In helping pre-service teachers work towards practices that are consistent with the wider goals of the educational system it is crucial that they explore their own beliefs about classroom practice. It is the responsibility of researchers to provide the framework and the results that inform both belief and action. The gains in the future are intertwined with the gains of the past.

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Identifying Tasks that Promote Creative Thinking in Mathematics: A Tool

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A *tool* constructed to examine a task's potential to stimulate and support creative mathematical thinking was derived from a *hierarchy* of cognitive activities students employ as they solve unfamiliar problems. These cognitive activities informed the formulation of categories of task features that support creative mathematical thinking. Use of this tool is illustrated in a study of the mismatch between potential student responses to a task and actual student responses to the task as implemented. The sufficiency of the tool is discussed.

> "The nation's SET [Science Engineering & Technology] base is held together by *ideas workers*… people who contribute to the creation of new ideas and the effective application of the existing global stock of ideas … Australia must build enterprise and innovation skills by … encouraging more innovative learning and teaching in Australian schools …" (Batterham & Miles, 2000).

To increase the likelihood of Australia sustaining a position on the cutting edge of research design in Science, Engineering, and Technology, school learning environments need to stimulate and nurture creative mathematical thinking. The characterisation of teaching and learning situations that promote creative mathematical thinking would support teacher educators and teachers in pursuit of this goal. The work in progress in this paper describes the construction of a tool to analyse tasks for the purpose of evaluating their potential to stimulate creativity. Contrast of the potential and actual student responses to a task found to trigger some creative thinking provided a way to study the mismatch and identify characteristics of the task as implemented that influenced the stimulation of creative thinking.

Literature Review and Construction of Theoretical Framework

Krutetskii (1976) described the mathematically insightful behaviour (mental activities) of highly capable students as they thought aloud to solve unfamiliar challenging problems. I used Krutetskii's empirical data to construct a hierarchy of Krutetskii's mental activities (Williams, 2000) and noted Krutetskii's language for describing these (mental) cognitive activities was consistent with the language used in Bloom's (1956) Taxonomy of Learning Objectives that has since been adapted to describe categories of higher-level thinking in educational settings (e.g., Tannenbaum, 1983). The cognitive activities in my hierarchy were: (a) repeat taught information (Recall); (b) understand a learnt concept (Comprehension); (c) apply a previously learnt procedure (Apply); (d) build on a previously known idea to solve a problem with a slight twist; recognise the need for extra information (Analysis); (e) work backwards; use more than one pathway; explain the need for extra information when insufficient information is provided to solve a problem (Analytic-synthetic); (f) combine concepts to create an original concept (Synthesis); and (g) recognise inconsistent information (Synthesising-evaluation, a subset of Evaluation).

Dreyfus, Hershkowitz and Schwarz (2001) have used three observable nested elements to study the process of abstraction (Recognising, Building-with and Constructing).

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 698-705). Sydney: MERGA. ©2002 MERGA Inc.

Recognising and Building-with are nested within Constructing. These nested elements of abstraction can be described in terms of the categories in my hierarchy. Recognising includes the process of identifying a context in which a previously abstracted entity applies (Comprehending) and recognising a previously abstracted entity can be applied to a given context (Applying). Building-with includes using previously abstracted entities as part of several different processes such as: applying a previously abstracted entity in a known context (Applying), or in a new context (Analysing), or applying several previously abstracted entities in a familiar order (Applying) or a different order (Analysing). The relationship between the observable elements of abstraction and my categories of cognitive activities suggests my term "hierarchy" should be replaced by nested categories of increasingly intellectually complex activity. Constructing—mathematically insightful behaviour—is a process of integrating previously abstracted entities to develop a new mathematical insight (Synthesising & Evaluating). Constructing may occur in a more expert-directed learning culture where suggestions, hints, and corrections are made or constructing may be a more autonomous activity where hints are not provided but questions are asked for the purpose of eliciting more complex thinking (Williams, 2000).

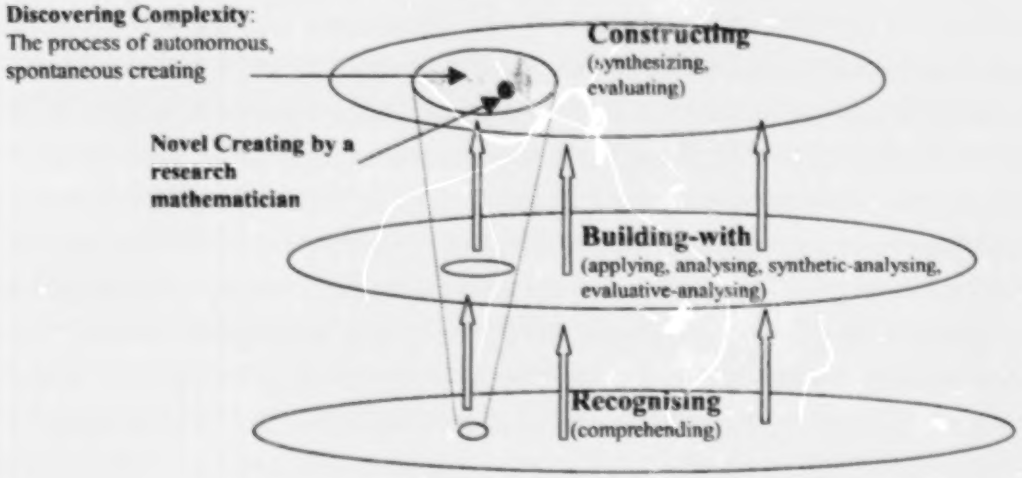
Discovering Complexity:
The process of autonomous, spontaneous creating
Constructing
(synthesizing, evaluating)
Novel Creating by a research mathematician
Building-with
(applying, analysing, synthetic-analysing, evaluative-analysing)
Recognising
(comprehending)

Figure 1. Relationship between observable nested elements in the process of abstraction, the process of discovering complexity and the process of employing cognitive activities.

Chick (1998) described her cognitive processes as a research mathematician "not only choosing the cues and concepts—and often unexpected cues and concepts—but even the very question" [p. 17]. Senior secondary mathematics students discovering complexity and spontaneously formulating their own questions exhibited characteristics of *flow* as they worked with unfamiliar mathematical ideas to resolve their questions (Williams, 2000). Csikszentmihalyi (1992) described flow as intense engagement where people lose all sense of self, time, and the world around. He found necessary conditions for flow included working autonomously just above the persons present skill level on a challenge almost out of reach. Figure 1 above and Table 1 below relate Dreyfus, Hershkowitz and Schwarz's (2001) elements of abstraction, my categories of cognitive activities, the process of discovering complexity (Williams, 2000) and Chick's (1998) cognition as a mathematician.

Table 1

Nested Cognitive Processes Associated with Mathematically Creative Behaviour

Cognitive Category	*Example of cognitive activity elicited and symbol for research source from which the activity was derived*
Evaluating. (Constructing)	* + Progressively reflect on the situation as a whole for the purpose of recognising inconsistent information and/or finding a more elegant solution pathway. + Reflect upon the process of problem solution for the purpose of recognising its limitations and applications to other contexts. ^ Reflect upon the solution pathway developed and its possible contribution to generic mathematical processes for future use.
Synthesising. (Constructing)	* + Explore the problem from many perspectives rather than just work towards a solution. + Formulate mathematical arguments to explain discovered patterns. + < Progressively explore the problem to continually develop new insights and simultaneously consolidated "fragile" new insights. * + Integrate concepts to create original concepts. Could vary in: • + Number of concepts involved • + # Diversity of the domains concepts were drawn from • + Size of the conceptual leap • + Spontaneity of the question pursued.
Evaluative-analysing (Consolidating)	+ Interconnect solution pathways for the purpose of identifying flaws and strengthening arguments. + Find fast ways to assess the reasonableness of a result.
Synthetic-analysing (Building-with, Consolidating)	* + Use more than one pathway to solve a problem. + Interconnect various representations, operations, and assumptions through comparison of different pathways. * + Explain why extra information is needed when insufficient information is provided to solve a problem. * + Work backwards to find what is usually an initial condition when given the answer and the other initial conditions. ^ Recognise and utilise previously known mathematical ideas in a new sequence to explore a situation.
Analysing (Building-with, Consolidating)	* Solve using a previously known process with a slight twist. + Familiarise self with problem using specific numerical examples. + Systematise the numerical results and search for patterns. * + Recognise insufficient mathematical information has been provided. ^ Apply previously known mathematical procedures in a new context.

Key: * Krutetskii (1976); # Chick (1998); + Williams (2000);
^ Dreyfus, Hershkowitz, & Schwartz (2001); < Dreyfus & Tamsir (2001)

My collaborative groups of senior secondary mathematics students met Csikszentmihalyi's (1992) conditions for flow because they continually posed and resolved their self-set questions involving intellectual challenges (mathematical intellectual complexity) by using mathematical ideas above their present skill and concept level (increased conceptual complexity) (Williams, 2000). Table 1 lists categories of cognitive activity in decreasing order of intellectual complexity in Column 1. Column 2 contains examples of student responses used to infer cognitive activity. These examples are drawn from research into students' thinking as they solved unfamiliar challenging problems (Krutetskii, 1976; Dreyfus, Hershkowitz, & Schwarz, 2001; Williams, 2000; Chick, 1998; Dreyfus & Tsamir, 2001). Evaluative-analysing is a new category associated with examining the reasonableness of findings as opposed to reflecting on the task overall (Evaluating). This category was empirically developed from responses of students in my classrooms and a similar category was proposed Anderson's in revisions to Bloom's Taxonomy (1998).

Table 2
Task Features that Promote Autonomous, Spontaneous, and Creative Thinking

Nested Student Responses	*Task Features*
Autonomous Access.	Variety of representations and/or initial assumptions. Work with specific cases or generalisations. Initial specific cases leading to later general arguments.
Spontaneous Pursuit.	Overarching task question sufficiently unusual to be unfamiliar to all. Over-arching question contains an *unusual twist*. Potential for idiosyncratic discovered complexity. Potential for successive discovered complexities (sustained challenge). Apparent inconsistencies or generic questions to elicit complex thinking.
Creative Execution.	Conceptual, general, or multiple solutions rather than an exact numerical solution provides scope for: Unusual solution pathways Non-linear progression Pursuit of elegance Simple elegant checks for inconsistencies Assembly and integration of unexpected mathematical ideas.

Table 2 (Column 1) displays nested categories of student responses associated with creative mathematical thinking (Autonomous Access, Spontaneous Pursuit, and Creative Execution). For example, autonomous access is nested within spontaneous pursuit. Task features relevant to each nested category of student response were developed by considering the cognitive activities described in Table 1 and identifying task features that support these types of activities (Table 2, Column 2). Autonomy for the purpose of this study refers to student choice in selecting: the representations to explore; the assumptions they apply; and to enter the task is through specific examples or general arguments. Access refers to the students' opportunity to find an entry point to the task and student opportunity for sustained exploration of relevant mathematical ideas. Spontaneity refers to student-focused inquiry triggered by curiosity. Pursuit refers to student activity related to resolving their self-focused questions. A high degree of creativity exists where students spontaneously formulate highly relevant questions and assemble and integrate

mathematical ideas from mathematical topics—which until now appeared to be unconnected—for the purpose of resolving these questions using elegant solution pathways. Execution refers to the quality of the creativity and is relative to (a) the prior mathematical background of the student; (b) the recentness of exposure to the mathematical ideas upon which they drew; (c) the students' prior knowledge of connections between the mathematical ideas they integrated, and (d) the elegance of the argument produced.

Rationale, Methodology, and Research Question

This research was undertaken in a Year 8 mathematics classroom in a large metropolitan city in Australia. The particular lesson and task studied were selected from a data set of fourteen consecutive lessons from a classroom that was part of the broader international Learners' Perspective Study of teachers seen to "display good teaching practice". The methodology for the broader study and a more detailed account of the lesson in which this task was situated has been provided in a previous paper (Williams, 2001) The classroom research undertaken used three video cameras to simultaneously capture: the whole class, the teacher, and two focus students. The two focus students took part in individual video-stimulated reconstructive interviews. Evidence was drawn from the video data, classroom field notes, and student and teacher interview data. The research questions discussed here are "Is this tool sufficient to account for the actual student responses to the task as implemented?" And "Does the contrast of potential and actual student responses inform my study of factors that promote or inhibit creative student thinking?"

The task was presented to the students in Lesson 3 after the topic of perimeters of rectangles. This task was selected as the illustrative example for this report because it contained many of the task features identified to promote creative thinking (Table 2) but during task implementation creative mathematical thinking was not evidenced. After task closure, two students (Leon and Pepe) displayed creative thinking in further exploration of the task. As Leon, and Pepe were not the focus students, classroom dialogue was captured in fragments by the video cameras and in the field notes. Leon's interview (Lesson 12) supplemented this data.

Two distinct learning cultures operated at different times in this classroom. One culture was based on teacher selection of direction and student contribution of ideas whereas the other culture was based on student selection of direction and teacher provision of assistance. In whole class discussions, the teacher selected the solution pathway and questioned students (video data) for the purpose of building on their ideas (teacher interview). The teacher decided which student comments to integrate into whole class discussion and acknowledged then discarded other comments. In small group work (usually pairs), students selected their own solution pathways and consulted the teacher when they experienced difficulties. The teacher generally provided hints or explanations and sometimes asked open questions. The task: 'What are the dimensions of the rectangle with perimeter 38 cm?' was undertaken as a whole class activity.

Analysis and Results

Table 3 contrasts the potential and actual student responses to the task. The task has the *potential* for exploration of unfamiliar mathematical ideas; the perimeter was given and working backwards was required to find the dimensions. As only one of three unknowns was given, substitution and algebraic manipulation would not provide a numerical solution. The existence of more than one rectangle with a perimeter of 38cm provided an 'unusual twist'. Some examples of the potential discovered complexities include (a) How does the pattern L + W = 19 relate to the perimeter formula or the context? (b) Why are the values of L and W restricted? (c) What mathematical arguments explain what happens? The conceptual ideas accessed may commence with generalisations or with tabulation of specific examples to find a pattern and then explain why it exists. These complexities involve student connection of solution pathways using various representations (specific numerical examples, tables of numbers, numerical patterns, algebraic patterns, algebraic variation to provide a general argument). Different assumptions are possible (number of decimal places) and elegant solutions may result (argument through variation using algebra).

In the *actual* task as implemented, the mixed ability of the students led to the teacher's decision to work at a pace that catered adequately for slow pace students but did not extend the 'brighter' students to their full potential (teacher interview). Most students (including 'brighter' students) reported how much they enjoyed being in this teacher's class and how much more they learnt than in most mathematics classes. The teacher selected a pathway that progressed from specific examples to tabulation to discussion of a student-recognised pattern to algebraic manipulation to confirm the pattern. The teacher integrated Susan's discovery of a pattern into the whole class discussion. She acknowledged the correctness Leon's recognition of the complimentary variation of dimensions to retain the same perimeter but she did not integrate this into the discussion. Leon's explained his use of the dimensions [9, 10] to find the width when the length was 9.6 by "moving the point six around the corner". This potentially could have been a precursor to a mathematical argument based on variation within the physical context.

After teacher questioning that explored the number of possibilities using one decimal place, the task was brought to closure with the question "How many possibilities if there were two decimal places?" (Posed but not answered). During the next few minutes (at the commencement of pair-work), Leon and Pepe exhibited an intensity of focus as they talked together at a fast pace and projected questions across the room to the teacher. This episode is described in more detail in a previous paper (Williams, 2001). Leon's interview suggests Leon and Pepe discovered a complexity focused around: What is the relationship between the number of rectangles possible using dimensions correct to different numbers of decimal places? Leon's interview descriptions provided evidence of creative thinking.

> '... it kept building and building and building- you sort of found a formula- a way to categorise it'

Although there is insufficient evidence to determine whether Leon and Pepe resolved their spontaneous question, they did assembled disparate mathematical ideas and attempt to formulate an algebraic expression to show the relationship between the terms (synthesising).

Table 3

Contrast of Potential Student Response and Actual Student Response

Student responses	*Potential task features (examples are illustrative not exhaustive)*	*Actual task as implemented*
Autonomous Access. Pathway choices	1) Representations: specific numbers, tabular, physical context, algebra. 2) Assumptions: integers, number of decimals places, all integers and decimals. 3) Conceptual complexity of pathway (specific dimensions, physical variation in context or general variation of formula). 4) Specific progressing to general through generation of specific dimensions, finding a pattern, explaining why this pattern exists.	No autonomy of pathway (teacher selected). Some autonomy in nature of student comments contributed to the class discussion. No autonomy in selection of student comments to explore further.
Spontaneous Pursuit. Idiosyncratic foci	1) Work in opposite direction but insufficient information to solve by substitution. 2)'Unusual twist': more than one solution. 3) Discover complexity: For example, how does the specific link with the physical and the algebraic? How can findings be checked or what mathematical arguments exist? 4) Successive discovery of complexity: progressive links between various complexities. 5) Task triggers to more complex thinking: Why was more than one rectangle possible?	The nested nature of autonomy within spontaneity accounts for lack of evidence of spontaneous questions. Several insightful comments (not questions) occurred: [0,19] will be a straight line; and Use of [9, 10] to get 9.4.
Creative Execution. Questions asked, strategies and ideas assembled, novelty and elegance of composition	1) Non-linear exploration: evidenced through discovered complexities previously discussed. 2) No exact numerical solution: existence of many possible rectangles and general arguments about why this is possible. 2) Unusual solutions: e.g., Developing an algebraic series from solutions to different numbers of decimal places. 3) Elegance (General arguments using variation in the physical context and/or algebra). 4) Fast check (connection of variation argument in context and algebra). 5) Unexpected concepts: restrictions linking algebra and context; arguments using variation.	Leon and Pepe worked autonomously after task closure to discover complexities encapsulated by the spontaneous questions: 'Why does the number of answers keep building and building?' 'How are they related? 'Can we use a formula to show the relationship?' They selected their own foci, and created an elegant and unusual pathway— algebra to relate 'answers'.

Discussion and Conclusions

This analysis tool has the potential to increase teacher awareness of the complex cognitive activities associated with creative student thinking and inform their pedagogical decisions about task selection and implementation. The sufficiency of this tool in accounting for student response to this task was demonstrated. The tool's usefulness in identifying the mismatch between potential and actual student responses was confirmed and has assisted in identifying factors that promoted and inhibited creative thinking. The provision of a task with the potential for creative thinking was not sufficient to trigger such thinking in the absence of student autonomy. The effect of the classroom culture on promotion of creative thinking was clearly demonstrated through the change of classroom culture during this lesson; the autonomy to explore aspects of this task after task closure was a significant aspect of the changed classroom culture that supported creative thinking. Areas for further study include testing the tool's sufficiency with a diverse range of tasks demonstrated to elicit creative thinking, and exploring triggers that may elicit increasingly more complex thinking.

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Spatial Reconstructions from Primary Children's Drawings

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In this paper geometric eigenproductions of primary school children are analysed in *spatial reconstruction experiments*: A *donor* draws an addressee- and purpose-specific picture of a spatial building, and an *acceptor* tries to reconstruct this object only by means of the partner's drawing. Comparing the original building with its reconstruction, the two partners evaluate the informational substance of the drawing. This paper discusses, in how far the drawings are reconstruction-efficient and how they are improved in the course of the interaction.

Eigenproductions in the Primary Mathematics Classroom

Eigenproductions are documents which children create whilst mastering (in the full sense of the word) a mathematical problem (Treffers, 1991; Selter, 1993). In them children establish their initial statements for the solutions, their strategies and/or results.

Time-based eigenproductions allow recognition of their history of origin; such documents are transcripts or representations which have the intention of presenting a chronological sequence of their elements. However, eigenproductions in which the time sequence of their composition is not intended, can occasionally still be "sequenced", that is, analysed with regard to their history of origin. The study of time-based documents in empirical didactical research frameworks therefore can shed light on the processes of learning mathematics—both with respect to individual learning as well as regarding to interaction processes.

Many of the primary school children's eigenproductions found in didactical analyses have been specifically created for these studies. Eigenproductions relating to the same task often portray a rather confusing variety, making systemisation difficult. In the experience of the author, eigenproductions are in each case *part of a learning history* and hence influenced by its development. Once the conditions of this learning history are considered, the eigenproductions can be interpreted more accurately.

Eigenproductions are messages. They carry information about the fictitious partner a child has in mind and the respective message or information this partner is assumed to be interested in. Children who are not given specific information regarding the addressee and his/her interests, often create eigenproductions for a fictitious partner, whom they sometimes even model as a kind of archetypal copy of themselves.

The author therefore differentiates between *purpose-specific eigenproductions* and non-purpose-specific ones. For example, the response to the task "draw a picture of this building" will in general not be a purpose-specific one, while the response to the task "draw a picture of this building that can be used as a construction plan" will most likely be purpose-specific.

The author further differentiates between *addressee-specific eigenproductions* and non-addressee-specific ones. For example, the verbal response to the task "describe how to start a car" differs depending on whether the person has or has not a drivers license.

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 706-713). Sydney: MERGA. ©2002 MERGA Inc.

Working environments in which children use purpose-specific and addressee-specific eigenproductions can additionally be provided with a *feed-back option* that can influence further eigenproductions. Thus, reconstruction tasks are created. The organisation of working environments as reconstruction tasks is a fundamental and effective method for both the empirical investigation of eigenproductions as well as for the organisation of classroom working environments.

An Investigation of Geometric Eigenproductions

One research focus of the author places particular emphasis on teaching and learning of spatial geometry in primary mathematics (Wollring, 2001; Wollring & Jungwirth, 2001).

Spatial geometric drawings of children, which are not purpose-focused and not addressed to specific partners have been analysed in a number of field studies. Pars pro toto the classical experiments of Lewis (1963), Mitchelmore (1978) and Ingram and Butterworth (1989) can be named. Lewis analysed children's drawings of a cube with special paintings on three of its sides and classified them by morphological patterns. The main distinction is *schematic versus realistic*. Mitchelmore analysed children's drawings of various solids in a rather behavioristic environment and classified the drawings by *series which characterise developments*. Both studies describe children's methods of encoding spatial objects. Ingram and Butterworth, in addition, investigated how kindergarten and primary school children draw configurations of two cubes. The smaller one could be placed either in front of, within, behind, on, or next to the bigger one. Beside morphologic strategies Ingram and Butterworth additionally described the drawing strategy of *partial occlusion*.

All three studies analysed drawings initiated by the neutral stimulus "draw what you see" which were neither purpose-focused nor specifically addressed. However, these studies provided categories of drawings for a new research project by the author investigating spatial reconstructions from drawings. This study, which provides the background of this paper, addresses the following research questions with respect to children's drawings: How do pairs of grade 3 children (8 to 9 years old) communicate the solution of a spatial geometric reconstruction on the basis of their eigenproduced drawings? Which effects does this communication have on succeeding reconstructions? What patterns of drawings and interaction arise? How successful are the pairs of children in solving the task?

Design of the Spatial Reconstruction Experiment

The study was organised as a set of eight experiments characterised by different working environments. All of these experiments, designed and directed by the author, were organised as a series of clinical interviews. The interviews were conducted by student teachers acting as teacher-researchers in interpretative classroom research (for details with respect to this approach see e.g., Wittmann, 1985; Peter-Koop & Wollring, 2001).

The special experiment reported in this paper, consisted of 16 clinical interviews with pairs of children from third grade classrooms. Each of the four student teachers involved in this part of the study conducted four of these interviews.

Each individual interview consisted of eight episodes in the form of *spatial reconstruction tasks*.

Each episode was designed as follows: One child acted as *donor*, the other child as *acceptor*. At the beginning only one child, the donor, was present, while the other child was waiting outside the room. The donor was given an *original spatial building* with the task to draw this building, bearing in mind that the partner, once entering the room would be confronted with the task to rebuild this object with given material merely on the basis of this drawing. Once the donor had completed the drawing, the original building was covered before the acceptor was asked to come into the room. The drawing was shown to the acceptor who was then asked to produce a *reconstruction*, that is, a copy of the original spatial building. During this reconstruction process the donor was present, however had to remain silent. After the acceptor had finished the reconstruction, it was compared with the original. The partners then assessed whether the drawing was reconstruction-efficient, discussed possible deficits of the drawing and, where necessary, jointly developed suggestions for improvements to be used in the drawings of the subsequent episodes. After this, the episode was completed. Before a new spatial building was introduced in the next episode, the two children changed roles (i.e., the donor became the acceptor and vice versa).

In the 16 interviews of this experiment, the spatial buildings were dice configurations similar to those used by Ingram and Butterworth (1989) in their experiments.

Each interview was documented by a video recording of the children's interaction and the original drawings of the children. On the basis of these documents the interviews were analysed in a quantitative and in a qualitative dimension:

Quantitatively, the reconstruction success was evaluated noting "1" if the reconstruction was successful and yielded a complete copy of the original spatial building and "0" if it was not.

Qualitatively, the interviews were analysed on the basis of *episode plans* arising from the video documents. For every episode, the episode plan portrays the roles of the partners, the original building, the drawing and the reconstruction (see Tables 1 and 2). Sections of special interest were additionally documented by transcripts. This data basis yielded information with respect to several aspects: The structure of the drawings was analysed and compared with the patterns reported in the literature. The sequences of the drawings were analysed with respect to specific developments and regarding the respective interaction between the two partners.

Discussion of Selected Results

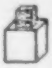
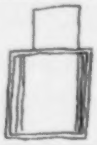
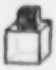
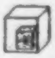
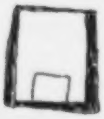
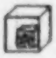
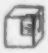
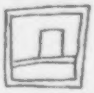
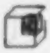
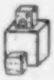
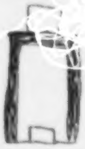
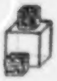
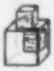
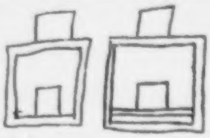
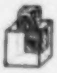
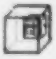
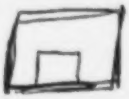
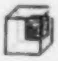
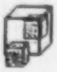
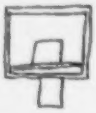
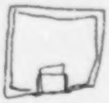
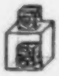
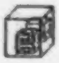
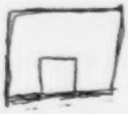
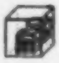
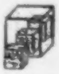
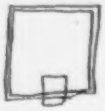
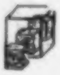
Key results of this study can be summarised both with respect to quantitative as well as qualitative categories. In this paper, primary focus is given to qualitative results describing the formation and the structure of the children's communication. However, the quantitative results were largely unexpected and are therefore briefly introduced.

Quantitative Results

- In the entire study 120 clinical interviews of spatial reconstruction experiments were conducted and video-taped. 16 of these interviews form the data basis of the reconstruction experiment using a specific working environment which is similar to that of Ingram and Butterworth (1989). In a "coarse" scoring system, in which successful reconstructions were counted as "1" and unsuccessful ones as "0", the pairs of children reached an average reconstruction success of 63% (standard deviation 10%). In a finer scoring system in which partial reconstruction success was scored "> 0", the pairs reached an average success of 78% (standard deviation 8%).

Table 1

Richard and Joachim (3rd grade) Reconstruct Dice Configurations According to their Eigenproduced Drawings. the Large Die is Transparent, the Small Ones are not. Left: Donor's Perspective, Middle: Donor's Drawing, Right: Acceptor's Reconstruction

Episode 1 2.34 – 5.41 *Donor:* Jo. *Acceptor:* Ri. Successful rec.			
Episode 2 6.06 – 8.16 *Donor:* Ri. *Acceptor:* Jo. Successful rec.			
Episode 3 8.41 – 12.17 *Donor:* Jo. *Acceptor:* Ri. Successful rec.			
Episode 4 12.43 – 14.53 *Donor:* Ri. *Acceptor:* Jo. Successful rec.			
Episode 5 15.18 – 18.50 *Donor:* Jo. *Acceptor:* Ri. Successful rec.			
Episode 6 19.20 – 21.15 *Donor:* Ri. *Acceptor:* Jo. Successful rec.			
Episode 7 21.40 – 27.00 *Donor:* Jo. *Acceptor:* Ri. Unsuccessful rec.		 	
Episode 8 27.25 – 29.27 *Donor:* Ri. *Acceptor:* Jo. Successful rec.			
Episode 9 29.48 – 32.34 *Donor:* Jo. *Acceptor:* Ri. Successful rec.			

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Similar results were found in the other spatial reconstruction experiments employing different working environments. (However, the finer scoring systems have to be closely adapted to the specific material used in this environment.) In general, the pairs of children were less successful in the first two reconstructions of a sequence than in the following ones. This indicates that a special starting phase is needed for effective communication.

Selected Qualitative Results: Two Case Studies

With respect to the qualitative results of the study the following two case studies were chosen in order to document typical effects. These are related to the formation of the drawings and their development in the children's interaction. Tables 1 and 2 show the episode plans of the reconstructions of two teams—Richard/Joachim and Sirkka/Martin.

The documents indicate that the drawings are for the most part *reconstruction-effective*. They enable the acceptor to produce an exact copy of the original dice configuration. Furthermore, the partners obviously learn through comparative feed-back. The donors organise their drawings according to schematic patterns, which in most cases the acceptor can correctly interpret. The picture sequences during this process indicate two forms of mutual understanding:

- The two children each develop their own drawing strategy but increasingly learn to understand those of their partner as in the case of Richard and Joachim.

Significant elements of the drawing strategies are mutually adopted by the partners – as in the case of Sirkka and Martin.

In the opinion of the author, however, a most remarkable finding is the fact that the series of the children's drawings do not develop in the sense of an increasingly realistic visual presentation, for example in the form of slanting pictures. Rather the contrary is the case. The drawings show an increasing tendency towards formal abstraction. Similar to parts of a mathematical formula, the donor only represents those elements which are considered necessary for effective interpretation by the acceptor. The drawings hardly show any redundancies.

Richard for example, from the second drawing onwards, only draws the "frame" formed by the front edges and in addition only the rear bottom edge. This he uses in order to designate the position of the smaller die unmistakably. Joachim learns to interpret this way of drawing, although he himself draws differently.

Sirkka "invents two signs" as she explains in the interview—the dark shading to signify "inside" and the arrow for the mental displacement to designate "behind this". Martin learns to read these signs and then adopts them with increasing success in his own drawings.

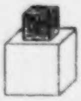
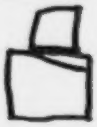
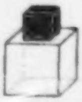
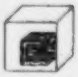
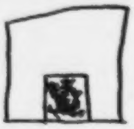
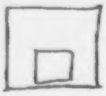
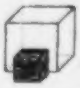
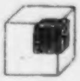
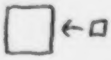
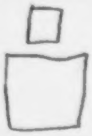
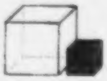
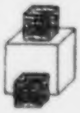
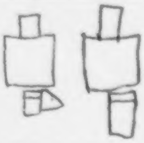
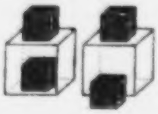
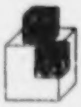
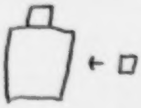
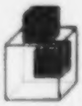
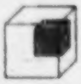
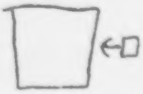
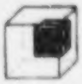
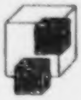
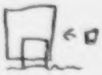
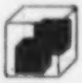
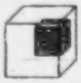
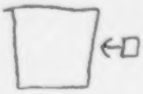
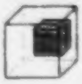
The findings of these two case studies correlate with the overall findings of the study. In their drawings all teams create elements of a situation specific pictorial language. In this, they generally do not only use optimised visual realistic pictures, they also invent formal elements similar to mathematical formulae and negotiate the meaning of these formal components. The correspondence itself always follows one of two different patterns:

- parallel development of individual encoding strategies accompanied by the understanding of the partner's strategy; or development of a common encoding system by exchanging invented "strategy elements" during the interaction process.

Table 2

Sirkka and Martin (3rd grade) Reconstruct Dice Configurations According to their Eigenproduced Drawings. All dice are Non-Transparent.

Left: donor's perspective, middle: donor's drawing, right: acceptor's reconstruction

Episode 1 2.04 – 3.32 *Donor:* Si. *Acceptor:* Ma. Successful rec.			
Episode 2 3.53 – 5.57 *Donor:* Ma. *Acceptor:* Si. Unsuccessful rec.		 	
Episode 3 6.16 – 8.57 *Donor:* Si. *Acceptor:* Ma. Unsuccessful rec.		 	
Episode 4 9.10 – 12.12 *Donor:* Ma. *Acceptor:* Si. Successful rec.			
Episode 5 12.29 – 13.50 *Donor:* Si. *Acceptor:* Ma. Successful rec.			
Episode 6 14.12 – 15.15 *Donor:* Ma. *Acceptor:* Si. Successful rec.			
Episode 7 15.34 – 17.25 *Donor:* Si. *Acceptor:* Ma. Unsuccessful rec.			
Episode 8 17.42 – 18.59 *Donor:* Ma. *Acceptor:* Si. Successful rec.			

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Conclusions

The children's drawings document genuine self-organised mathematical learning processes—the interactive development of an efficient means of communication in a geometric working situation.

Working environments such as those used in this study not only yield important research results, they also serve as models for working environments in the classroom.

Even if, in many cases, the expressions and procedures conventionally used in mathematics are not found or invented, it can still be assumed that children who have worked in environments as described above become increasingly sensitive to respective problem posing and, influenced by the need of invention, they also approach the corresponding conventions differently.

While it is certainly not suggested that the entire primary mathematics curriculum can be organised according to such environments, the main idea behind these experiments can inform classroom practice. With respect to a constructivistic understanding of mathematical teaching and learning, hence the teacher should be sensitive to children's eigenproductions and foster social learning by encouraging purpose- and addressee-specific working environments. Thus, children's original approaches and productions are not overwhelmed by norms too early.

In this context, the analyses of eigenproductions carried out by the student teachers involved in this project can serve as effective elements of a teacher education programme, since they yield foundations rather than simple action patterns for didactical decision-making, provide encouragement and self-confidence, reinforce design-competence and can in some cases even lead to a more positive re-evaluation of mathematics course contents (Wittmann, 1985).

Eigenproductions are communicable. In interactive learning processes they are optimised and support systems of communication in a self-organising fashion. Hence, especially geometric eigenproductions seem to be suitable in order to stimulate and to further develop a culture of interaction in the classroom. With respect to teacher education, the study of efficient interactive eigenproductions can provide an authentic constructivistic perspective on classroom discourse.

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Boys' Motivational Goals in Mathematics During the Transition from Single Sex Education to Co-Education

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Boys' task involvement and ego orientation motivational goals in mathematics were measured prior, during and one year after the introduction of co-education to a single sex boys' school. All primary and secondary boys enrolled in Grades 3 to 10 in the single sex school were followed over the three-year period. Comparisons were made over time across grades and cohorts with analysis of variance and hierarchical linear modelling. While there were significant effects for boys' task involvement in the transition from single sex to co-education, particularly at the primary school level, there were no significant changes in boys' ego orientation goals.

Gender differences have long been of concern in mathematics education (Forgasz, 1997). Numerous studies have investigated boys' and girls' achievement, participation rates, motivation, self concept, self esteem, confidence, goal orientations, attributional style, reactions to success and failure in mathematics, as well as a range of other teacher and classroom variables (McLeod, 1992). Several societal, contextual, affective and cognitive variables have been identified that interact with gender to influence the mathematics learning outcomes of boys and girls (Barnes & Horne, 1996: Leder, Forgasz & Solar, 1996). Single sex and co-educational contexts of learning have also been considered, including the introduction of single sex mathematics classes within co-educational systems and the transition from single sex education to co-education. However, in these contextual and transitional studies, boys' motivational mathematics goal orientations have not been investigated.

Motivational Goal Orientations in Mathematics

"Motivation is a theoretical construct used to explain the initiation, direction, intensity and persistence of behaviour, especially goal-directed behaviour" (Brophy, 1998, p.3). Goal orientation theory posits a relationship between beliefs about the causes of school success, and students' engagement and persistence in academic learning. Students' espoused goals affect the quality of their motivation (Dweck, 2000), which in turn influences behavioural, cognitive and affective outcomes (Urdan, 1997). Task involvement and ego orientation goals are important motivational constructs in mathematics, as they reflect students' reasons to achieve mastery of the subject matter, to be competitive with their fellow students or to do both (Nicholls, Cheung, Lauer & Patashnick, 1989; Nicholls, Cobb, Wood, Yackel & Patashnick, 1990).

Students with task involvement goals strive to achieve understanding and gain mastery in their learning, while students holding ego oriented goals focus on their performance relative to others (Duda & Nicholls, 1992). Ego orientated students need to appear successful, to be better than others and to avoid failure in the classroom. They are motivated only when their performance is being evaluated and they choose tasks and expend effort accordingly (Stipek, 1996). They are less likely to attempt difficult tasks, as the need to expend effort is tantamount to an admission of a lack of ability (Covington &

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 714-721). Sydney: MERGA. ©2002 MERGA Inc.

were collected from all boys in Grades 3 to 10 in the fourth term of that year (T1) and from the same boys for the following two years (T2 and T3 respectively). At T2 the boys were in Grades 4 to 11 and at T3, Grades 5 to 12. Table 1 presents the numbers participating and their respective grades at T1, T2 and T3.

Table 1
Number of Boys by Grade Level at Time 1, Time 2 and Time 3

Grs	3,4,5	4,5,6	5,6,7	6,7,8	7,8,9	8,9,10	9,10,11	10,11,12	Total
T1	44	37	42	40	67	76	63	73	442
T2	33	40	37	56	59	65	55	65	410
T3	42	39	38	38	49	64	53	50	373

There is some variability in the student numbers over the three year period. Student ID numbers used for data entry in SPSS (Norusis, 1993) were generated from school class lists at T1. Some students were absent at T1, T2 or T3 or may have left the school altogether at T2 or T3. There are, however, sufficient numbers of students at each grade level on each occasion for the statistical analyses to be conducted.

The task involvement and ego orientation dimensions of student goal orientations towards mathematics were measured with *Your Feelings in Mathematics: A Questionnaire* (Yates, 2000b: Yates, Yates & Lippett, 1995), a 25 item variant of the *Motivation Orientation Scales* (Nicholls, Cobb, Wood, Yackel, & Patashnick; Duda and Nicholls, 1992). Student competitiveness was sampled by six ego orientation items while the extent to which they were interested in and engaged in learning mathematics was measured by fifteen task involvement goal items. The questionnaire also contained four filler items in random order. Each item commenced with the stem *Do you really feel pleased in maths when* ... followed by a statement that related to student mathematics behaviour. Typical task involvement items were: (Item 1) *you get really busy with the work* and (Item 15) *something you learn makes you want to find out more*. Ego orientation in mathematics items included: (Item 12) *you are the only one who can answer a question*; and (Item 23) *you score better on a test than others*. Students rated their motivation towards mathematics on a five point scale ranging from (1) a *strong no* to (5) a *strong yes*.

Procedure

Your Feelings in Mathematics: A Questionnaire (Yates, Yates & Lippett, 1995) was administered to the boys either by their classroom teachers or by a research assistant on the same day in the fourth term of each year of the study.

Analyses

Factor analysis of *Your Feelings in Mathematics: A Questionnaire* indicated two separate subscales of *Task Involvement* (TK) and *Ego Orientation (*EGO) (Yates, 1997; Yates & Yates, 1996). TK and EGO were calibrated separately with the Rasch scaling procedure (Rasch, 1966) to bring each to a logit scale with interval properties, which then allows for comparison between grade levels as well as across time. Thirteen TK and five EGO items met item response theory requirements, with misfitting items 20 and 21 deleted from the TK scale and item 14 from the EGO scale.

Omelich, 1979). They are also more likely to believe that ability is fixed (Dweck, 2000; Pintrich & De Groot, 1990). By early adolescence some students have begun to purposefully withdraw effort and avoid seeking help in mathematics (Turner, Midgley, Meyer, Gheen, Anderman, Kang & Patrick, 2002). By contrast, task involved students espouse learning goals which motivate them to learn, improve, seek challenges, persist in the face of difficulty and to focus on mastery of the topic or task (Nicholls & Miller, 1984). They are also more likely to believe that ability is incremental (Dweck, 2000) and to seek appropriate assistance (Butler & Neuman, 1995). In a recent longitudinal study boys were found to have less task involvement and lower achievement in mathematics than girls. (Yates, 2000a; 2002).

Single Sex Education and Co-Education

Differences have been found in boys' and girls' ratings of their achievement (Foon, 1988), attitudes and beliefs (Forgasz, 1995) in mathematics in single sex and co-educational settings. A substantial research literature suggests that in co-educational classrooms boys receive more academic support from teachers, different patterns of teacher feedback and different forms of assigned academic tasks than girls (Good, Nichols & Sabers, 1999). Recent studies of single sex and mixed sex mathematics classrooms in co-educational secondary schools found both teachers and students perceived single sex classes to be generally less supportive for boys than girls (Rennie & Parker, 1997; Jackson & Smith, 2000). However, other evidence has indicated that school type does not benefit either boys or girls once other variables are controlled (Marsh, 1989; Marsh & Rowe, 1996).

No significant differences in boys' and girls' mathematics' achievement were found during and after the merger of two single sex schools into two co-educational secondary schools (Marsh, 1989; Marsh, Smith, Marsh & Owens, 1988). A small decrease in self-concept was evident for students in co-educational classes in the transitional year, but overall multidimensional self-concepts increased in the co-educational setting. In a similar study, girls from a single sex secondary school maintained positive attitudes towards mathematics following amalgamation with a single sex boys' school, but reported feeling intimidated, hesitant, uncomfortable and dumb during mathematics lessons in the mixed sex classrooms (Steinbeck and Gwizdala, 1995). They also perceived that teachers gave more attention to the boys. Perceptions of disadvantage within co-educational settings have been mirrored by girls in other studies (Milligan & Thomson, 1992; Parker & Rennie, 1995; Forgasz & Leder, 1996; Gill, 1996). However, boys' and girls' goal orientations in mathematics have not been considered in studies of single sex and co-educational school contexts.

Aim

To investigate, through a longitudinal design, the goal oriented motivational beliefs of primary and secondary school boys during the transition from single sex education to co-education.

Method

The study, conducted over three consecutive years, commenced in the year prior to the introduction of co-education to a non-government single sex boys' school in South Australia. The school was located on two separate primary and secondary campuses. Data

Students' responses to the TK and EGO scales were equated concurrently across the three occasions as this yields stronger case estimates than common item difference or anchor item equating methods (Mohandos, 1996). Rasch case estimate scores were used in the analysis of variance (ANOVA) (Norusis, 1993), and Hierarchical Linear Modelling (HLM5) (Raudenbush, Bryk, & Congdon, 2000) procedures.

Results

Results are examined in terms of grade level and group differences in Task involvement and Ego orientation in the context of single sex education at T1, in the transition to co-education at T2 and following co-education at T3. Figures 1 and 3 present TK and EGO across Grades 3 to 12 at Time 1, 2 and 3 respectively, with Figures 2 and 4 presenting TK and EGO respectively for each Grade level cohort (group) across the three years of the study.

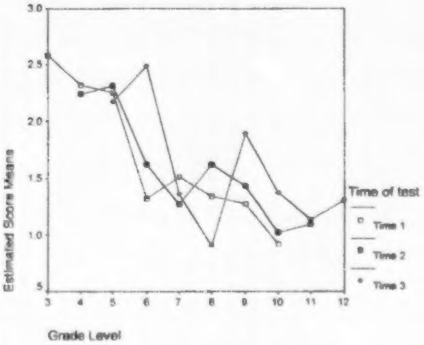

Figure 1. Task involvement at time 1, 2 & 3.

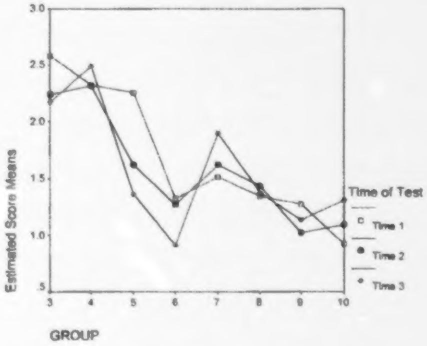

Figure 2. Task involvement by grade cohort.

Table 2

Significant Coefficients for Intercepts for Task Involvement by Grade at T1, T2 & T3

	Coefficient	Std Error	T-ratio	Approx d.f.	p-value
Grade 3	1.41	0.32	4.38	5	0.008
Grade 4	1.07	0.22	4.77	5	0.004
Grade 5	1.00	0.15	6.45	5	0.000
Grade 9	0.40	0.133	3.00	5	0.032

Table 3

Significant Coefficients for Intercepts for Task involvement by Grade Cohort

	Coefficient	Std Error	T-ratio	Approx d.f.	p-value
Grade 3	1.23	0.16	7.19	4	0.000
Grade 4	1.17	0.16	7.30	4	0.000
Grade 5	1.29	0.34	3.8	4	0.032

Task Involvement

In Figures 1 and 2 there is an overall pattern of a decrease in TK across the grade levels for all three years of the study. However, the impact of the transition to co-education is quite variable across the school both in relation to the Grade levels from T1 to T2 and T3 in Figure 1 and the Grade cohorts in Figure 2. Univariate ANOVA confirmed these Grade level differences ($\bullet^2 = 35.37$, $F = 17.26$, $p < 0.000$), and differences between Grade cohorts as significant ($\bullet^2 = 26.97$, $F = 13.16$, $p < 0.000$). With respect to the differences between the grades at Times 1, 2 and 3 presented in Figure 1 there was also an interaction between Grade and time ($\bullet^2 = 4.29$, $F = 2.10$, $p < 0.015$).

However, the data in this study is hierarchical in nature, with the boys clustered each year within different mathematics classes and grade levels across the two campuses of the school. Some students also changed from one campus to the other during the course of the study. Hierarchical Linear Modelling (HLM5) was therefore used to take the cross-level nature of the data into account (Osborne, 2000). Table 2 presents the significant effects for TK in the HLM5 analyses for the intercepts for Grades 3, 4, 5 and 9. There is also a significant interaction effect for the time slope for Grade 6 ($\beta = 0.33$, T-ratio = 5.33, $p < 0.000$). Significant effects for the intercepts for the Grade 3, 4 and 5 cohorts for TK are presented in Table 3. There are also significant interaction effects for the time slopes for Grade 5 ($\beta = -0.37$, T- ratio = -2.46, $p < 0.056$) and Grade 7 ($\beta = 0.21$, T-ratio = 3.44, $p < 0.024$).

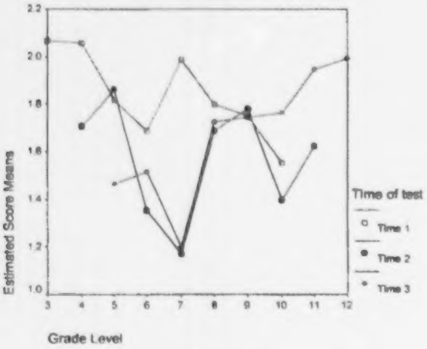

Figure 3. Ego orientation at time 1, 2 & 3.

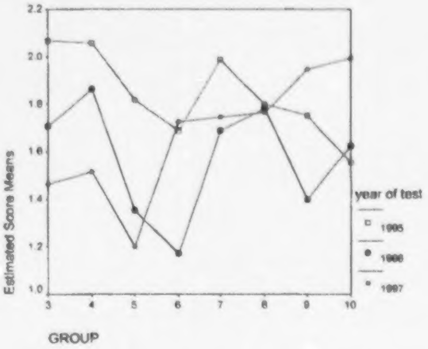

Figure 4. Ego orientation by grade cohort.

Ego Orientation

There are no significant effects in either the ANOVA or HLM analyses for EGO. However, in Figures 3 and 4 there is an overall trend for EGO to change with the introduction of co-education, with the exception of Grade 9 in Figure 3, which barely alters from T1 to T2 and T3. Analyses of the cohort effects in Figure 4 indicate a general reversal of levels of competitiveness across the school from T1 to T3. In general lower grades record higher levels of EGO at Time 1, with these levels of EGO decreasing across the school grades. In the year following co-education at T3, the lower grades reveal decreases in their levels of competitiveness, while the higher grades record corresponding rises in EGO. In Figure 4 there is an interesting decrease in EGO for Grade 6 during the transition year of T2, but this is reversed at T3 to almost the identical value recorded at T1 prior to the introduction of co-education. However, all of these trends are not statistically significant.

Discussion

Motivation encompasses students' subjective experiences, particularly their willingness to engage in lessons and learning activities and the reasons they cite for doing so (Brophy, 1998). In this study motivation was measured directly in primary and secondary mathematics classrooms from the perspectives of boys' views of their own task or ego goal directed motivation immediately prior to, in the transitional year and in the year following the introduction of co-education to a single sex boys' school. These constructs of task involvement and ego orientation were chosen as they directly tapped boys' expressed goals for their mathematics learning during a period of significant change within their school.

Results show that there was considerable variability within and across the grades in boys' task involvement and ego orientation in mathematics in the transition from single sex education to co-education. HLM analyses of both Grade level differences at Time 1, 2 and 3 and cohort effects indicate that the transition process had a significant effect on boys' task involvement goals in the primary grades 3, 4, 5, and 6 and middle school grades 7 and 9, but not on their ego orientation. During the course of this study a number of other changes took place across the school. In the same year that co-education was introduced, Junior Primary classes from Reception to Grade 2 began on the primary school campus. A Middle School program embracing Grades 7 to 10 also commenced. Students in Grades 3 to 6 remained at the primary school campus, but the Grade 6 cohort were shifted to the Middle School which was established at the senior school campus. These moves had the effect of giving the Grade 4 cohort from Time 1 the status of being the senior students in the primary school at Time 3. However, the Grade 6 cohort from Time 1 were denied their final year in the primary school setting at Time 2 where they would have customarily enjoyed certain responsibilities and privileges as the "senior" primary students.

The trend for task involvement to decrease as students move through the grades has been reported elsewhere (Yates, 2002), particularly in relation to early adolescence (Turner, Midgley, Meyer, Gheen, Anderman, Kang & Patrick, 2002). However, these findings would suggest that conclusions from previous studies that secondary school boys fared less well in single sex contexts (Rennie & Parker, 1997; Jackson & Smith, 2000) and were advantaged in co-educational settings (Good, Nichols & Sabers, 1999) can be extended to include boys at the primary school level. Results from this study take into account the hierarchical, nested nature of the student sample, as well as particular situational and environmental variables that may be operating at the class and school levels. Previous studies of single sex and co-educational contextshave not incorporated cluster sample designs.

In comparison with the single sex classes at T1, the results indicate that the boys generally became less ego oriented in the co-educational settings at T2 and T3. The decrease in competitiveness was particularly evident in the primary school grades and in the period of early adolescence when peer relationships are paramount. Although these trends are not statistically significant, they are nevertheless interesting as they substantiate findings from a previous study in which primary and secondary students' perceptions of competitiveness decreased during and following the transition to co-education in a single sex boys' school (Yates, 2001). They also support studies of the introduction of single sex classes in mathematics within co-educational high schools (Parker & Rennie, 1995; Rennie & Parker, 1997) in which teachers perceived boys, in absence of girls, to be unruly, disruptive, apparently unwillingly and unable to work co-operatively with each other. Teachers also observed greater co-operation in co-educational settings in high schools, with less able boys in particular more likely to request and receive help from girls (Parker & Rennie, 1995; Rennie & Parker, 1997).

The findings from the present study in relation to the reduction in competitiveness with the introduction of co-education would suggest the need for future studies to compare differences between competitive and co-operative learning in mathematics in single sex and co-educational settings at the primary school level. Differences between boys and girls' task involvement goal orientations could also be considered. The problem of the relatively small number of ego items in *Your Feelings in Mathematics: A Questionnaire* could also be addressed.

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Validation of the New Zealand Number Framework

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This paper reports the patterns of performance of almost 10,000 Year 4 to 6 students whose teachers participated in a project (ANP) designed to improve their professional knowledge, skills, and confidence. An important component of the project was the number framework, informed by work with *Count Me In Too*, and extended to cater for students at Year 4 and beyond. Data collected at the beginning and end of the project provided validation for the upper levels of the framework. A hierarchical sequence of development was evident, with additive part-whole thinking acquired before multiplicative part-whole thinking, which came before proportional reasoning. Support was also found for the idea that knowledge about numbers provides a necessary foundation for strategies.

It now seems to be generally accepted that national initiatives in mathematics education need to be based on sound research. In the UK, the Numeracy Taskforce claimed that its National Numeracy Strategy was research-based, and this claim was carefully scrutinised by Margaret Brown and colleagues (1998). In Australia, researchers involved in the development of numeracy initiatives have documented the extensive research base underpinning the initiatives. For example, Wright and Gould (2000) have provided a comprehensive account of the research behind *Count Me In Too* in NSW, while Clarke and his colleagues (2000, 2001) have done a similar exercise for the Early Numeracy Research Project (ENRP) in Victoria.

In 2000, New Zealand began developing a national number framework. The number framework was developed to give teachers a way of describing students' attainment on the basis of their number knowledge and problem-solving strategies. The framework has two main components: Strategy and Knowledge (see Ministry of Education 2001a, b). The Strategy component focuses on *how* students solve number problems, and the extent to which they use mental processes as part of their solution strategies. The Knowledge component encompasses key items of knowledge about the number system, including the identification and ordering of whole numbers and fractions, as well as grouping by tens. The two components are seen as interdependent, with Strategy creating new knowledge through use, and Knowledge providing the foundation upon which new strategies are built.

At the lower end of the framework are the counting stages, beginning with Emergent (a stage at which students are unable to count consistently), progressing through a series of increasingly sophisticated counting-all stages (One-to-one Counting, Counting from One on Materials, Counting from One by Imaging), to the highest counting strategy – Advanced Counting that incudes counting on and back, and skip counting. The lower end of the framework was informed by a pilot project with *Count Me In Too*, and is based on the work of researchers such as Les Steffe and Bob Wright (see Steffe & Cobb, 1988; Wright & Gould, 2000). The lower end of the framework focuses particularly on the transition from increasingly sophisticated counting strategies to using knowledge about the additive composition of numbers (ie, part-whole strategies). The first of the part-whole strategies, Early Additive Part-Whole, corresponds roughly to the Facile Number Sequence stage in *Count Me In Too*, and to the Derived Facts stage in the addition and subtraction framework

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of *Cognitively Guided Instruction* (Carpenter, et al., 1999). Part-whole strategies then become increasingly complex and more powerful as students progress from additive to multiplicative thinking, and then from multiplicative to proportional reasoning. The upper (part-whole) stages of the number framework were developed from the work of several internationally recognised mathematics educators (e.g., Behr et al., 1994; Confrey & Harel, 1994; Lamon, 1994; Pitkethly & Hunting, 1996), but unlike the lower (counting) stages, the upper (part-whole) stages have not been validated empirically on any large scale.

In 2001, the number framework became the basis for several initiatives, including professional development programmes for teachers at Years 1 to 3 (5-8 yr olds; Early Numeracy Project: ENP) and Years 4 to 6 (8-11 yr olds; Advanced Numeracy Project: ANP). Exploratory work with teachers at Years 7 to 10 (11-15 yr olds; Numeracy Exploratory Study: NESt) also began. The Advanced Numeracy Project provided an opportunity to examine the internal consistency of the framework, particularly at the upper levels. Data was gathered by teachers using individual diagnostic interviews with each child at the beginning and end of the project. Data gathered at the beginning of the project would have been affected little, if at all, by the teaching approaches taken in the project. On the other hand, teachers were inexperienced with using the framework and assessment tasks at this stage, and this may have affected the reliability of the data. Data gathered at the end of the project may have been more reliable than the initial data, but the teaching approaches taken in the project may have had some influence. For this reason, data from both the initial and final assessments was analysed to check for consistency.

This paper reports on the analysis of data gathered at the beginning and end of the project, and explores the interrelationships among different aspects of the framework.

Method

Participants

Just under ten thousand students (9964) from across New Zealand were assessed at the beginning of ANP. There were approximately equal numbers of girls and boys. The majority of students were of European descent (61.6%), approximately one fifth were Maori (19.5%), and the others consisted of Pacific Islands (10.2%), Asian (5.8%), and other ethnic groups (3.0%). Most of the students attended schools with catchment areas serving families of low (38.8%) or medium (40.3%) socio-economic status. Only one fifth (20.9%) of the participants came from schools in the high socio-economic decile band (deciles 8 to 10). Over the course of the project, just under a fifth of the initial sample (18.8%) were lost from the study.

Procedure

Each child was assessed using the Advanced Numeracy Project Assessment (ANPA) interview at the beginning and end of the project (see Ministry of Education 2001b). The questions and a brief description of the characteristics of each aspect of Strategy and Knowledge for levels 3 to 8 is presented in Appendix A and Appendix B. In assigning strategy stages, teachers focused on how the student solved the given problems rather than only attending to the correctness of answers.

Results

Particular subgroups of students were selected and their performance on the various Strategy and Knowledge components of the framework examined (see Table 1).

Table 1

Percentage of Children at Particular Levels on the Framework by Level and Component

	Strategies				Knowledge		
Level	A/S	M/D	Ratio	Level	WNIDS	FRIDS	Group
Advanced Proportional Thinkers at the Start of ANP (n = 43)							
3				3		....	
4	2.3			4			4.7
5	9.3	2.3		5	4.7	16.3	11.6
6	88.4	23.3		6	95.3	30.2	18.6
7	....	74.4		7	....	14.0	23.3
8	....	....	100.0	8	....	39.5	41.9
Advanced Proportional Thinkers at the End of ANP (n = 176)							
3				3		....	
4				4			1.1
5	4.0			5	1.7	4.5	2.3
6	96.0	8.5		6	98.3	12.5	12.5
7	....	91.5		7	....	21.0	19.9
8	....	....	100.0	8	....	61.9	64.2
Advanced Multiplicative Thinkers at the Start of ANP (n = 424)							
3			0.2	3		....	0.7
4	1.2		4.0	4	.5	10.1	7.5
5	14.9		15.6	5	11.3	31.8	16.3
6	84.0		22.9	6	88.2	26.4	34.7
7	....	100.0	49.8	7	....	17.0	21.0
8	....	....	7.5	8	....	14.6	19.8
Advanced Multiplicative Thinkers at the End of ANP (n = 1098)							
3			0.1	3		....	
4	.1		1.2	4		0.7	1.1
5	6.2		8.6	5	3.1	13.9	10.8
6	93.7		24.2	6	96.9	32.8	28.5
7	....	100.0	51.3	7	....	25.5	23.0
8	....	....	14.7	8	....	27.0	36.5
Advanced Additive Thinkers at the Start of ANP (n = 1569)							
3		1.0	3.8	3		....	1.6
4		6.4	13.8	4	1.9	22.8	18.7
5		27.5	33.4	5	22.9	41.9	29.3
6	100.0	42.4	25.0	6	75.2	21.6	31.6
7	....	22.7	21.5	7	....	8.9	10.5
8	....	....	2.4	8	....	4.8	8.3
Advanced Additive Thinkers at the End of ANP (n = 3004)							
3		0.1	0.8	3		....	0.1
4		2.5	7.4	4	0.3	2.7	5.3
5		20.4	28.6	5	12.1	31.8	23.8
6	100.0	42.7	30.8	6	87.6	36.2	36.7
7	....	34.3	26.9	7	....	17.4	16.4
8	....	....	5.6	8	....	11.9	17.8

The analysis began at the highest strategy level, focusing on those students who were classified by their teachers as Advanced Proportional thinkers, progressing though Multiplicative and Additive stages. The relationship of Knowledge to Strategy was explored by examining the performance of students who had low levels of Knowledge to see whether this limited the type of strategy that was available to the students.

Advanced Proportional Thinkers

Only a very small number of students at Years 4 to 6 were categorised as Advanced Proportional (0.4% initially, and 2.2% finally). If the Advanced Proportional stage builds on the Additive and Multiplicative stages, then the students at this top level should be proficient with addition/subtraction and multiplication/division. At both time points and for both aspects, most of the students were at the highest possible stage (see Table 1). A small number were operating just one stage below this. Measurement error may explain why one child was initially judged as two steps below the highest stage for Addition/Subtraction and for Multiplication/Division. The Advanced Proportional thinkers performed well on tasks designed to assess their Knowledge. All were judged to be at level 5 or above on Identification and Ordering of Whole Numbers and Fractions at both time points. All were at level 5 for Grouping, with the exception of two students (initially & finally) who needed to count by tens to 100 in order to find the number of tens.

Advanced Multiplicative Thinkers

A small number of students (4.3%) were classified initially as Advanced Multiplicative, but this had increased to 13.6% by the end of the project. Virtually all of the Advanced Multipliers were at the Advanced or Early stage for Addition/Subtraction (see Table 1). A small number (five initially, one finally) were classified as being at the Advanced Counting stage. The majority of the Advanced Multipliers were at level 5 or above on the various Knowledge domains. An exception to this was the 10.1% of students who could not identify unit fractions at the beginning of the project. However, knowledge of fractions was not necessary for the multiplication and division tasks the students were given.

Advanced Additive Thinkers

Only 15.7% of students were classified initially as Advanced Additive, but this had increased to more than a third (37.1%) by the end of the project. Virtually all of these students (98.1% initially, 99.7% finally) could identify and order numbers to 1000, and at least three quarters (75.2% initially, 87.6% finally) could do this with numbers to a million (see Table 1).

Students who use Counting Strategies for Addition/Subtraction

Table 2 shows the percentages of children who used counting strategies to solve addition and subtraction problems. Almost half (46.8%) of the students used counting strategies (either Count All or Count On) to solve addition problems initially. This had more than halved (20.8%) by the end of the project. The majority of these students used counting strategies to solve multiplication, division, and fractional problems. Of the students who used higher level strategies, virtually all used simple adding strategies (i.e., level 5).

Table 2

Percentage of Children who use Counting Strategies or can't Find Ten for Addition by Level and Component

		Strategies				**Knowledge**	
Level	**A/S**	**M/D**	**Ratio**	**Level**	**WNIDS**	**FRIDS**	**Group**
Children who Count All to solve Addition Problems at the Start of ANP (n = 698)							
3	100.0	75.8	76.4	3	21.8	....	44.0
4		21.1	18.8	4	54.9	86.7	50.7
5		3.0	4.7	5	20.8	12.6	4.7
6		0.1	0.1	6	2.6	0.7	0.6
7	....			7	....		
8	....	....		8	....		
Children who Count All to solve Addition Problems at the End of ANP (n = 131)							
3	100.0	73.3	64.9	3	19.8	....	39.7
4		24.3	27.5	4	56.5	62.6	51.9
5		2.3	7.6	5	23.7	34.4	8.4
6				6		3.1	
7	....			7	....		
8	....	....		8	....		
Children who Count On to solve Addition Problems at the Start of ANP (n = 3966)							
3		19.1	27.6	3	1.9	....	9.9
4	100.0	61.6	45.5	4	37.3	71.3	71.0
5		16.5	25.0	5	48.9	26.8	16.5
6		2.6	1.7	6	11.9	1.8	2.3
7	....	0.1	0.1	7	....	0.1	0.2
8	....	....	0.0	8	....	0.0	0.1
Children who Count On to solve Addition Problems at the End of ANP (n = 1550)							
3		11.8	18.2	3	1.0	....	3.0
4	100.0	64.3	49.3	4	30.3	35.6	63.9
5		21.7	30.3	5	54.8	57.4	28.5
6		2.1	2.0	6	13.9	6.2	4.1
7	....	0.7	0.3	7	....	0.7	0.4
8	....	....		8	....	0.7	0.1
Children who can't Find Tens in Numbers to 100 at the Start of ANP (n = 823)							
3	37.3	57.4	60.9	3	19.7	....	100.0
4	47.5	31.1	27.5	4	44.7	92.1	
5	12.2	9.0	10.1	5	28.6	7.3	
6	3.0	2.2	1.2	6	7.0	0.6	
7	....	0.4	0.4	7	....		
8	....	....		8	....		
Children who can't Find Tens in Numbers to 100 at the End of ANP (n = 113)							
3	46.0	60.2	57.5	3	24.8	....	100.0
4	40.7	36.3	31.0	4	50.4	72.6	
5	11.5	2.7	10.6	5	23.0	26.5	
6	1.8	0.9	0.0	6	1.8	0.9	
7	....		0.9	7	....		
8	....	....		8	....		

Students with Limited Knowledge about Numbers

An analysis was done of students who were unable to find tens in numbers to 100 (8.3% initially, 1.4% finally). Most (85% or more) relied on counting strategies to solve problems involving operations (see Table 2). Of the few who did use a part-whole strategy to solve these kinds of problems, virtually all were at the initial stage (level 5).

Discussion

The analysis gives strong support to the idea that the complexity of unit structures involved in number problems is the most significant indication of difficulty. Students appear to acquire strong control of additive unit structures before getting strong control of multiplicative structures then, in turn, proportional structures. The clarity of this pattern was surprising given the nature of the problems posed. In designing questions, the variables of number size and solution steps were constrained. This was done for two reasons. Firstly, students were asked to solve the problems mentally. Secondly, the problems were designed to be readily accessible to students who understood the unit structure. For example, to be classified as being at the Advanced Additive Part-Whole stage for multiplication and division, students needed to solve one of the following problems by deriving the answer from the given result:

$17 \times 6 = 102$ so $18 \times 6 = ?$ $\quad$ $27 \times 2 = 54$ so $27 \times 4 = ?$

In each case the strong control of additive structures was not a prohibitive factor in students solving the problems. $102 + 6 = 108$, and, $54 + 54 = 108$, are easy mental calculations. Yet only 2% of Advanced Counting (Counting on) students, and 22% of Early Additive Part-Whole (Derived addition facts) students could connect the given multiplication result with the unknown (final assessment). By contrast, 77% of Advanced Additive students could do so.

It appears that students who employ counting strategies on simple addition and subtraction problems apply the same strategies to multiplication, division, and fractional number problems. Early Additive Part-Whole students apply their preferred unit structures to multiplication, division, and fraction problems in the form of repeated addition strategies.

The findings provide substantial support for the hierarchical organisation of the Number Framework, with multiplicative thinking building on additive thinking, and proportional reasoning building on multiplicative thinking. The data also support the idea that having some minimal level of knowledge about numbers is a prerequisite for the development of part-whole strategies. Knowledge of the rudimentary place value conventions of "-teen", and "-ty" numbers (e.g., $10 + 4 = 14$, $6 \times 10 = 60$) seems critical to the development of early additive part-whole strategies. Further trialing in 2002 will provide more data on the power of the framework hierarchy as a predictive instrument of students' strategies across a range of problems, and also show the validity of the framework as a pedagogical guide to teachers.

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Appendix A

Questions & Criteria to be met at each level for selected components and Characteristics of the Framework

Ratio

8 *It takes 10 balls of wool to make 15 mittens. How many balls of wool does it take to make 6 mittens* (correct); or *There are 21 boys and 14 girls in Anna's class. What percentage of Ana's class are boys?* (correct)

7 *Of every 8 apples in the box, 3 are bad. There are 40 apples in the box. How many are bad?* (correct); or *There are 28 beans under the card. You need to take 3/4 of them. How many is that?* (solved using division & multiplication)

6 The 3/4 of 28 beans problem described above solved using addition and multiplication

5 *Here are 12 beans. You have to get one half of them. How many should you take?* (solved using addition facts)

Multiplication & Division

7 *There are 24 muffins in each basket. How many muffins are there altogether?* (solved using the distributive property or compensation); or *At the car factory, they need 4 wheels to make each car. How many cars can they make with 72 wheels?* (solved using the distributive property or compensation)

6 *17 x 6 = 102 so what does 18 x 6 equal?* or *27 x 2 = 54 so what does 27 x 4 equal?* (solved by deriving from known multiplication facts)

5 *Here is a field of cows. There are 5 cows in each row and there are 6 rows. How many cows are there in the field altogether? Here are 15 more cows. If they join the others in the rows of 5, how many rows will there be then? How many cows will that be altogether?* (solved using repeated addition)

Addition & Subtraction

6 *There are 53 people on the bus, and 26 people get off. How many people are left on the bus?* (solved using a part-whole strategy)

5 *On this page there are 15 birds and 27 cats. How many animals is that altogether?* or *I have 9 beans under this card and another 7 beans under here. How many is that altogether?* (2 bags of 10 beans added) *Now I have 29 beans under here and there are still 7 under here. How many is that altogether?* (solved using standard place value partitioning, adjusted doubles, or bridging through 10)

Grouping

8 *Here are some decimal numbers* (3.2, 1506.9). *How many tenths are in each number?*

7 *Suppose you had to make this much money* ($7 815, $253 000) *using only $100 notes. How many notes would you get?*

6 *Suppose you had to make this much money* ($60, $230, $4 520, $82 600) *using only $10 notes. How many notes would you get?*

Appendix B

Characteristics of each Aspect of the Framework for Levels 3 to 8

Strategies

Level	Addition/Subtraction (A/S)	Multiplication/Division (M/D)	Ratio
3	Counts all from one	Counts all from one	Uses equal sharing of objects
4	Counts On	Skip counts	Uses equal sharing of objects
5	Uses limited range of P/W strategies	Uses repeated addition	Uses addition facts
6	Uses full range of P/W strategies for add/sub'n	Derives from known multiplication facts	Uses multiplication & addition facts
7	……	Uses full range of P/W strategies for mult/div'n	Uses multiplication & division facts
8	……	……	Uses full range of P/W strategies for fractions

Knowledge

Level	ID & Order Whole Nos. (WNIDS)	ID & Order Fractions (FRIDS)	Grouping/Place Value (Group)
3	……	……	……
4	IDs & orders numbers to 100	……	Finds tens by skip counting
5	IDs & orders numbers to 1000	IDs unit fractions	Finds tens by using 10 tens = 100
6	IDs & orders numbers to a million	IDs decimals & orders unit fractions	Knows tens in any whole number
7	……	Orders decimals & nonunit fractions	Finds 10s & 100s, finds 10ths by ten 10ths = 1
8	……	Orders fractions, decimals & percentages	Finds 10ths in decimal numbers

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Short Communication Abstracts

Children's Conceptual Understanding of Decimal Fractions

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This paper reports three case studies of Year 5 students drawn from a longitudinal study of students' multiplicative reasoning. The analysis shows wide differences in students' conceptual understanding and strategy use across tasks involving decimal fractions. The high ability student demonstrated a strong link between decimal understanding and the base 10 numeration system. By comparison the lower ability student showed conceptual difficulties with decimals based on inappropriate utilization of base 10 material. The average ability student displayed conceptual confusion based on her treatment of decimals as whole numbers.

The Impact of the Intermediate Number Project on Teacher Classroom Practice - A Case Study

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This session discusses work in progress that aims to monitor the impact of the Intermediate Number Project on the classroom practice of year 7 and 8 teachers in a New Zealand Intermediate School. The levels of the teachers' mathematical content knowledge, pedagogical content knowledge, strategic thinking, the availability and effective use of equipment and grouping versus whole class teaching are some of the issues to be considered. Preliminary research results related to the impact of the Early Number Project on teachers' classroom practice are encouraging, but will this be as effective at the upper ends of the primary school system? The ability of a school and individual teachers to implement the approaches inherent in the numeracy projects independently is to be explored in this research.

The Role of Mathematical Processes: Issues for Pacific Island NESB Students in New Zealand.

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This paper reports on a study that investigated the experiences and perceptions of a group of Pacific Island students whose first experience of studying mathematics in New Zealand came in a university bridging programme. The importance given to problem-solving, logical reasoning, and communication was quite different from their experiences in their home countries. The study indicated some difficulties that the mathematical processes approach posed for newcomers adapting to the New Zealand system, and especially for newcomers from non-English-speaking backgrounds (NESB).

Does the Use of Spreadsheets in Year 4 Number Programmes Enhance Children's Numeracy Development?

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Little is known about how integrating ICT actually enhances the learning process. This study investigated the link between integrating spreadsheets into mathematics programmes and the development of numeracy. While a correlation between the two is difficult to establish, significant development occurred and insights into the way spreadsheets facilitate that development were gained.

A Review of Textbooks in Integrating The Use of Technology in Teaching Mathematics: The Case of Hong Kong

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A new secondary mathematics curriculum emphasizing the use of technology was introduced in Hong Kong in September 2001. This study summarizes how technology is used in textbook materials for the new curriculum. The study found that when use of technology is mandated in the curriculum technology can be used simply as a marketing initiative, without much relevance to pedagogy. It is noted that support of teachers in a technologically rich work environment becomes the greatest concern of teachers.

Rethinking Curriculum: A Values Perspective

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The most important issue for practising teachers is what content will they be teaching. Curriculum developers also concentrate most efforts on this issue. But policy makers speak more often about long term goals including values, but then revert to very specific cognitive skills that they think students should be able to perform. Interesting questions are why teachers don't seem to concern themselves with values, and why policy makers seem to think there is a direct and immediate linkage between values and immediate skill type cognitive behaviour. These questions will be considered in the light of the ongoing work of the Values and Mathematics Project (VAMP), which has been discussed at earlier MERGA conferences.

A Learning Support Program for First Year Tertiary Mathematics Students

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This short communication describes the work in progress at QUT's Mathematics Access Centre (MAC). The rationale of the project is to provide first year students of mathematics with face-to-face programmes that support their learning needs. This is done with weekly enabling tutorials, examination preparation workshops, and a drop-in centre. Approximately 1000 students are eligible for MAC assistance. Initial data which will be presented suggest that MAC programmes are having a positive impact on qualitative and quantitative learning measures.

Development of Effective Web Materials for the Mathematics Learning Centre

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This paper seeks to explore issues inherent in the development of effective web materials and the manner in which these have been addressed at the Mathematics Learning Centre (MLC) at Central Queensland University (CQU). The MLC is a support unit which provides assistance to students who are experiencing difficulty with the mathematics or the quantitative component of their course at CQU. The paper also presents a number of guiding principles for the design and development of web-based study materials and their implementation by the MLC for its students.

Learning Multiplication in Year 5 and 6 Classrooms

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Multiplicative thinking represents a qualitative change in children's understanding of number and operations. We know how a child's thinking changes in terms of their approach to problems. We also know that children's increasing sophistication in strategy use does not follow a linear pathway. This research aims to trace the emergence of multiplicative thinking through an examination of the classroom experience of children. Initial studies have been undertaken with teachers and children. The next phase of the research brings these two together to examine the dynamics of teaching and learning.

Working Mathematically: A Transatlantic Comparison

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Part of the presenter's PhD research involved examining curriculum documents from England and America. Two, which were chosen because of their international prominence, were the Mathematics: the National Curriculum for England and the American Principles and Standards for School Mathematics,(NCTM). The focus of this paper is the way in which the processes of mathematics learning and thinking are described and incorporated into the two curricula. Comparisons of the two documents revealed a very different rationale behind the writing of each document and the need for further research into what is meant by "working mathematically".

Enactivism, Post-Modernism and Curriculum

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Traditionally, curriculum dialogue revolves around consideration of aims/intentions, content, teaching methods and assessment. These concepts are interpreted within a modernist paradigm that objectifies knowledge, privileges product over process and separates self from other and environment. Enactivism, a learning theory embracing post-modernist thought, sees knowing as (*emergent*) process and as embodied and embedded within broader contexts. Self, other and environment are co-defining and co-emerging and can be considered as 'open' (*transforming and creating*) systems. From this perspective, the curriculum dialogue necessarily changes. This paper explores the implications, for curriculum dialogue, of an enactivist view.

Adult Students Learning Mathematics – Some Case Studies

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As part of a larger study, 11 adult students studying in a Bridging Mathematics course were interviewed on three occasions about their experiences and feelings. The interviews were early in the course, towards the end of the course, and the following semester. Their interviews were analysed looking at their past experiences of learning mathematics in schools, issues of confidence, their experiences learning mathematics again as adult learners, and their ideas as to the usefulness of mathematical tools in real life.

An Investigation into Undergraduate Examinations

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Examinations are the main form of assessment in undergraduate mathematics and statistics courses and examination papers tend to follow a very rigid format, but very little research has been carried out on the structure of examination papers or on their impact on learning. Recent work has focussed on ways of incorporating technology into the standard examination format making as little change as possible to the format itself. This paper demonstrates firstly how little research exists on examinations and their effects on learning and secondly how a small change in the structure of an examination can significantly alter the results. This leads to the question, "Are we testing the right things in the right way or are we just perpetuating a ritual without questioning its effectiveness?"

Teaching Multiplication of Negative Numbers

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Current approaches to teaching mathematics in New Zealand place emphasis on the use of meaningful contexts. One method of teaching multiplication of two negative numbers is described here. In a class of year 9 students a video camera was used to record the motion of a toy car travelling forwards and backwards, then the videotape replayed in both directions. Stimulated recall interviews were conducted with four of the students, whose responses are analysed with reference to the Pirie-Kieren model.

Multiplicative Skills of Year 9 Students

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The New Zealand initiative to improve numeracy skills was extended to students in years 7 - 10 (age 11-14) in 2001. Results of initial assessment and reassessment after one term of intervention showed that such a programme was needed and that it was effective for this age group. This paper focuses on the results of assessments of multiplicative scales for year 9 students, demonstrating initial results that concerned educators and a positive response of the students involved to attention to these skills.

Facilitating the Teaching of Space Mathematics: An Evaluation

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The implementation of a NSW teacher development program to improve teaching the space strand of mathematics has been evaluated in terms of students' learning and whether the teachers did indeed implement the intended program. Results indicate a successful program was established. The study identified the role of school facilitators and the comprehensive support package as effective features of the teachers' professional development. The package included the purpose for teaching space written in terms of students' expected learning with background theoretical notes, assessment tasks, lesson plans, and supporting videotapes which illustrated the theory and classroom implementation.

Gender Bias in Mathematics Textbooks at Secondary Level in Mauritius

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There are several factors that affect acquisition of mathematical knowledge at school level. It is generally held that the method of teaching mathematics through concept formation is probably the most simple technique of teaching. Our research team is proposing that using 'core constructs' as a pedagogical paradigm to teach mathematical knowledge is more useful and coherent. The relevance of core constructs is well embodied in the way we perceive the nature and structure of mathematical knowledge. This is then followed by the next overall purpose of our study which examines "Gender bias in mathematics textbooks" produced locally. Indeed our study has shown that our curriculum materials are significantly gender biased and that there is an urgent need to reconsider, rethink and rewrite mathematics curricular materials by ensuring much more fairness and balance in terms of gender notions and gender concepts.

How Crucial is a Clear Understanding of Algebraic Symbols?

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This paper presents research data, assembled from three countries, to support the view that students with better levels of understanding of the meaning of algebraic symbols are more likely to have higher degrees of success with algebraic tasks. Misunderstanding alphabetic symbols, especially by regarding them as representing objects, can severely handicap those studying the algebra of generalised arithmetic. Awareness of the importance of a numbers view of such symbols alerts educationists to take great care regarding methods used to introduce students to symbolic algebra.

Children's Mathematical Communication During Play

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The importance of language for communicating mathematically has long been recognised. This study of young children determines the purposes for which children communicate as they play and confirms the significance of mathematical language development in the early years. Playing together in early childhood settings, children communicated their understandings and experiences to each other using language that was rich in mathematical ideas, fantasy and social construction.

Calculators and Number Sense: Teachers' Responses

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This project introduced TI-15 calculators into five older primary classrooms. Classes were used where children had already undertaken work with standard algorithms for calculating. The need for children to 'learn the basics' is one reason for teachers not introducing calculators to many lower primary classrooms. This paper which reports part of a larger project noted a change in general teaching style in mathematics, more talk and exploration by children, an acknowledgment of different mathematical abilities, and a renewed enthusiasm for learning mathematics with calculators.

Development of a Number Framework

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Over the past two decades mathematics education researchers have created learning frameworks for various aspects of number. In New Zealand an attempt has been made to develop an integrated number framework that links students' mental strategies across different operational domains. This paper discusses the theoretical underpinnings for the number framework and gives reasons for the directions taken in its development.

The Influence of Language Factors on the Teaching and Learning of Equations and Inequations

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An investigation into the influence of language factors on the learning of equations and inequations in two schools in Chiang Mai, Thailand, is described. 152 Year 9 students, in four classes, were administered a *Language of Equations and Inequations Test* (LEIT) based on nine categories of language. On average, students answered 12 of the 27 LEIT questions correctly. Questions in the "graph to word" category were best answered, and those in the "word to graph" category were the most difficult.

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